Article

Heisenberg Group & Mannheim Curves

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Abstract

We study non-geodesic biharmonic curves in the Heisenberg group Heis^3 to characterize the Mannheim curves in terms of their biharmonic partner curves in Heis^3.

Keywords: Heisenberg group, biharmonic curves, Mannheim curves.

1. Introduction

Recently, there has been a growing interest in the theory of biharmonic maps [1-10], which can be divided into two main research directions. On one hand, the differential geometric aspect has driven attention to the construction of examples and classification results. On the other hand, the analytic aspect from the point of view of PDE are solutions of a fourth order strongly elliptic semilinear PDE.

Let (N,h) and (M,g) be Riemannian manifolds; denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention $R^N(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$, $X,Y\in\Gamma(TN)$.; for a smooth map $\phi:N\to M$, the Levi-Civita connection ∇ of (N,h) induces a conncetion ∇^ϕ on the pull-back bundle $\phi^*TM =_{p\in N} T_{\phi(p)}M$. The section $\mathrm{T}(\phi):=\mathrm{tr}\nabla^\phi d\phi$ is called the tension field of ϕ; a map ϕ is said to be harmonic if its tension field vanishes identically.

A smooth map $\phi:N\to M$ is said to be biharmonic if it is a critical point of the bienergy functional $E_2(\phi)=\int_N \frac{1}{2}|\mathrm{T}(\phi)|^2 dv_h$. The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi)=0$. Here the section $\mathrm{T}_2(\phi)$ is defined by:

$$\mathrm{T}_2(\phi)=\Delta_\phi \mathrm{T}(\phi)+\mathrm{tr}R(\mathrm{T}(\phi),d\phi)d\phi, \tag{1.1}$$

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and called the bitension field of ϕ. The operator Δ_ϕ is the rough Laplacian acting on $\Gamma(\phi^* TM)$ defined by $\Delta_\phi := -\sum_{i=1}^{n}\left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla^N_{e_i}e_i}\right)$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N.

In particular, if the target manifold M is the Euclidean space E^m, the biharmonic equation of a map $\phi : N \to \mathsf{E}^m$ is $\Delta_h\Delta_h\phi = 0$, such that Δ_h is the Laplace--Beltrami operator of (N,h). Clearly, any harmonic map is biharmonic; however, the converse is not true. Nonharmonic biharmonic maps are said to be proper. It is well known that proper biharmonic maps, that is, biharmonic functions, play an important role in elasticity and hydrodynamics.

Here we study non-geodesic biharmonic curves in the Heisenberg group Heis^3, and we characterize the Mannheim curves in terms of their biharmonic partner curves in Heis^3.

2. Heisenberg group Heis^3

Heisenberg group Heis^3 can be seen as the space R^3 endowed with the following multiplication:

$$(\bar{x},\bar{y},\bar{z})(x,y,z) = (\bar{x}+x, \bar{y}+y, \bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}) \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group. The Riemannian metric g is given by $g = dx^2+dy^2+(dz+\frac{y}{2}dx-\frac{x}{2}dy)^2$. The Lie algebra of Heis^3 has an orthonormal basis:

$$e_1 = \frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products $[e_1,e_2]=e_3, [e_2,e_3]=[e_3,e_1]=0$ with $g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1$. Hence $\nabla_{e_1}e_1=\nabla_{e_2}e_2=\nabla_{e_3}e_3=0,\quad \nabla_{e_1}e_2=-\nabla_{e_2}e_1=\frac{1}{2}e_3,$

$$\nabla_{e_1}e_3=\nabla_{e_3}e_1=-\frac{1}{2}e_2,\qquad \nabla_{e_2}e_3=\nabla_{e_3}e_2=\frac{1}{2}e_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by $R(X,Y)Z = -\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z+\nabla_{[X,Y]}Z$, while the Riemannian curvature tensor is given by $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, where X,Y,Z,W are smooth vector fields on Heis^3. The components $\{R_{ijkl}\}$ of R relative to $\{e_1,e_2,e_3\}$ are defined by $g\left(R(e_i, e_j)e_k, e_l\right) = R_{ijkl.}$

The non-vanishing components of the above tensor fields are

$$R_{121} = -\frac{3}{4}e_2, \quad R_{131} = \frac{1}{4}e_3, \quad R_{122} = \frac{3}{4}e_1, \; R_{232} = \frac{1}{4}e_3, \quad R_{133} = -\frac{1}{4}e_1, \quad R_{233} = -\frac{1}{4}e_2,$$
$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \tag{2.3}$$

3. Biharmonic curves in the Heisenberg group Heis^3

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to (N, h)$ be a curve parametrized by arc length on a Riemannian manifold. Putting $\mathbf{T} = \gamma'$, we can write the tension field of γ as $\tau(\gamma) = \nabla_{\gamma'}\gamma'$ and the biharmonic map equation (1.1) reduces to:

$$\nabla^3_{\mathbf{T}}\mathbf{T} + R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0. \tag{3.1}$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve, which is recalled in the following.

Let $\gamma : I \to Heis^3$ be a curve on $Heis^3$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to Heis^3 along γ defined as follows: T is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and B is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet-Serret formulas:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \qquad \nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} - \tau\mathbf{B}, \qquad \nabla_{\mathbf{T}}\mathbf{B} = \tau\mathbf{N}, \tag{3.2}$$

such that $\kappa = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ is its torsion. Therefore, we can write $\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3$, $\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3$, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3$.

