# Lorentz-Invariant Gravitation Theory

## Chapter 11. Solution of the Kepler problem in the framework of LIGT

## Introduction

In present chapter, based on results of previous chapter 9, we consider the solution of the Kepler problem, i.e., the solution of the problem of motion of two bodies in a centrally symmetric gravitational field of a stationary source. It is shown that this solution coincides with that obtained in GR.

As the motion equation of LITG we use the Hamilton-Jacobi equation (Chapter 8). According to Chapter 7, the equation of motion of Hamilton-Jacobi has a one-to-one connection with the square of the interval (square of arc element of trajectory) in framework of LITG. Therefore, as we will show below, it is not necessarily to find an appropriate interval to write the corresponding Hamilton-Jacobi equation for particle motion in gravitation field.

## 1.0. Effects of Lorentz transformation

A consequence of the previously adopted axiomatics (chap. 3) of Lorentz-invariant gravitation theory (LIGT) is the assertion that all features of the motion of matter in the gravitational field owed their origin to effects associated with the Lorentz transformations. This means that the elaboration of the equations of Newton's gravitation must follow from considering of these effects.

As is well known (Becker, 2013), these questions can be considered without special relativity theory, using only the Maxwell equations.

Effects, that owe their existence to the Lorentz transformations are discussed in many textbooks devoted to the EM theory or SRT (Becker, 2013; Pauli, 1981; et al.). We will not dwell on their withdrawal, and we will only briefly mention some of them.

From the Lorentz transformations follows the velocity transformation, showing that no body can overcome the speed of light. From the Lorentz speed transformations follow the time dilation and length contraction in a moving frame of reference, as well as the transformation of energy and momentum. The use of invariance properties of the wave phase with respect to the Lorentz transformations, allows to obtain the relativistic formula of Doppler effect, aberration, reflection from a moving mirror, Wien's displacement law, etc.

### 1.1. The transition from Newtonian mechanics to the Lorentz-invariant mechanics

Let us try (Беккер, 2013) to alter the Newtonian equations so that they satisfy the Lorentz transformations. We begin by considering the motion of a particle in a given force field (e.g., electromagnetic or gravitational). Newtonian equations of motion read as follows:

$$m\frac{d\vec{\upsilon}}{dt} = \vec{F}_L\,, \tag{1.1}$$

where $\vec{F}_L$ is, e.g., the Lorentz force :

$$\vec{F}_L = q\vec{E} - \frac{q}{c}\vec{\upsilon}\times\vec{H}\,, \tag{1.2}$$

Now we will try to give this equation the Lorentz-invariant form. Obviously, the Lorentz-invariant version of the equation (1.1) instead of the classical time $t$ must contain the proper time $\widetilde{t}$ :

$$m\frac{d\vec{\upsilon}}{d\tau}=\vec{F}_L\,, \tag{1.1'}$$

In order to find this version of the equation, we replace in (1.1') its proper time in line with the ratio for the Lorentz time dilation $d\widetilde{t}=dt\sqrt{1-\beta^2}$ on $dt\sqrt{1-\beta^2}$ :

$$m_0\frac{d}{dt}\frac{\vec{\upsilon}}{\sqrt{1-\beta^2}}=q\vec{E}-\frac{q}{c}\vec{\upsilon}\times\vec{H}\,, \tag{1.3}$$

As is known, the equation (1.3) is the Lorentz-invariant equation of motion of a charged particle in an EM field.

Below we will consistently apply this method to obtain the relativistic equations of gravitation in the form of Hamilton-Jacobi equations.

## 2.0. Solution of the Kepler problem in the framework of LIGT

Two of the most important effects from the point of view of mechanics that arise due to the Lorentz transformations, are the Lorentzian time dilation and contraction of lengths:

$$d\widetilde{t}=dt\sqrt{1-\beta^2}\,,\quad d\widetilde{r}=\frac{dr}{\sqrt{1-\beta^2}}\,, \tag{2.1}$$

where, as shown previously, $\beta^2=r_S/r$, and $r_S$ is the Schwarzschild radius.

The free particle motion is described by the Hamilton-Jacobi equation Landau and Lifshitz, 1971):

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\vec{\nabla}S\right)^2=m^2c^2\,, \tag{2.2}$$

In a spherical coordinate system (taking into account both relativistic effects) it takes the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial\widetilde{t}}\right)^2-\left(\frac{\partial S}{\partial\widetilde{r}}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2-\frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial\varphi}\right)^2=m^2c^2\,, \tag{2.3}$$

where $\widetilde{t}$ and $\widetilde{r}$ are measured in a fixed coordinate system associated with a stationary spherical mass $M$ .

