Lorentz-Invariant Gravitation Theory

Chapter 7. The mathematical apparatus of LIGT

1.0. Introduction.

Using the results of chapter 2 in this chapter, we will list the quantum equations and relationships which correspond to our goal - the construction of the LIGT. Then we consider the transition from quantum mechanical equations of motion to the motion equations of classical mechanics.

These mathematical tools will be the base for the solution of specific problems in the theory of gravity, expounded in the following chapters.

1.1. The Bases for selection of LIGT equations

From the equivalence of inertial and gravitational masses follows that the field of gravity is generated simultaneously with inertial mass. This means that the equations of massive elementary particles describe also the gravitational field equations.

Electron is the simplest stable massive particle. Since, according to axioms of the LIGT, the gravitational field is a small part of the electric field, it can be assumed that the simplest candidate for the gravitational equation must be a modification of the nonlinear equation of the electron. In this case, the mass of this equation is the gravitational mass, i.e., the source of the gravitational field.

On the other hand we have the equation of the neutral "massive photon", which we can also - and with a significant reason - consider as a gravitation source equation. The following facts are the arguments in favor of this choice:

1) "massive photon" is the primary massive particle;
2) it is an electrically neutral particle;
3) fermions are not the interaction carriers in the microworld, but bosons are;
4) the "massive photon" equation and the lepton equation are related through operations of decomposition of first equation and squaring of second equation. From this it follows that the first or the second choice of the equations of gravitation is a matter of convenience.

Hence, the "massive photon" equation may be an advantageous variant of the gravitation source equation. Nevertheless, the close relationship of theories of "massive photon", and electron, implies the possibility of the use of the Dirac equation, as the basis of our approach.

There is another indirect argument. As noted by Richard Feynman, a direct transition from the quantum to the classical form of the fermion equation is difficult. In the case of bosons, such a transition is quite simple: we can say that it is the same equation (Feynman, 1964, 21-4. *The meaning of the wave function*).

"…*In the situation in which we can have very many particles in exactly the same state, there is possible a new physical interpretation of the wave functions. The charge density and the electric current can be calculated directly from the wave functions and the wave functions take on a physical meaning which extends into classical, macroscopic situations*" (see also Chapter 6).

2.0. The photon equation

The classical EM wave equation of motion has the form:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\vec{\nabla}^2\right)\Phi = 0\,, \tag{2.1}$$

wherein c is light velocity, Φ is a matrix, which contains the components of the wave function of an electromagnetic field $\vec{E}, \vec{H}$: $\Phi^+ = \left(\mathrm{E}_x \quad \mathrm{E}_z \quad -i\mathrm{H}_x \quad -i\mathrm{H}_z\right)$

This wave is a superposition of two waves with plane polarization: $\Phi_1 = \begin{pmatrix} E_x \\ H_z \end{pmatrix}$ and $\Phi_2 = \begin{pmatrix} E_z \\ H_x \end{pmatrix}$, which also satisfy (2.1). In this sense, the wave with the flat polarization, and not cyclic polarization, can be regarded as the primary particle of EM field.

3.0. The current and mass of massive particles

Equation (2.1) can be represented as a system of two equations for massless electron and positron:

$$\left(\hat{\alpha}_o\hat{\varepsilon} + c\hat{\vec{\alpha}}\,\hat{\vec{p}}\right)\psi' = 0\,, \tag{3.1}$$

$$\psi'^{+}\left(\hat{\alpha}_o\hat{\varepsilon} - c\hat{\vec{\alpha}}\,\hat{\vec{p}}\right) = 0\,, \tag{3.1'}$$

where the wave function of these equations we denoted as ψ'.

Self-interaction of the photon fields leads to the appearance in the photon of two displacement currents of different directions (chapter 2 or (Kyriakos, 2009)). In the mathematical description of this process, in the equations (3.1) an additional term arises.

