Article

On the Horndeski's Formula for the Lanczos Invariant

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Abstract

Horndeski wrote the Lagrangian $\sqrt{-g}\ {}^{*}R^{*\,\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ as an exact divergence; here we exhibit that this formula is useful to study the existence of non-null constant vectors in R_4.

Keywords: Lanczos scalar, Horndeski's formula, Lanczos potential.

1. Introduction

Here we consider the Lanczos invariant [1]:

$$K_2 = {}^{*}R^{*\,\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\ , \tag{1}$$

which is the contraction of the Riemann tensor with its double dual [2]. If we employ the Lagrangian $L = \sqrt{-g}\,K_2$ in the Hilbert type variational principle $\delta\int L\,d^4x = 0$, we obtain 0 = 0, hence one suspects that the density L is an exact divergence for any R_4:

$$\sqrt{-g}\,K_2 = (\sqrt{-g}\,C^{\mu})_{,\mu} \tag{2}$$

where $,\mu = \frac{\partial}{\partial x^{\mu}}$. This suspicion turned out to be correct because Buchdal [5, 6] and Goenner-Kohler [7] got non-tensorial expressions for C^{ν}. From (2) Lanczos [8] proved that the Weyl tensor is generated by a potential of third order.

In Sec. 2 we indicate the Horndeski's tensorial expression [9] for C^{α} and we exhibit that it is useful to analyze the presence of non-null constant vectors in a given space-time.

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2. Horndeski's formula for $\sqrt{-g}\, K_2$

It is interesting to note that the vector field C^{μ} obtained by Horndeski [9] contains an arbitrary non-null vector A^{ν}, that without loss of generality we take unitary and time-like, that is, $A^{\alpha}A_{\alpha} = 1$, then:

$$C^{\mu} = 8\,(\,{}^{*}R^{*\,\mu\nu}{}_{\alpha\beta} + \tfrac{1}{3}\delta^{\lambda\nu\theta\mu}_{\tau\beta\varphi\alpha}\, A^{\tau}{}_{;\lambda}\, A^{\varphi}{}_{;\theta})A^{\beta}{}_{;\tau}\, A^{\alpha} \,, \qquad (3)$$

verifying (2), with the participation of the generalized Kronecker delta [4]. We observe that $(\sqrt{-g}\, C^{\nu})_{,\nu} = = (\sqrt{-g}\, C^{\nu})_{;\nu}$ because (3) has tensorial character.

Therefore, when $K_2 \neq 0$ the space-time does not accept non-null constant vectors, in other words, the existence of a non-null vector such that $A^{\mu}{}_{;\nu} = 0$ implies $K_2 = 0$ via (2) and (3) (which occurs, for example, in the Gödel solution [2, 10, 11]). The metrics of Schwarzschild, Taub, C, Kerr, …. have $K_2 \neq 0$ [2], hence these space-times do not admit non-null constant vectors.

Thus, the Lagrangian $\sqrt{-g}\, K_2$ is an ordinary divergence, however, it can contribute [12, 13] to the gravitational energy-momentum distribution. For empty spaces, in [14] there is a tensor formula for C^{μ} in terms of the Lanczos potential [8].

Received Feb. 23, 2016; Accepted March 3, 2016

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