We will start with the account of the first effect

### 2.1. The equation of motion of a particle in a gravitational field, taking into account the relativistic effect of time dilation

Taking into account that the motion of a particle around the source occurs in the plane, we define this plane by condition $\theta=\pi/2$. In this case, the equation (2.3) takes the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial\widetilde{t}}\right)^2-\left(\frac{\partial S}{\partial\widetilde{\vec{r}}}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial\varphi}\right)^2=m^2c^2\,, \tag{2.4}$$

Taking into account only the transformation of time $d\tilde{t} = dt\sqrt{1-\beta^2}$ (see (2.1)), equation (2.4) can be rewritten as follows:

$$\frac{1}{1-\beta^2}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4, \quad (2.5)$$

Substituting $1-\beta^2 = 1 - r_s/r$, we obtain:

$$\frac{1}{1-\frac{r_s}{r}}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4 \ , \quad (2.6)$$

Let us simplify this equation, taking into account the expansion $1/(1-x) = 1 + x + x^2 + ... + x^n$ for $x << 1$. Since for the actual sizes of the planets and Sun and the distances between them, value $r_s/r << 1$, we can be limited by first two terms of the expansion. At the same time $\frac{1}{1-r_s/r} \cong 1 + r_s/r$, and the equation (2.4) takes the form:

$$\left(1+\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4 \ , \quad (2.7)$$

We will show that L-invariant time dilation leads to the appearance of Newton's gravitational field.

### 2.1.1 *Newton's approximation*

Let us present this equation to the nonrelativistic mind, using the transformation $S = S' - mc^2 t$:

$$\left(\frac{\partial S}{\partial t}\right)^2 = \left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + m^2c^4 .$$

Substituting this in (7), we find

$$\left(1+\frac{r_s}{r}\right)\left[\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + m^2c^4\right] - c^2\left(\frac{\partial S'}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4 .$$

Expanding the brackets, we obtain:

$$\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + \frac{r_s}{r}\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{r_s}{r}\frac{\partial S'}{\partial t} + \frac{r_s}{r}m^2c^4 - c^2\left(\frac{\partial S'}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = 0 .$$

Dividing this equation by $2mc^2$, we find:

$$\frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 - \frac{\partial S'}{\partial t} + \frac{1}{2mc^2}\frac{r_s}{r}\left(\frac{\partial S'}{\partial t}\right)^2 - \frac{r_s}{r}\frac{\partial S'}{\partial t} + \frac{1}{2}\frac{r_s}{r}mc^2 - \frac{1}{2m}\left(\frac{\partial S'}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = 0 , \quad (2.8)$$

Taking into account that $r_s = \dfrac{2\gamma M}{c^2}$, we obtain $\dfrac{1}{2}\dfrac{r_s}{r}mc^2 = \dfrac{\gamma mM}{r} = m\varphi_N = -U$, where $U$ is the energy of the gravitational field in the Newtonian theory. In the nonrelativistic case we put $c \to \infty$. Furthermore, for real distances $r$ of the body movement around source with Schwarzschild radius $r_s$, we have $\dfrac{r_s}{r} << 1$ and $\dfrac{r_s}{r}\dfrac{\partial S'}{\partial t} << \dfrac{\partial S'}{\partial t}$, and then we can ignore the term $\dfrac{r_s}{r}\dfrac{\partial S'}{\partial t}$ .

В пределе при $c \to \infty$ уравнение (2.8) переходит в известное классическое уравнение Гамильтона-Якоби для гравитационного поля Ньютона:

In the limit as $c \to \infty$, equation (2.8) goes over into the classical Hamilton-Jacobi equation for Newton gravitation field

$$\frac{\partial S'}{\partial t} + \frac{1}{2m}\left(\frac{\partial S'}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = -U \,, \qquad (2.9)$$

As is known, the solution of this problem leads to a closed elliptical (not preccesing) satellite orbit around the spherical central body.

From this it follows that the inclusion only of Lorentz time dilation into the free Hamilton-Jacobi equation leads to the Kepler problem in non-relativistic theory of gravitation.