The emerging particle, which we conditionally call "massive photon", is unstable and breaks into massive particle-antiparticle, particularly, electron and positron (this fact allows us to consider a "massive photon" as an intermediate boson).

$$\left(\hat{\alpha}_o\hat{\varepsilon} + c\hat{\vec{\alpha}}\,\hat{\vec{p}} + \hat{\beta}\, m_e c^2\right)\psi = 0\,, \tag{3.2}$$

$$\psi^{+}\left(\hat{\alpha}_o\hat{\varepsilon} - c\hat{\vec{\alpha}}\,\hat{\vec{p}} - \hat{\beta}\, m_e c^2\right) = 0\,, \tag{3.2'}$$

4.0. The equation of "massive photon"

At a time when the system of equations (3.1) obtains current (mass) terms, the photon ceases to move at the speed of light and becomes massive :

$$\left(\frac{\partial^2}{\partial t^2} - c^2\vec{\nabla}^2\right)\Phi = \frac{m_{ph}^2 c^4}{\hbar^2}\Phi\,, \tag{4.1}$$

Since the currents have a different direction, the photon remains a neutral vector boson.

The equation of neutral "massive photon" (4.1) can be rewritten in the view:

$$\left[\left(\hat{\alpha}_o\hat{\varepsilon}\right)^2 - c^2\left(\hat{\vec{\alpha}}\hat{\vec{p}}\right)^2\right]\Phi = m_{ph}^2 c^4\Phi\,, \tag{4.1'}$$

or

$$\left(\hat{\varepsilon}^2 - c^2\hat{\vec{p}}^2 - m_{ph}^2 c^4\right)\Phi = 0, \tag{4.2}$$

From equation (4.1) follows the conservation equation for the elementary particles:

$$\varepsilon^2 - c^2\vec{p}^2 - m_{ph}^2 c^4 = 0, \tag{4.3}$$

Note that this equation is valid both in quantum mechanics and in classical mechanics for all particles.

In connection with general relativity, the different form of equation (4.1) could be interesting for us. The expression for the current was obtained by the rotation transformation, the radius of which was equal to $r_C = \hbar/m_e c$. For this reason, the current (mass) j^e contains curvature $\kappa = 1/r_C$ through which the mass term $m_{ph}^2 c^4/\hbar^2 = 1/4r_C^2 = \kappa^2/4$ can be expressed. In other words, the equation (4.1) can be expressed as:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\vec{\nabla}^2\right)\Phi = \frac{1}{4r_C^2}\Phi, \tag{4.4}$$

This equation is similar to the equation obtained by Schrödinger as the generalization of the Dirac equation on Riemannian space (see below).

5.0. The generally covariant equation of "massive photon"

The generally covariant equation of "massive photon" Schroedinger (Schroedinger, 1932), was the first to obtain by squaring of Dirac equation, written for the curved space:

$$\frac{1}{\sqrt{g}}\nabla_k \sqrt{g} g^{kl}\nabla_l - \frac{R}{4} - \frac{1}{2} f_{kl} S^{kl} = \mu^2, \tag{5.1}$$

Here $\mu = \frac{m_e c}{\hbar} = \frac{1}{r_C}$, where r_C is the Compton wave length of electron, R is the invariant curvature.

In the first term is easy to find a regular operator of the Klein second order equation in the Riemann geometry. In the third term on the left is recognized well-known term associated with the spin magnetic and electric moments of the electron (tensor S^{kl}).