Theorem 3.1. *(see [11]) Let $\gamma : I \to Heis^3$ be a non-geodesic curve on $Heis^3$ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if*

$$\kappa = \text{constant} \neq 0, \quad \kappa^2 + \tau^2 = \frac{1}{4} - B_3^2, \quad \tau' = N_3 B_3. \tag{3.3}$$

Theorem 3.2. *(see [11]) Let $\gamma : I \to Heis^3$ be a non-geodesic curve on the Heisenberg group $Heis^3$ parametrized by arc length. If κ is constant and $N_1 B_1 \neq 0$, then γ is not biharmonic.*

4. Mannheim curves in Heisenberg group Heis^3

Definition 4.1. *Let $\gamma, \beta : I \to Heis^3$ be a unit speed non-geodesic curve. If there exists a corresponding relationship between the space curves γ and β such that, at the corresponding points of the curves, the principal normal lines of β coincides with the binormal lines of β, then β is called a Mannheim curve, and γ a Mannheim partner curve of β. The pair $\{\gamma, \beta\}$ is said to be a Mannheim pair.*

Theorem 4.2. *Let $\beta : I \to Heis^3$ be a Mannheim curve and γ its biharmonic partner curve. Then, the parametric equation of Mannheim curve β in terms of its biharmonic partner curve γ of β are*

$$\begin{aligned}
&x_\beta(s) = \frac{\lambda}{\kappa}\sin\varphi(\cos\varphi - \Re)\cos[\Re s + \rho](\cos\varphi + \frac{1}{2\Re}\sin^2\varphi) \\
&+\frac{1}{\Re}\sin\varphi\sin[\Re s + \rho], \\
&y_\beta(s) = \frac{\lambda}{\kappa}\sin\varphi(\cos\varphi - \Re)(\cos\varphi + \frac{1}{2\Re}\sin^2\varphi)\sin[\Re s + \rho] \\
&-\frac{1}{\Re}\sin\varphi\cos[\Re s + \rho], \\
&z_\beta(s) = (\cos\varphi + \frac{1}{4\Re}\sin^2\varphi)s - \sin\varphi,
\end{aligned} \tag{4.1}$$

where ρ is constant of integration and $\Re = \dfrac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}$.

Proof. The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1^{'} + T_2T_3)\mathbf{e}_1 + (T_2^{'} - T_1T_3)\mathbf{e}_2 + T_3^{'}\mathbf{e}_3. \tag{4.2}$$

Thus using Theorem 3.2, we find:

$$\mathbf{T} = \sin\varphi\cos[\Re s + \rho]\mathbf{e}_1 + \sin\varphi\sin[\Re s + \rho]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \tag{4.3}$$

such that $\Re = \dfrac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}$.

Using (2.2) in (4.3) we obtain:

$$\mathbf{T} = (\sin\varphi\cos[\Re s + \rho], \sin\varphi\sin[\Re s + \rho],$$
$$\cos\varphi - \frac{1}{2}y(s)\sin\varphi\cos[\Re s + \rho] + \frac{1}{2}x(s)\sin\varphi\sin[\Re s + \rho]).$$

From (2.2), we get: $\mathbf{T} = (\sin\varphi\cos[\Re s + \rho], \sin\varphi\sin[\Re s + \rho],$ and $\cos\varphi + \dfrac{1}{2\Re}\sin^2\varphi\cos^2[\Re s + \rho] + \dfrac{1}{2\Re}\sin^2\varphi\sin^2[\Re s + \rho]).$ On the other hand, suppose that $\beta(s)$ is a Mannheim curve, then by the definition we can assume that:

$$\beta(s) = \gamma(s) + \lambda\mathbf{B}(s). \tag{4.4}$$

From (4.2) and (4.3), we deduce $\nabla_{\mathbf{T}}\mathbf{T} = \sin\varphi(\cos\varphi - \Re)(\sin[\Re s + \rho]\mathbf{e}_1 - \cos[\Re s + \rho]\mathbf{e}_2).$

By the use of Frenet-Serret formulas, we get:

$$\mathbf{N} = \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T} = \frac{1}{\kappa}[\sin\varphi(\cos\varphi - \Re)(\sin[\Re s + \rho]\mathbf{e}_1 - \cos[\Re s + \rho]\mathbf{e}_2)]. \tag{4.5}$$

Substituting (2.2) in (4.5), we have $\mathbf{N} = \dfrac{1}{\kappa}\sin\varphi(\cos\varphi - \Re)(\sin[\Re s + \rho], -\cos[\Re s + \rho], 0).$ Noting that $\mathbf{T}\times\mathbf{N} = \mathbf{B}$, we deduce:

$$\mathbf{B} = \frac{1}{\kappa}\sin\varphi(\cos\varphi - \Re)(\cos[\Re s + \rho](\cos\varphi + \frac{1}{2\Re}\sin^2\varphi), \tag{4.6}$$
$$(\cos\varphi + \frac{1}{2\Re}\sin^2\varphi)\sin[\Re s + \rho], -\sin\varphi).$$

Finally, we substitute (4.3) and (4.6) into (4.4), we get (4.1). The proof is completed.

Corollary 4.3. *Let* $\gamma : I \to \mathrm{K}$ *be a unit speed non-geodesic biharmonic partner curve of Mannheim curve* β. *Then, the parametric equations of* γ *are*

$$x(s)=\frac{1}{\Re}\sin\varphi\sin[\Re s+\rho],\ \ y(s)=-\frac{1}{\Re}\sin\varphi\cos[\Re s+\rho],$$

$$z(s)=(\cos\varphi+\frac{1}{4\Re}\sin^2\varphi)s,\ \ \Re=\frac{\cos\varphi\pm\sqrt{5(\cos\varphi)^2-4}}{2}. \tag{4.7}$$

The theory of biharmonic functions is an old and rich subject [12-19], and they were studied since 1862 by Maxwell to describe a mathematical model in elasticity.

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