Note also that equation (2.9) is a consequence of the L-invariant HJE with the Newton potential field:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + U\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2 \,, \qquad (2.10)$$

Thus, the equations (2.6), (2.9) and (2.10) are equivalent from point of view of their results.

## 2.2. The equation of motion of a particle in a gravitational field with the Lorentz time dilation and length contraction

Now in order to take into account the length contraction effect along with the effect of time dilation, we will use the Hamilton-Jacobi equation (2.3) in form:

$$\left(\frac{\partial S}{\partial \tilde{t}}\right)^2 - c^2\left(\frac{\partial S}{\partial \tilde{r}}\right)^2 - \frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] = m^2c^2 \,, \qquad (2.11)$$

Substituting in (2.11) not only $d\tilde{t} = dt\sqrt{1-\beta^2}$, but also $d\tilde{r} = dr\big/\sqrt{1-\beta^2}$, we obtain:

$$\frac{1}{1-\beta^2}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(1-\beta^2\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] = m^2c^4 \,, \qquad (2.12)$$

Taking into account that in our theory $1-\beta^2 = 1 - r_s/r$, we obtain from (2.12) the well-known Hamilton-Jacobi equation for general relativity in the case of the Schwarzschild-Droste metric (Schwarzschild, 1916; Droste, 1917):

$$\frac{1}{1-\frac{r_s}{r}}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] = m^2c^4, \tag{2.13}$$

As is well known (Landau and Lifshitz, 1971), the solutions of this equation are three well-known effects of general relativity, well confirmed by experiment: the precession of Mercury's orbit, the curvature of the trajectory of a ray of light in the gravitational field of a centrally symmetric source and the gravitational frequency shift of EM waves.

As we noted, in the Kepler problem solution, based on this equation, there is an additional term in the energy, which is missing in Newton's theory:

$$U(r) = -\frac{\gamma_N m M_s}{r} + \frac{M^2}{2mr^2} - \frac{\gamma_N M_s M^2}{c^2 m r^3}, \tag{2.14}$$

which is responsible for the precession of the orbit of a body, rotating around a spherically symmetric stationary center. From the above analysis it follows that the appearance of this term is provided by Lorentz effect of the length contraction.

We found above that the term $\frac{1}{1-r_s/r}\left(\frac{\partial S'}{\partial t}\right)^2$ containing the Lorentz time dilation effect in the classical approximation leads to the equation of Newton gravitation with Newton's gravitational energy. From this it follows that the precession of the orbit ensure the introduction of an additional term $c^2\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2$.

## 2.3. Gravitational deflection of light ray trajectory

The path of a light ray (Landau and Lifshitz, 1971, p. 308-309) in a centrally symmetric gravitational field is determined by the eikonal equation which differ from Hamilton-Jacobi equation only in having $m = 0$, at the same time, in place of the energy $\varepsilon_p = -\partial S/\partial t$ of the particle we must write the frequency of the light $\omega_\lambda = -\partial\Psi/\partial t$.

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0, \tag{2.15}$$

The solution show that under the influence of the field of attraction the light ray is bent: its trajectory is a curve, which is concave toward the center (the ray is 'attracted' toward the center), so that the angle between its two asymptotes differs from $\pi$ by

$$\delta\varphi = \frac{2r_s}{\rho} = \frac{4\gamma_N M_S}{c^2\rho}, \tag{2.16}$$

In other words, the ray of light, passing at a distance $\rho$ from the center of the field, is deflected through an angle $\delta\varphi$.

## 2.4. Gravitational time dilation and red shift of the frequency

Within the framework of general relativity, these gravitational effects are took into consideration on the basis of the Schwarzschild-Droste metric. In the framework of LIGT, this solution is based on the account of effects resulting from the Lorentz transformations, and has no relation to the metric. Nevertheless, the indicated effects are easily solved here.

We are able to prove a general statement regarding the influence of a gravitational field on clocks (Pauli, 1981). Let us take a reference system *K* which rotates relative to the Galilean system $K_0$ with angular velocity $\omega$. A clock at rest in *K* will then be slowed down the more, the farther away from the axis of rotation the clock is situated, because of the transverse Doppler effect. This can be seen immediately by considering the process as observed in system $K_0$. The time dilatation is given by

$$t = \frac{\tau}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \frac{\tau}{\sqrt{1-\frac{\omega^2 r^2}{c^2}}}, \tag{2.17}$$

The observer rotating with *K* will not interpret this shortening of the time as a transverse Doppler effect, since after all the clock *is* at rest relative to him. But in *K* a gravitational field (field of the centrifugal force) exists with potential $\varphi = -\frac{1}{2}\omega^2 r^2$.