"*To me, the second term seems to be of considerable theoretical interest. To be sure, it is much too small by many powers of ten in order to replace, say, the term on the r.h.s. For μ is the reciprocal Compton length, about $10^{11} cm^{-1}$. Yet it appears important that in the generalised theory a term is encountered at all which is equivalent to the enigmatic mass term.*"

This term can be associated with the free term of the equation of Dirac's electron μ. According to Gauss, on a curved surface $R = \kappa_1 \cdot \kappa_2$, where κ_1, κ_2 are the normal curvature of the surface. If, $\kappa_1 = \kappa_2 = \kappa'$ then $R = \kappa'^2$. Assuming by Schrodinger that $R/4 = \mu^2$, we obtain that $\kappa' = 2\mu = \frac{2m_e c}{\hbar} = \frac{m_{ph} c}{\hbar}$

6.0. Quantum equations of particles' motion in the external field

For a complete accordance with the electromagnetic theory of matter (EMTM), the energy ε_{ex} and momentum $\vec{p}_{ex}$ in the equation (3.3) must be expressed as the EM values. We can include the electromagnetic potentials $\varphi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$, using the fact that φ and $(1/c)\vec{A}$ have the same

Lorentz-transformation properties as ε and $\vec{p}$ (here φ is scalar potential, $\vec{A}$ is the vector potential of the EM field, and the dimension of $\varphi(\vec{r},t)$ is energy per unit charge, and the dimension of $(1/c)\vec{A}$ is equal to the momentum per unit charge).

As is known, the total momentum and the total energy of a charged particle in an electromagnetic field is determined by the following expressions:

$$\vec{p}_{ful} = \vec{p} + \frac{q}{c}\vec{A}\,, \quad \varepsilon_{ful} = \varepsilon + q\varphi\,, \tag{6.1}$$

where q is charge, $\vec{p} = \dfrac{m\vec{\upsilon}^2}{\sqrt{1-\vec{\upsilon}^2/c^2}}$ and $\varepsilon = \dfrac{mc^2}{\sqrt{1-\vec{\upsilon}^2/c^2}}$ are the momentum and energy of a free particle, $\vec{\upsilon}$ is particle velocity, $\vec{p}_{ex} = \dfrac{q}{c}\vec{A}_{ex}$ and $\varepsilon_{ex} = q\varphi_{ex}$ are the potential momentum and energy of some external source (charged particles), obtained in the EM field.

Hence, (4.1) can be rewritten as the Dirac equation with an external EM field

$$\left[\hat{\alpha}_0\left(\hat{\varepsilon} \mp e\varphi_{ex}\right) + c\hat{\vec{\alpha}}\cdot\left(\hat{\vec{p}} \mp \frac{q}{c}\vec{A}_{ex}\right) + \hat{\beta}m_e c^2\right]\psi = 0\,, \tag{6.2}$$

The corresponding differential equations for the "massive photon" will be:

$$\left[(\varepsilon + \varepsilon_{ex})^2 - c^2(\vec{p} + \vec{p}_{ex})^2 - m^2c^4\right]\Phi = 0\,, \tag{6.3}$$

$$\left[(\varepsilon + q\varphi_{ex})^2 - c^2\left(\vec{p} + \frac{q}{c}\vec{A}_{ex}\right)^2 - m^2c^4\right]\Phi = 0\,, \tag{6.3'}$$

(here and from now on we omit the subscript “ph” in mass of “massive photon”)

From this we can obtain the equations of energy-momentum conservation of a particle in an EM field:

$$(\varepsilon + \varepsilon_{ex})^2 - c^2(\vec{p} + \vec{p}_{ex})^2 - m^2c^4 = 0\,, \tag{6.4}$$

$$(\varepsilon + q\varphi_{ex})^2 - c^2\left(\vec{p} + \frac{q}{c}\vec{A}_{ex}\right)^2 - m^2c^4 = 0\,, \tag{6.4'}$$

From the above it follows that the values $\dfrac{q}{c}\vec{A}_{ex}$ and $q\varphi_{ex}$ completely characterize the external field source. Below we will find the expression for the force, with the source acts on the particle.

