Article

The (*t/z*) Considerations of Plane Wave Solutions of Gauge-Invariant Generalizations of Field Theories

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Abstract

In the present paper, we have considered the gauge-invariant generalization of Einstein field equations in our chosen metric for (*t/z*)-type waves. The results obtained are written in the form of theorems which resemble to those obtained by Katore and Rane (2008).

Keywords: General relativity, generalization, gauge-invariant, plane gravitational waves.

1. Introduction

Gauge Invariant Generalization of Field Theories is an important class of an asymmetric field theories introduced by Buchdahl (1957)**[1]**, by combining two nonsymmetric geometries of Weyl (1919)**[9]** and that of Einstein (1951)**[10]** and is based upon an asymmetric covariant tensor g_{ij} and a covariant vector K_i relating those to an asymmetric linear connection Γ^i_{jk} in such a way that the geometry could be regarded equivalently as 'Gauge Invariant generalization of Einstein's theory' propounded by Weyl. Lal and Srivastava (1972) **[5]** obtained the plane wave solutions of these theories for asymmetric fundamental tensor in Bondi (1959) **[14]** space-time. S. D. Katore and R.S. Rane (2008)**[7]** have generalized these solutions in the plane symmetry. In the present paper we have considered the Gauge invariant generalization of Einstein field equations in our chosen metric for (t/z) -type waves. The results obtained are written in the form of theorems which resemble to those obtained by Katore and Rane (2008).

2. The Line element and field equations

The metric chosen for the investigation is

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - CZ^2dz^2 + Cdt^2 \ , \tag{2.1}$$

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which was obtained by Lal, Ali (1969)**[2]** by using certain transformations in Bondi space-time. Here A, B, C, D are all the functions of $Z(= t/z)$ and the solutions correspond to the (t/z) -type waves. Buchdahl's Gauge Invariant generalization of field theories are based upon an asymmetric tensor $g_{ij} = h_{ij} + f_{ij}$, a covariant vector K_i and a linear connection L^i_{js} defined by

$$\Gamma^m_{is} g_{mj} + \Gamma^m_{sj} g_{im} = g_{ij,s} \tag{2.2}$$

$$\gamma^m_{is} g_{mj} + \gamma^m_{sj} g_{im} = g_{ij} K_s \tag{2.3}$$

$$L^i_{js} = \Gamma^i_{js} - \gamma^i_{js} \tag{2.4}$$

Here h_{ij},the symmetric part of g_{ij} coincides with the fundamental tensor of space-time which is minkowskian and f_{ij} is the skew-symmetric part of g_{ij} satisfying the electromagnetic field. Γ^i_{jk} is the linear connection of usual asymmetric theory of Einstein. Buchdahl provided an asymmetric fundamental tensor g_{ij}, a linear connection L^i_{js} and a covariant vector K_i , defined in such a way that L^i_{js} and all the basic tensors derived from the g_{ij}, K_i , L^i_{js} and their derivatives possess the property of Gauge invariance. The indices i, j, k take values 1, 2, 3, 4 and a comma (,) before an index s denotes its partial derivative w.r.to x^s .

Buchdahl defined the following simplest field equations as

$$L_i = L^s_{[is]} = 0 \tag{2.5}$$

$$G_{ij} = P_{ij} + \overset{0}{\gamma}{}^{s}_{\underset{+-}{ij;s}} - (K_{\underset{+}{i,j}} + K_{\underset{-}{j,i}}) + \gamma^s_{im}\gamma^m_{sj} - 2\gamma^s_{ij} K_s = 0 \tag{2.6}$$

where P_{ij} is defined by

$$P_{ij} = -\Gamma^s_{ij,s} + \frac{1}{2}\left[\Gamma^s_{(is),j} + \Gamma^s_{(sj),i}\right] + \Gamma^s_{im}\Gamma^m_{sj} - \Gamma^s_{ij}\Gamma^m_{sm} \tag{2.7}$$

Here G_{ij} is the Gauge invariant Hermitian Einstein tensor and P_{ij} is the usual tensor of Einstein asymmetric theory. A semicolon (;) represent a covariant differentiation anda sign (+), (-) or (o) below the index fixes the position of covariant index k in connections as $\Gamma^{\cdot}_{\cdot k}$, $\Gamma^{\cdot}_{k\cdot}$, $\Gamma^{\cdot}_{(\cdot k)}$ and a pair of parenthesis ( ) and [ ] including two indices represents the symmetry and skew symmetry between them.