Thus the observer hi *K* will come to the conclusion that the clocks will be slowed down the more, the smaller the gravitational potential at the particular spot. In particular, taking into account that $\upsilon^2/c^2 = \beta^2 = r_s/r = 2\varphi/c^2$, the time dilatation $\Delta t$ is given, to a first approximation, by

$$t = \frac{\tau}{\sqrt{1+\frac{2\varphi}{c^2}}} \sim \tau\left(1-\frac{\varphi}{c^2}\right); \quad \frac{\Delta t}{\tau} = -\frac{\varphi}{c^2}, \tag{2.18}$$

Einstein[293] applied an analogous argument to the case of uniformly accelerated system. We thus see that the transverse Doppler effect and the time dilatation produced by gravitation appear as two different modes of expressing the same fact, namely that a clock will always indicate the proper time $\tau = \frac{1}{ic}\int ds$.

Relation (2.18) has an important consequence which can be checked by experiment. The transport of clocks can also be effected by means of a light ray, if one regards the vibration process of light as a clock.

If, therefore, a spectral line produced in the sun is observed on the earth, its frequency will, according to (2.18), be shifted towards the red compared with the corresponding terrestrial frequency. The amount of this shift will be

$$\frac{\Delta\nu}{\nu} = -\frac{\varphi_E - \varphi_S}{c^2}, \tag{2.19}$$

where $\varphi_E$ is the value of the gravitational potential on the earth, $\varphi_S$ that on the surface of the sun. The numerical calculation gives $\frac{\Delta\nu}{\nu} = 2{,}12\cdot 10^{-6}$ , corresponding to a Doppler effect of 0,63 km/sec.

Einstein (Einstein, 1911) applied an analogous argument to the case of uniformly accelerated system.

Let us assume (Sivukhin, 2005) that the clock *A* relatively to the system *S* is moving with constant acceleration $a$. We will count the time $t$ from the moment when the velocity was zero. Then $\upsilon = \sqrt{2ax}$, where *x* is the distance that the clock A covered during the time $t$. Therefore:

$$dt = dt_0 \Big/ \sqrt{1 - 2ax/c^2}\ , \tag{2.20}$$

Now let us introduce an accelerated reference frame $S_0$, which moves together with the clock $A$. In this system the clock $A$ is immobile, but there are inertial forces. If all the phenomena will be described, taking $S_0$ as a reference frame, then as the cause of time dilation $t_0$ the inertial forces should be considered. The inertial force per unit mass of the moving body is $-a$. But, according to the principle of equivalence, the inertial forces are indistinguishable from the gravitational field, the intensity of which in our case is $\vec{g} = -\vec{a}$. Then the gravitational potential is $\varphi = -gx$ and the formula (2.20) becomes:

$$dt = dt_0 \Big/ \sqrt{1 - 2\varphi/c^2} \approx dt_0 \left(1 - \varphi/c^2\right), \tag{2.21}$$

or

$$\frac{dt - dt_0}{dt_0} = -\frac{\varphi}{c^2}, \tag{2.22}$$

As zero gravitational potential, the potential of point is considered, at which the moving and stationary clocks run equally fast. Therefore, in formulas (2.21) and (2.22), the time interval $dt$ can be counted not by the clock of the inertial system $S$, but by the clock that is in rest in system $S_0$, which is located at the point $B$ with zero potential. In general, we can set the initiation of count of gravitational potential at any point, if the formula (2.22) has the form:

$$\frac{dt_{0A} - dt_{0B}}{dt_{0A}} = -\frac{\varphi_B - \varphi_A}{c^2}, \tag{2.23}$$

where the time intervals $dt_{0A}$ and $dt_{0B}$ are counted by two clocks, which are in rest in an accelerated reference frame $S_0$ at points $A$ and $B$ with gravitational potentials $\varphi_A$ and $\varphi_B$.

## Conclusion

Thus, we can say that, in the case of centrally symmetric gravitational field, within the framework of LIGT we get the same results as in the framework of general relativity. It is noteworthy that in order to obtain these results minor adjustments in equation of motion are required, which are ensured by two effects following from the Lorentz transformations.