7.0. The transition from quantum mechanical equations of motion to the motion equations of classical mechanics

There are three main methods of transition from the quantum mechanical equations of motion to the classical equations (Schiff, 1955; Levich, Myamlin and Vdovin, 1973, Landsman, 2005; Anthony, 2014).

a) theorem of Ehrenfest,

b) on the basis of Hamilton's canonical equations, using Poisson brackets,

c) the transition from the wave equation to the Hamilton-Jacobi equation.

We shall illustrate this transition based on the methods a) and b).

7.1. Ehrenfest's theorem in the case of the Lorentz-invariant quantum theory

Let us use the Lorentz-invariant quantum wave equation of "massive photon" in external EM field (6.3), obtained in the above section:

In this case (Anthony, 2014) the wave function has the form

$$\psi = \psi_0 \exp \frac{i}{\hbar}\left[\left(\vec{p} - \frac{q}{c}\vec{A}\right)\vec{r} - (\varepsilon + q\varphi)t\right], \tag{7.1}$$

Now we want to see whether that equation gives us a description of Reality that conforms to the classical theory. To that aim we will calculate the expectation value of the rate at which a particle's linear momentum changes with the elapse of time.

Using the relativistic formula for the probability density, we have

$$\frac{d}{dt}\langle \vec{p} \rangle = \frac{i\hbar}{2mc^2}\int\left[\psi^+\left(-i\hbar\frac{d}{dt}\vec{\nabla}\right)\frac{\partial \psi}{\partial t} - \psi\left(i\hbar\frac{d}{dt}\vec{\nabla}\right)\frac{\partial \psi^+}{\partial t}\right]d\tau\,, \tag{7.2}$$

In that equation the operators extract the argument of the wave function and differentiate it, so we have

$$-i\hbar\frac{d}{dt}\vec{\nabla}\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial t}\left[\frac{d}{dt}\vec{\nabla}\left(\vec{p}\cdot\vec{r} - \frac{q}{c}\vec{A}\cdot\vec{r}\right) - \frac{d}{dt}\vec{\nabla}(\varepsilon\, t + q\varphi\, t)\right], \tag{7.3}$$

The vector variables $\vec{r}$ and $\vec{p}$ do not represent fields, but rather represent points in phase space that the particle occupies as time elapses, so we take the spatial derivatives of those variables as equal to zero. Further, if we do not want to have the complications with radiation fields, then with respect to the source of the potential fields we must take $d\varphi/dt = 0$ and $d\vec{A}/dt = 0$.

Carrying out the differentiations thus gives us:

$$\begin{aligned} -i\hbar\frac{d}{dt}\vec{\nabla}\frac{\partial\psi}{\partial t} &= \frac{\partial\psi}{\partial t}\left[q\frac{d\vec{r}}{dt}\times\left(\vec{\nabla}\times\vec{A}\right) - q\left(\frac{d}{dt}\vec{\nabla}\right)\vec{A} - \vec{\nabla}U - q\vec{\nabla}\varphi\right] = \\ &= \frac{\partial\psi}{\partial t}\left[q\vec{\upsilon}\times\left(\vec{\nabla}\times\vec{A}\right) - q\left(\frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t}\right) - \vec{\nabla}U - q\vec{\nabla}\varphi\right] \end{aligned}, \tag{7.4}$$

Substituting that result and its complex conjugate into Equation 18 then gives us:

$$\frac{d}{dt}\langle\vec{p}\rangle = q\left\langle\vec{\upsilon}\times\left(\vec{\nabla}\times\vec{A}\right) - \frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\right\rangle + \left\langle -\vec{\nabla}U\right\rangle, \tag{7.5}$$

which describes the Lorentz electromagnetic force plus the force due to any other static potentials of the particle interaction. Thus we gain strong evidence that the relativistic quantum theory, like its non-relativistic version, has the classical limit.

