**Article**

# Laplace Transform of Laguerre Associated Polynomials

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## Abstract

The Laplace operator is applied to the Laguerre equation to obtain this Transform for the corresponding associated polynomials.

**Keywords**: Laguerre polynomials, Laplace transform.

## 1. Introduction

The Laguerre polynomials $L_n(x)$ [1] verify the differential equation [2-5]:

$$x\frac{d^2}{dx^2}L_n + (1-x)\frac{d}{dx}L_n + nL_n = 0, \quad L_n(0) = 1, \qquad [\frac{d}{dx}L_n]_{x=0} = -n, \tag{1}$$

and the corresponding associated polynomials $L_n^\alpha(x)$ satisfy [2, 6]:

$$x\frac{d^2}{dx^2}L_n^\alpha + (\alpha+1-x)\frac{d}{dx}L_n^\alpha + nL_n^\alpha = 0, \quad L_n^0 = L_n, \qquad L_0^\alpha = 1, \tag{2}$$

$$L_n^\alpha(0) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} = \frac{(n+\alpha)!}{n!\,\alpha!} = \binom{n+\alpha}{n}, \qquad [\frac{d}{dx}L_n^\alpha]_{x=0} = -\binom{n+\alpha}{n-1},$$

with the explicit expression:

$$L_n^\alpha(x) = (n+\alpha)!\,\sum_{k=0}^{n}\frac{(-1)^k}{k!\,(n-k)!\,(k+\alpha)!}\,x^k. \tag{3}$$

In [7] the formula (3) is employed for the direct calculation of the Laplace transform [8-10]:

$$\mathcal{L}[x^{p+\alpha}\,L_n^\alpha(x)] = \frac{(p+\alpha)!}{s^{p+\alpha+n+1}}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k+\alpha}\binom{p}{k}(s-1)^{n-k}, \;\; p\geq 0, \;\; \alpha>-1, \;\; s>0, \tag{4}$$

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which implies the particular cases [11, 12]:

$$\mathcal{L}[x^{\alpha}\, L_n^{\alpha}(x)] = \frac{(n+\alpha)!}{n!\, s^{\alpha+1}}\,(1-\tfrac{1}{s})^n, \qquad \mathcal{L}[L_n(x)] = \frac{1}{s}\,(1-\tfrac{1}{s})^n, \tag{5}$$

$$\mathcal{L}[x^p\, L_n(x)] = \frac{p!}{s^{p+n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{p}{k} (s-1)^{n-k}.$$

The relation (4) can be deduced [13] using the Gregory [14]-Newton [15] infinite expansion for equidistant interpolation [12], and it is useful to determine radial matrix elements for hydrogen-like atoms [16].

## 2. Laplace transform via the Laguerre differential equation

We apply the Laplace operator to (1) to obtain the differential equation:

$$s(1-s)\frac{d}{dx}\mathcal{L}[L_n] + (1-s+n)\mathcal{L}[L_n] = 0, \qquad \mathcal{L}[L_0] = \frac{1}{s}, \tag{6}$$

whose solution is immediate:

$$\mathcal{L}[L_n] = \frac{(s-1)^n}{s^{n+1}}, \tag{7}$$

in according with (5).

This procedure can be employed in (2) with $\alpha \neq 0$ to deduce the equation:

$$s(1-s)\frac{d}{dx}\mathcal{L}[L_n^{\alpha}] + [1+n+(\alpha-1)s]\,\mathcal{L}[L_n^{\alpha}] = \alpha\binom{n+\alpha}{n}, \tag{8}$$

therefore:

$$\mathcal{L}[L_n^{\alpha}] = \binom{n+\alpha}{n}\frac{\alpha}{s^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{(s-1)^k}{n+\alpha-k}, \qquad \alpha \neq 0, \tag{9}$$

where is clear the fulfillment of $\mathcal{L}[L_0^{\alpha}] = \frac{1}{s}$, thus:

$$\mathcal{L}[L_n^1] = 1-(1-\tfrac{1}{s})^{n+1}, \qquad \mathcal{L}[L_n^2] = (n+2) - s\left[1-(1-\tfrac{1}{s})^{n+2}\right], \text{ etc.} \tag{10}$$