3. Calculations of nonsymmetric tensors

Following the method of Takeno (1961)**[8]** and Lal and Shrivastav (1972) **[5]** the non-symmetric tensors g_{ij} as used by S.W.Bhaware, D.D.Pawar and A.G. Deshmukh(2012) **[11]** are obtained as follows. On the lines of V.B.Johari(1966) **[6]**the components of electromagnetic field tensors F_{ij} for the (t/z) -type line element (2.1) are:

$$\left(F_{ij}\right)=\begin{bmatrix} 0 & 0 & \frac{-\sigma_1}{z} & \frac{\sigma_1}{t} \\ 0 & 0 & \frac{\rho_1}{z} & \frac{-\rho_1}{t} \\ \frac{\sigma_1}{z} & \frac{-\rho_1}{z} & 0 & 0 \\ \frac{-\sigma_1}{t} & \frac{\rho_1}{t} & 0 & 0 \end{bmatrix}, \tag{3.1}$$

where σ_1, ρ_1 are arbitrary functions of $Z(=t/z)$.

The components of h_{ij} ,the symmetric part of g_{ij} are

$$\left(g_{(ij)}\right)=(h_{ij})=\begin{bmatrix} -A & -D & 0 & 0 \\ -D & -B & 0 & 0 \\ 0 & 0 & -CZ^2 & 0 \\ 0 & 0 & 0 & C \end{bmatrix} \tag{3.2}$$

Using the Ikeda (1954)**[13]** relations

$$F_{ij}=\frac{1}{2}\varepsilon_{ijkl}\sqrt{-g}\,g^{kl} \tag{3.3}$$

where ε_{ijkl} = +1 or -1 according as i, j, k, l have even or odd permutations, we find the fundamental metric tensor (g_{ij}) for the metric (2.1) as

$$\left(g_{ij}\right)=\begin{bmatrix} -A & -D & \frac{\rho}{z} & \frac{-\rho}{t} \\ -D & -B & \frac{\sigma}{z} & \frac{-\sigma}{t} \\ \frac{-\rho}{z} & \frac{-\sigma}{z} & -CZ^2 & 0 \\ \frac{\rho}{t} & \frac{\sigma}{t} & 0 & C \end{bmatrix}, \tag{3.4}$$

$$g=-(AB-D^2)\,C^2Z^2=-mC^2Z^2 \tag{3.5}$$

where

$$m=(AB-D^2),\ \rho=\frac{(A\rho_1+D\sigma_1)}{\sqrt{m}} \text{ and } \sigma=\frac{(D\rho_1+B\sigma_1)}{\sqrt{m}} \tag{3.6}$$

The conjugate metric tensors are

$$\left(g^{ij}\right)=\begin{bmatrix} -\frac{B}{m} & \frac{D}{m} & U & ZU \\ \frac{D}{m} & \frac{-A}{m} & V & ZV \\ -U & -V & (-\frac{1}{CZ^2}+W) & ZW \\ -ZU & -ZV & ZW & (\frac{1}{C}+Z^2W) \end{bmatrix}, \tag{3.7}$$

where

$$U=\frac{B\rho-D\sigma}{zZ^2mC}\ ,\quad V=\frac{A\sigma-D\rho}{zZ^2mC}\ ,\ W=\frac{A\sigma^2-2D\rho\sigma+B\rho^2}{z^2Z^4mC^2} \tag{3.8}$$

According to Hlavaty (1957)**[4]** f_{ij} belongs to the class III if $K=k=0$,
where

$$K=\frac{1}{4}f_{ij}f^{ij} \text{ and } k=\frac{f}{h}=\frac{\det.(f_{ij})}{\det.(h_{ij})} \tag{3.9}$$