7.2. Derivation of generally covariant classical equation of motion on the base of Ehrenfest theorem

An interesting application of the theory (see chapter 4) is to establish an analogue of Ehrenfest's theorem for the Dirac equation, generalized to the Riemann geometry (Sokolov and Ivanenko, 1952; pp. 650-651). In addition to the results obtained above, by squaring of the Dirac equation, for the center of gravity of the wave packet (provided $\hbar \to 0$), we obtain the equation of relativistic mechanics of point:

$$\frac{d}{dx^4}\left(\gamma^4 p_\alpha\right) = \Gamma^\sigma_{\alpha\rho} p_\alpha + \gamma^\rho \frac{e}{c} F_{\rho\alpha}, \tag{7.6}$$

where γ^4 is the fourth Dirac matrix, γ^ρ corresponds to the particle velocity in fraction of the speed of light c, $\Gamma^\sigma_{\alpha\rho}$ is the Christoffel brackets $\{\mu\nu,\alpha\} = \Gamma^\sigma_{\alpha\rho} = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} + \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right)$, $F_{\rho\alpha}$ is the electromagnetic field tensor. The first term on the right of equation is the force of gravity, and the second term is the Lorentz force.

7.3. Derivation of classical Hamilton-Jacobi equation of motion on the base of quantum wave equation

The Hamilton-Jacobi equation (HJE) in the classic mechanics is usually obtained by postulating the action in the form of:

$$S = S_{free} + S_{\text{int}} + S_{ext}, \tag{7.7}$$

where S_{free} is the action of a free particle in the absence of other particles; S_{int} is the action of the interaction between the free particle and other particles; S_{ext} is the action of other particles in the absence of the first particle.

In quantum physics HJE can be obtained, if we postulate that the action is equal to phase of the de Broglie wave (as Schrödinger did for the derivation of the Schrödinger equation (Schroedinger, 1932).

The particle wave function, in general, has the form:

$$\psi = \psi_0 \exp i\theta, \tag{7.8}$$

where θ is the phase of the wave function. In the case of a free particle the wave function has the form:

$$\psi = \psi_0 \exp\frac{i}{\hbar}\left(\varepsilon t - \vec{p}\vec{r} + \varphi_0\right), \tag{7.9}$$

Substituting this function in the equation (4.1), we obtain the law of conservation of energy and momentum for a free particle (5.3):

$$\varepsilon^2 - c^2\vec{p}^2 = m^2c^4, \tag{4.3}$$

In the case of a particle in an external field with the energy and momentum $\varepsilon_{ex}, \vec{p}_{ex}$ the wave function has the form:

$$\psi = \psi_0 \exp\frac{i}{\hbar}\left[\left(\vec{p} - \vec{p}_{ex}\right)\vec{r} - \left(\varepsilon + \varepsilon_{ex}\right)t + \varphi_0\right], \tag{7.10}$$

Substituting these functions in the equation (6.3), we obtain the conservation law for a particle in an external field (6.4):

$$\left(\varepsilon - \varepsilon_{ex}\right)^2 - c^2\left(\vec{p} - \vec{p}_{ex}\right)^2 = m^2c^4, \tag{6.4}$$

According to Schrödinger in case of a free particle we take:

$$S = \theta\hbar = \varepsilon t - \vec{p}\vec{r} + \varphi_0 \ , \tag{7.11}$$

and in case of a particle in external field:

$$S = \left[(\vec{p} - \vec{p}_{ex})\vec{r} - (\varepsilon + \varepsilon_{ex})t + \varphi_0\right] , \tag{7.12}$$

Hence we have in the first case for the energy and momentum $\frac{\partial S}{\partial t} = \varepsilon$, $\frac{\partial S}{\partial \vec{r}} = \vec{p}$, and in the second case $\frac{\partial S}{\partial t} = \varepsilon + \varepsilon_{ex}$, $\frac{\partial S}{\partial \vec{r}} = \vec{p} - \vec{p}_{ex}$.