On the other hand, we know the relation [2]:

$$L_n^{\alpha}(x) = \sum_{k=0}^{n}\binom{\alpha+k-1}{k} L_{n-k}(x), \tag{11}$$

which allows to obtain an alternative expression to (9):

$$\mathcal{L}[L_n^\alpha] = \frac{1}{s}\sum_{k=0}^{n}\binom{\alpha+k-1}{k}(1-\tfrac{1}{s})^{n-k}\,, \tag{12}$$

in harmony with (10). It is interesting to note that (12) is valid for negative values of $\alpha$, for example, $\alpha = -N = -1, -2, \ldots$, then:

$$\mathcal{L}[L_{N+m}^{-N}] = \frac{1}{s}(1-\tfrac{1}{s})^m \sum_{r=0}^{N}\binom{-r-1}{N-r}(1-\tfrac{1}{s})^r = \frac{(-1)^N}{s}(1-\tfrac{1}{s})^m \sum_{r=0}^{N}\binom{N}{r}[-\left(1-\tfrac{1}{s}\right)]^r,$$

hence:

$$\mathcal{L}[L_{N+m}^{-N}(x)] = \frac{(-1)^N}{s^{N+1}}(1-\tfrac{1}{s})^m, \tag{13}$$

and for $m = 0$:

$$\mathcal{L}[L_N^{-N}(x)] \equiv \mathcal{L}\left[\frac{(-x)^N}{N!}\right] = \frac{(-1)^N}{s^{N+1}}\,. \tag{14}$$

Let's remember the formula [17]:

$$x^N L_m^N(x) = \frac{(-1)^N\,(N+m)!}{m!}\, L_{N+m}^{-N}\,, \tag{15}$$

then (13) permits to deduce the following Laplace transform:

$$\mathcal{L}[x^N L_m^N(x)] = \frac{(N+m)!}{m!\, s^{N+1}}\,(1-\tfrac{1}{s})^m\,, \tag{16}$$

in according with (5). The relation [2]:

$$L_m = \frac{m!}{(N+m)!}\,\frac{d^N}{dx^N}(x^N L_m^N), \tag{17}$$

allows other proof of (16), in fact:

$$\mathcal{L}[L_m] = \frac{m!}{(N+m)!}\left\{s^N \mathcal{L}[x^N L_m^N] - \sum_{k=1}^{N} s^{N-k}\left[\frac{d^{k-1}}{dx^{k-1}}(x^N L_m^N)\right](0)\right\},$$

therefore $\mathcal{L}[x^N L_m^N] = \frac{(N+m)!}{m!\, s^N}\,\mathcal{L}[L_m] = \text{eq.}\,(16).$

The direct method of integration and the interpolation technique are important to deduce the Laplace transform of the Laguerre associated polynomials, but here we show that also is useful to apply the Laplace operator to the Laguerre differential equation with the corresponding boundary conditions.

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## References

1. E. N. Laguerre, *Sur l'intégrale* $\int_x^\infty \frac{e^{-x}}{x}dx$, Bull. Soc. Math. France **7** (1879) 72-81.
2. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions*, Wiley & Sons, New York (1972) Chap. 22.
3. J. López-Bonilla, A. Lucas-Bravo, *A note on the finite Taylor series applied to Laguerre polynomials,* Educatia Mathematica **4**, No. 1 (2008) 53-55.
4. R. Cruz-Santiago, J. López-Bonilla, H. Torres-Silva, *On a hypergeometric formula for orthogonal polynomials*, Prespacetime Journal **7**, No. 15 (2016) 2023-2026.
5. R. Cruz-Santiago, J. López-Bonilla, S. Yáñez-San Agustín, *A note on the Laguerre polynomials,* Open J. Appl. Theor. Maths. **2**, No. 4 (2016) to appear.
6. A. Sommerfeld, *Partial differential equations*, Academic Press, New York (1964) Chap. 5.
7. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of integral transforms.I,* McGraw-Hill, New York (1954) Chap. 4.
8. G. Doetsch, *Introduction to the theory and application of the Laplace transformation,* Springer-Verlag, New York (1974).
9. J. López-Bonilla, D. Romero-Jiménez, A. Zaldívar-Sandoval, *Laplace transform of matrix exponential function,* Prespacetime Journal **6**, No. 12 (2015) 1410-1413.
10. J. López-Bonilla, R. López-Vázquez, O. Puente-Navarrete, *Laplace transform and the error function,* Prespacetime Journal **7**, No. 8 (2016) 1214-1216.
11. C. Lanczos, *Discourse on Fourier series,* Oliver & Boyd, Edinburgh (1966) Chap. 3.
12. C. Lanczos, *Linear differential operators*, Dover, New York (1997) Chap. 1.
13. A. Bucur, B. Carvajal, J. López-Bonilla, *Laguerre polynomials: Their Laplace transform via equidistant interpolation*, J. Sci. Res. (India) **53** (2009) 257-258.
14. J. Gregory, *Geometriae pars Universalis* (1668)
15. I. Newton, *Principia* (1687) Book III, Lemma V.
16. M. Bautista, J. López-Bonilla, J. Morales, *Matrix elements* $< n_2 l_2 \parallel r^k \parallel n_1 l_1 >$ *for the Coulomb interaction*, Apeiron **13**, No. 1 (2006) 34-42.
17. J. D. Talman, *Special functions: A Group theoretic approach,* W. A. Benjamin Inc, New York (1968) Chap. 13.