Here we note that, the magnetic components of g_{ij} coincide with electromagnetic field tensor f^{ij} given by

$$f^{ij}=h^{i\alpha}h^{j\beta}f_{\alpha\beta} \tag{3.10}$$

4. Solutions of field equations (2.3) and (2.4)

We use the Hlavaty,s method to solve equation (2.3). Mishra (1963)**[12]** has proved that γ^s_{ij} can be put in the form

$$\gamma^s_{ij}=H^s_{ij}+S^s_{ij}+U^s_{ij} \tag{4.1}$$

Where

$$U^s_{ij}=2h^{sn}S^m_{n[i}f_{j]m} \tag{4.2}$$

$$H^s_{ij}=\frac{1}{2}h^{sm}(K_ih_{jm}+K_jh_{im}-K_mh_{ij}) \tag{4.3}$$

$$S^s_{ij}=\gamma^s_{[ij]}=h^{sm}(K'_{ijm}+2U^n_{m[ifj]n}) \tag{4.4}$$

$$K'_{ijs}=K_{ijs}+2f_{l[i}H^l_{j]s} \tag{4.5a}$$

$$K_{ijs} = \frac{1}{2}(K_j f_{is} + K_s f_{ij} - K_i f_{js}) \tag{4.5b}$$

As f_{ij} given by (3.1) belongs to the third class in the sense of Hlavaty (1957) we have solution of (4.4)

$$S^s_{ij} = h^{sm}(K'_{ijm} - 2 f^n_{[j} K'_{i]nl}\ f^l_n)\ . \tag{4.6}$$

Using above equations the components of H^s_{ij} $(= H^s_{ji})$ and those of K_{ijs} $(= -K_{jis})$ are obtained as follows:

$$H^s_{11} = \left[\frac{K_1}{2} + \frac{\psi D}{2m}, \frac{-\psi A}{2m}, \frac{-K_3 A}{2CZ^2}, \frac{K_4 A}{2C}\right]$$
$$H^s_{22} = \left[\frac{B\mu}{2m}, \frac{K_2}{2} - \frac{\mu D}{2m}, \frac{-K_3 B}{2CZ^2}, \frac{K_4 B}{2C}\right]$$
$$H^s_{33} = \left[\frac{Z^2 C\mu}{2m}, \frac{-Z^2 C\psi}{2m}, \frac{K_3}{2}, \frac{Z^2 K_4}{2}\right]$$
$$H^s_{44} = \left[\frac{-C\mu}{2m}, \frac{C\psi}{2m}, \frac{K_3}{2Z^2}, \frac{K_4}{2}\right]$$
$$H^s_{12} = \left[\frac{B\psi}{2m}, \frac{-A\mu}{2m}, \frac{-K_3 D}{2CZ^2}, \frac{K_4 D}{2C}\right]$$
$$H^s_{13} = \left[\frac{K_3}{2}, 0, \frac{K_1}{2}, 0\right]$$
$$H^s_{14} = \left[\frac{K_4}{2}, 0, 0, \frac{K_1}{2}\right]$$
$$H^s_{23} = \left[0, \frac{K_3}{2}, \frac{K_2}{2}, 0\right]$$
$$H^s_{24} = \left[0, \frac{K_4}{2}, 0, \frac{K_2}{2}\right]$$
$$H^s_{34} = \left[0, 0, \frac{K_4}{2}, \frac{K_3}{2}\right] \tag{4.7}$$

$$K_{12s} = \left[0, 0, \frac{1}{2z}(K_2\rho - K_1\sigma), \frac{1}{2t}(-K_2\rho + K_1\sigma)\right]$$
$$K_{13s} = \left[\frac{1}{z}(K_1\rho), \frac{1}{2z}(K_2\rho + K_1\sigma), \frac{1}{z}(K_3\rho), \frac{\rho}{2}\left(\frac{K_4}{z} - \frac{K_3}{t}\right)\right]$$
$$K_{14s} = \left[-\frac{K_1\rho}{t}, \frac{-1}{2t}(K_2\rho + K_1\sigma), \frac{\rho}{2}\left(\frac{K_4}{z} - \frac{K_3}{t}\right), \frac{-K_4\rho}{t}\right]$$