Substituting partial derivatives of first type in the conservation law of energy-momentum without an external field, we obtain the relativistic HJE without an external field:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = m^2c^2, \tag{7.13}$$

Substituting second partial derivatives of second type in the conservation law of energy-momentum with an external field, we obtain the relativistic HJE with the external field:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + \varepsilon_{ex}\right)^2 - \left(\frac{\partial S}{\partial x} - p_{x\ \mathrm{ex}}\right)^2 - \left(\frac{\partial S}{\partial y} - p_{y\ \mathrm{ex}}\right)^2 - \left(\frac{\partial S}{\partial z} - p_{z\ \mathrm{ex}}\right)^2 = m^2c^2, \tag{7.14}$$

In the case of the electromagnetic field we have:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - \left(\frac{\partial S}{\partial x} - \frac{q}{c}A_x\right)^2 - \left(\frac{\partial S}{\partial y} - \frac{q}{c}A_y\right)^2 - \left(\frac{\partial S}{\partial z} - \frac{q}{c}A_z\right)^2 = m^2c^2, \tag{7.15}$$

The action for the interaction can be obtained as an instantaneous change of action:

$$dS_{\mathrm{int}} = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial \vec{r}}d\vec{r} = \frac{\partial S}{\partial t}dt - \frac{\partial S}{\partial \vec{r}}\frac{d\vec{r}}{dt}dt = \frac{\partial S}{\partial t}dt - \frac{\partial S}{\partial \vec{r}}\vec{\upsilon}dt = (\varepsilon - \vec{p}\vec{\upsilon})dt, \tag{7.16}$$

i.e., $dS_{\mathrm{int}} = (\varepsilon - \vec{p}\vec{\upsilon})dt = L_{\mathrm{int}}dt$; in the case when the external field is organized by electrical charged particles, we have: $dS_{\mathrm{int}} = \left(q\varphi - \frac{q}{c}\vec{A}\right)dt$.

Here

$$L_{\mathrm{int}} = (\varepsilon - \vec{p}\vec{\upsilon}) = q\varphi - \frac{q}{c}\vec{A}\vec{\upsilon}, \tag{7.17}$$

is the interaction Lagrangian (the so-called, minimal connection). As is known, by variation of this action gives the expression for the Lorentz force.

8.0. The interaction law of gravitation field in framework of LIGT

In the case of electrodynamics it is necessary to use not the classical potential energy, but the generalized (and depending on the speed) potential energy (energy of interaction)

$$U = q\varphi - \frac{q}{c}\vec{\upsilon}\cdot\vec{A} = \int\left(\rho\varphi - \frac{1}{c}\vec{j}\cdot\vec{A}\right)dxdydz, \tag{8.1}$$

This interaction energy corresponds to the above interaction Lagrangian (7.17).

From this Lagrangian follows the equation for the Lorentz force. In terms of EM vectors it has the form:

$$\vec{F} = q\vec{E} - \frac{q}{c}\vec{\upsilon} \times \vec{H} \ , \tag{8.2}$$

Lorentz force in terms of potentials:

$$\vec{F} = q\vec{\nabla}\varphi - \frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\vec{\upsilon} \times \left(\vec{\nabla} \times \vec{A}\right) = q\vec{\nabla}\varphi - \frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\left[\vec{\nabla}\left(\vec{\upsilon} \cdot \vec{A}\right) - \left(\vec{\upsilon} \cdot \vec{\nabla}\right)\vec{A}\right], \tag{8.3}$$

9.0. Conclusion

Thus, we have shown that the Lorentz force occurs at the transition from quantum mechanics of massive particle to classical mechanics of this particle, as a reflection of the unique relation of the inertial mass with internal and external fields of the particle. According to our axioms, we must conclude that the Lorentz force law or its modifications should be responsible for the description of the gravitational force or energy.

In addition, the connection of inertial mass with gravitational charge becomes clear, as well as the relationship between the electric charge and gravity charge (mass), which allow us to proceed from Coulomb equation to the Newton equation of gravitation.