$$K_{23s}=\left[\frac{1}{2z}(K_1\sigma+K_2\rho),\frac{K_2\sigma}{z},\frac{K_3\sigma}{z},\frac{\sigma}{2}(\frac{K_4}{z}-\frac{K_3}{t})\right]$$
$$K_{24s}=\left[-\frac{1}{2t}(K_1\sigma+K_2\rho),-\frac{K_2\sigma}{t},\frac{\sigma}{2}(\frac{K_4}{z}-\frac{K_3}{t}),-\frac{K_4\sigma}{t}\right]$$
$$K_{34s}=\left[-\frac{\rho}{2}(\frac{K_4}{z}+\frac{K_3}{t}),-\frac{\sigma}{2}(\frac{K_4}{z}+\frac{K_3}{t}),0,0\right] \tag{4.8}$$

and $K_{ijs}=0$ for i=j.

Using (3.4),(4.7),(4.8) in (4.5 a) we find the values of $K'_{ijs}(=-K'_{jis})$ as follows

$$K'_{12s}=\left[\frac{1}{2Ct^2}(zK_3+tK_4)\lambda,\frac{1}{2Ct^2}(zK_3+tK_4)\nu,0,0\right]$$
$$K'_{13s}=\left[-\frac{\psi\lambda}{2zm},-\frac{\psi\nu}{2zm},\frac{K_4\rho t}{2z^2},-\frac{K_4\rho}{2z}\right]$$
$$K'_{14s}=\left[\frac{\psi\lambda}{2mt},\frac{\psi\nu}{2mt},\frac{K_3\rho}{2t},-\frac{K_3\rho z}{2t^2}\right]$$
$$K'_{23s}=\left[-\frac{\mu\lambda}{2zm},-\frac{\nu\mu}{2zm},\frac{K_4\sigma t}{2z^2},\frac{K_4\sigma}{2z}\right]$$
$$K'_{24s}=\left[\frac{\mu\lambda}{2mt},\frac{\nu\mu}{2mt},\frac{\sigma K_3}{2t},-\frac{K_3\sigma z}{2t^2}\right]$$
$$K'_{34s}=\left[0,0,\frac{CZ}{2zm}(\rho\mu-\sigma\nu),-\frac{C}{2z}(\rho\mu-\sigma\nu)\right] \tag{4.9}$$

where

$$\psi=(K_2A-K_1D)$$
$$\mu=(K_2D-K_1B)$$
$$\lambda=(\rho D-\sigma A)$$
$$\nu=(\rho B-\sigma D) \tag{4.9a}$$

Substituting from equations (3.1) and (4.9) in equation (4.6) we see that the last term on RHS is identically zero and consequently equation(4.6) reduces to

$$S^s_{ij}=h^{sm}K'_{ijm} \tag{4.10}$$

Using (3.2) and (4.9) in equation (4.10) we find the components of $S^s_{ij}\,(=-S^s_{ji})$ as follows:

$$S^s_{12}=\left[\frac{\sigma}{2Ct^2}(zK_3+tK_4),-\frac{\rho}{2Ct^2}(zK_3+tK_4),0,0\right]$$
$$S^s_{13}=\left[-\frac{\sigma\psi}{2zm},\frac{\rho\psi}{2zm},-\frac{\rho K_4}{2Ct},-\frac{\rho K_4}{2Cz}\right]$$

$$S_{14}^{s} = \left[\frac{\sigma\psi}{2mt}, -\frac{\rho\psi}{2mt}, -\frac{\rho K_3}{2CZ^2 t}, -\frac{\rho K_3}{2CZ^2 z} \right]$$
$$S_{23}^{s} = \left[-\frac{\sigma\mu}{2zm}, \frac{\rho\mu}{2zm}, -\frac{\sigma K_4}{2Ct}, -\frac{\rho K_4}{2Cz} \right]$$
$$S_{24}^{s} = \left[\frac{\sigma\mu}{2mt}, -\frac{\rho\mu}{2mt}, -\frac{\sigma K_3}{2CZ^2 t}, -\frac{\sigma K_3}{2CZ^2 z} \right]$$
$$S_{34}^{s} = \left[0, 0, -\frac{(\rho\mu - \sigma\psi)}{2tm}, -\frac{(\rho\mu - \sigma\psi)}{2zm} \right] \tag{4.11}$$

$S_{ij}^{k} = -S_{ji}^{k}$ and $S_{ij}^{k} = 0$ for i = j (i.e., S_{ij}^{k} is antisymmetric in i and j).

Using equation (3.1) and (4.11), the equation (4.2) reduces to

$$U_{11}^{s} = 0 \quad . \tag{4.12}$$

Then by using equations (4.7), (4.11) and (4.12) in equation (4.1) the components of $\gamma^{s}{}_{ij}$ are obtained as follows:

$$\gamma_{11}^{s} = \left[\frac{K_1}{2} + \frac{\psi D}{2m}, -\frac{\psi A}{2m}, -\frac{K_3 A}{2CZ^2}, -\frac{K_4 A}{2C} \right] = H_{11}^{s} \qquad (\because \gamma_{ij}^{k} = H_{ij}^{k} + S_{ij}^{k} \text{ and } S_{ii}^{k} = 0)$$
$$\gamma_{22}^{s} = \left[\frac{B\mu}{2m}, \frac{K_2}{2} + \frac{-\mu D}{2m}, -\frac{K_3 B}{2CZ^2}, \frac{K_4 B}{2C} \right] = H_{22}^{s}$$
$$\gamma_{33}^{s} = \left[\frac{Z^2 C\mu}{2m}, -\frac{CZ^2\psi}{2m}, \frac{K_3}{2}, \frac{Z^2 K_4}{2} \right] = H_{33}^{s}$$
$$\gamma_{44}^{s} = \left[-\frac{C\mu}{2m}, -\frac{C\psi}{2m}, \frac{K_3}{2Z^2}, \frac{K_4}{2} \right] = H_{44}^{s}$$
$$\gamma_{\substack{12\\21}}^{s} = \left[\frac{B\psi}{2m} \pm \frac{\sigma}{2Ct^2}(zK_3 + tK_4), \frac{-A\mu}{2m} \pm \frac{\rho}{2Ct^2}(zK_3 + tK_4), -\frac{K_3 D}{2CZ^2}, \frac{K_4 D}{2C} \right]$$
$$\gamma_{\substack{13\\31}}^{s} = \left[\frac{K_3}{2} \pm \frac{-\sigma\psi}{2zm}, \pm\frac{\rho\psi}{2zm}, \frac{K_1}{2} \mp \frac{\rho K_4}{2Ct}, \mp\frac{\rho K_4}{2Cz} \right]$$
$$\gamma_{\substack{14\\41}}^{s} = \left[\frac{K_4}{2} \pm \frac{\sigma\psi}{2mt}, \mp\frac{\rho\psi}{2mt}, \mp\frac{\rho K_3}{2CZ^2 t}, \frac{K_1}{2} \mp \frac{\rho K_3}{2CZt} \right]$$
$$\gamma_{\substack{23\\32}}^{s} = \left[\mp\frac{\sigma\mu}{2zm}, \frac{K_3}{2} \pm \frac{\rho\mu}{2zm}, \frac{K_2}{2} \mp \frac{\sigma K_4}{2Ct}, \mp\frac{\sigma K_4}{2Cz} \right]$$
$$\gamma_{\substack{24\\42}}^{s} = \left[\pm\frac{\sigma\mu}{2mt}, \frac{K_4}{2} \mp \frac{\rho\mu}{2mt}, \mp\frac{\sigma K_3}{2CZ^2 t}, \frac{K_2}{2} \mp \frac{\sigma K_3}{2CZt} \right]$$
$$\gamma_{\substack{24\\42}}^{s} = \left[0, 0, \frac{K_4}{2} \mp \frac{(\rho\mu - \sigma\psi)}{2mt}, \frac{K_3}{2} \mp \frac{(\rho\mu - \sigma\psi)}{2mt} \right] \tag{4.13}$$

Next we find the affine connections Γ_{ij}^{k} by using the relations

$$\Gamma_{ij}^{k} = \Gamma_{ij}^{k} + \Gamma_{ij}^{k} = p_{ij}^{k} + q_{ij}^{k} \tag{4.14}$$

$$p_{ij}^{k} = \{_{ij}^{k}\} + h^{kl}(q_{li}^{m} f_{jm} + q_{lj}^{m} f_{im}) \tag{4.15}$$

The values of Γ_{ij}^{k} already obtained in (**[11],**(3.15)) are

$$\Gamma_{11}^{k} = \left[0,0,\frac{\overline{A}}{2Ct},\frac{\overline{A}}{2Cz}\right]$$

$$\Gamma_{12}^{k} = \left[0,0,\frac{\overline{D}}{2Ct},\frac{\overline{D}}{2Cz}\right] = \Gamma_{21}^{k}$$

$$\Gamma_{13}^{k} = \left[aZ,bZ,\alpha,\alpha Z\right]$$

$$\Gamma_{14}^{k} = \left[-a,-b,-\frac{\alpha}{Z},-\alpha\right]$$

$$\Gamma_{22}^{k} = \left[0,0,\frac{B}{2Ct},\frac{\overline{B}}{2Cz}\right]$$

$$\Gamma_{23}^{k} = \left[dZ,eZ,\beta,Z\beta\right]$$

$$\Gamma_{24}^{k} = \left[-d,-e,-\frac{\beta}{Z},-\beta\right]$$

$$\Gamma_{24}^{k} = \left[-d,-e,-\frac{\beta}{Z},-\beta\right]$$

$$\Gamma_{31}^{k} = \left[aZ,bZ,-\alpha,-\alpha Z\right]$$

$$\Gamma_{32}^{k} = \left[dZ,eZ,-\beta,-Z\beta\right]$$

$$\Gamma_{33}^{k} = \left[0,0,-(\frac{1}{z}+\frac{\overline{C}Z}{2Cz}),(\frac{Z}{z}+\frac{\overline{C}Z^{2}}{2Cz})\right]$$

$$\Gamma_{34}^{k} = \left[0,0,(\frac{1}{t}+\frac{\overline{C}}{2Cz}),-\frac{\overline{C}Z}{2Cz}\right] = \Gamma_{43}^{k}$$

$$\Gamma_{41}^{k} = \left[-a,-b,\frac{\alpha}{Z},\alpha\right]$$

$$\Gamma_{42}^{k} = \left[-d,-e,\frac{\beta}{Z},\beta\right]$$

$$\Gamma_{44}^{k} = \left[0,0,-\frac{\overline{C}}{2Ct},-\frac{\overline{C}}{2Cz}\right] \tag{4.16}$$

where

$$a = \frac{1}{2mz}(\overline{D}D - \overline{A}B)$$

$$b=\frac{1}{2mz}(D\overline{A}-A\overline{D})$$
$$d=\frac{1}{2mz}(D\overline{B}-B\overline{D})$$
$$e=\frac{1}{2mz}(D\overline{D}-A\overline{B})$$
$$(a+e)=-\frac{(\overline{A}B-2\overline{D}D+A\overline{B})}{2mz}=\frac{-\overline{m}}{2mz}$$

and
$$\alpha=\frac{1}{Ct}(\rho a+\sigma b)-\frac{\rho}{ct^2}+\frac{1}{C^2tz}(\overline{\rho}C-\rho\overline{C})=\frac{1}{Ct}\left[(\rho a+\sigma b)-\frac{\rho}{t}+\frac{1}{Cz}(\overline{\rho}C-\rho\overline{C})\right]$$
$$\beta=\frac{1}{Ct}(\rho d+\sigma e)-\frac{\sigma}{ct^2}+\frac{1}{C^2tz}(\overline{\sigma}C-\sigma\overline{C})=\frac{1}{Ct}\left[(\rho d+\sigma e)-\frac{\sigma}{t}+\frac{1}{Cz}(\overline{\sigma}C-\sigma\overline{C})\right] \quad (4.17)$$

$$p_{11}^k=\left[0,0,\frac{\overline{A}}{2Ct},\frac{\overline{A}}{2Cz}\right]$$
$$p_{12}^k=\left[0,0,\frac{\overline{D}}{2Ct},\frac{\overline{D}}{2Cz}\right]$$
$$p_{13}^k=\left[aZ,bZ,0,0\right]$$
$$p_{14}^k=\left[-a,-b,0,0\right]$$
$$p_{22}^k=\left[0,0,\frac{\overline{B}}{2Ct},\frac{\overline{B}}{2Cz}\right]$$
$$p_{23}^k=\left[dZ,eZ,0,0\right]$$
$$p_{24}^k=\left[-d,-e,0,0\right]$$
$$p_{33}^k=\left[0,0,-\frac{1}{z}-\frac{\overline{C}Z}{2Cz},\frac{Z}{z}+\frac{\overline{C}Z^2}{2Cz}\right]$$
$$p_{34}^k=\left[0,0,\frac{1}{t}+\frac{\overline{C}}{2Ct},-\frac{\overline{C}Z}{2Cz}\right]$$
$$p_{44}^k=\left[0,0,-\frac{\overline{C}}{2Ct},\frac{\overline{C}}{2Cz}\right] \quad (4.18)$$

$$q_{14}^k=\left[0,0,\frac{-\alpha}{Z},-\alpha\right]$$
$$q_{22}^k=\left[0,0,0,0\right]$$
$$q_{23}^k=\left[0,0,\beta,\beta Z\right]$$
$$q_{24}^k=\left[0,0,\frac{-\beta}{Z},-\beta\right] \quad (4.19)$$

By substituting the value of Γ_{ij}^{s} from (4.16) and γ_{ij}^{s} from (4.13) and using equation(2.4) we get the component of L_{ij}^{s}.

5. Calculations of tensors G_{ij} and P_{ij}:-

The Gauge invariant Hermition-Einstein tensor as given by Buchdahl(1958) is

$$G_{ij} = P_{ij} - (K_{i,j} + K_{j,i} - 2K_m \Gamma_{ij}^{m}) - 2\gamma_{ij}^{m} K_m + \gamma_{ij,m}^{m} + \Gamma_{(mn)}^{m}\gamma_{ij}^{n} - \Gamma_{im}^{n}\gamma_{nj}^{m} - \Gamma_{mj}^{n}\gamma_{in}^{m} + \gamma_{im}^{n}\gamma_{nj}^{m} \quad (5.1)$$

and the Einstein tensor P_{ij} of usual asymmetric theory with a linear connection Γ_{ij}^{s} is given by

$$P_{ij} = -\Gamma_{ij,s}^{s} + \frac{1}{2}\left[\Gamma_{(is),j}^{s} + \Gamma_{(sj),i}^{s}\right] + \Gamma_{im}^{s}\Gamma_{sj}^{m} - \Gamma_{ij}^{s}\Gamma_{(sm)}^{m} \quad (5.2)$$

where Γ_{ij}^{s} and γ_{ij}^{s} are given by (4.16) and (4.13) respectively.

The component of P_{ij} are obtained by using (4.16) and the relation

$$\begin{aligned} \Gamma_{(ij)}^{j} &= \Gamma_{2j}^{j} = 0,\ j = 1,2 \\ \Gamma_{13}^{3} &= -\Gamma_{14}^{4} = \alpha \\ \Gamma_{23}^{3} &= -\Gamma_{24}^{4} = \beta \\ \Gamma_{33}^{3} &= -Z\Gamma_{43}^{3} = -\left(\frac{1}{z} + \frac{\overline{C}Z}{2Cz}\right) \end{aligned} \quad (5.3)$$

These are

$$\begin{aligned} P_{11} &= P_{12} = P_{21} = P_{22} = 0 \\ P_{32} &= -P_{23} = \frac{2\beta}{z} \\ P_{44} &= \left[(a^2 + 2bd + e^2) + \frac{(a+e)\overline{C}}{Cz} - \frac{(\overline{a} + \overline{e})}{z}\right] \\ P_{33} &= Z^2 P_{44} + \frac{(a+e)Z}{z} \\ P_{34} &= P_{43} = -ZP_{44} - \frac{a+e}{2z} \end{aligned} \quad . \quad (5.4)$$

Using (5.4), (4.13) and (4.16) in (5.1) we get the components of Gauge invariant Einstein tensor G_{ij} as follows:

$$G_{11} = \left[-2K_{1,1} + \frac{\bar{A}}{C}(\frac{K_3}{Z} + K_4) \right] - 2\left[\frac{K_1^2}{2} + \frac{-\psi^2}{2m} + \frac{A}{2C}\{(K_4^2) - (\frac{K_3}{Z^2})^2\} \right]$$
$$+ \left[(a + e - \frac{1}{t} - \frac{\bar{C}}{Cz})(\frac{-A}{2C})(\frac{K_3}{Z} + K_4) \right] - \left[(\frac{K_3}{Z} + K_4)(\frac{\bar{A}}{2Cz} - \frac{Aa}{C} - \frac{bD}{C} \right]$$
$$+ \left[3(\frac{K_1}{2})^2 + \frac{K_1 D\psi}{2m} + \frac{D^2\psi^2}{4m^2} - \frac{AB\psi^2}{2m^2} + \frac{A}{2C}(K_4^2 - (\frac{K_3}{Z^2})^2) + \frac{A^2\mu^2}{4m^2} - \rho(\frac{K_3}{Z} + K_4)^2 \right]$$
$$G_{22} = -2K_{2,2} + \frac{\mu^2}{m} + \frac{1}{4}K_2^2 + (\frac{-K_3 B}{2CZ^2})_{,3} + (\frac{K_4 B}{2C})_{,4} + \beta(\frac{\sigma K_4}{Ct} + K_2)$$
$$+ \frac{1}{4m^2}(B^2\psi^2 - 2AB\mu^2 - \mu^2 D^2) + \frac{\sigma}{2C^2}(\frac{K_3 K_4}{Zt^2})$$
$$G_{33} = \left[Z^2(a^2 + 2bd + e^2 - \frac{(\bar{a} + \bar{e})}{z}) + \frac{(a+e)Z}{z}(\frac{Z\bar{C}}{C} + 1) \right] - 2K_{3,3} - 2K_3(\frac{1}{z} + \frac{\bar{C}Z}{2Cz}) - \frac{Z^2 C}{2m}(\mu K_1 - \psi K_2)$$
$$-\frac{1}{2}Z^2 K_4^2 + (\frac{1}{2}K_{3,3} + \frac{Z^2}{2}K_{4,4} + \frac{ZK_4}{z}) - (\frac{1}{z^2} + \frac{1}{t^2})\left[\frac{1}{4m^2}(\sigma\psi - \rho\mu)^2 \right]$$

$$G_{44} = \left[(a^2 + 2bd + e^2 + \frac{(a+e)\bar{C}}{Cz} - \frac{(\bar{a} + \bar{e})}{z}) \right] + \frac{1}{2}(a + e - \frac{1}{t} - \frac{\bar{C}}{Cz})(\frac{K_3}{Z} + K_4)$$
$$+ \left[(\frac{1}{2Z^2}(K_{3,3} + \frac{2K_3}{z}) + \frac{1}{2}K_{4,4} \right] - \frac{1}{2m^2 t^2}(\rho\mu - \sigma\psi)^2 + (\frac{\mu C}{2m}K_1 - \frac{\psi C}{2m}K_2 - \frac{K_3^2}{2Z^2} + \frac{\bar{C}}{Cz}K_4) \tag{5.5}$$
$$G_{12} = -(K_{1,2} + K_{2,1}) + \frac{1}{m}(A\mu K_2 + B\psi K_1) + (\frac{-K_3 D}{2CZ^2})_{,3} + (\frac{K_4 D}{2C})_{,4}$$
$$+ \frac{1}{4m^2}(2K_1 B\psi + BD\psi^2 - K_2 A\mu m + AD\mu^2 - 2AB\mu\psi)$$
$$+ \left[\begin{array}{l} (\frac{K_3}{2} - \frac{\rho\mu}{2zm})(\frac{-K_3 D}{2CZ^2} - \frac{\bar{D}}{2Ct}) + (\frac{K_2}{2} + \frac{\sigma K_4}{2Ct})(\frac{K_1}{2} - \frac{\sigma K_4}{2Ct} - \alpha) \\ + (\frac{K_4}{2} + \frac{\rho\mu}{2mt})(\frac{K_4 D}{2C} - \frac{\bar{D}}{2Cz}) + (\frac{K_2}{2} + \frac{\sigma K_3}{2CZt})(\frac{K_1}{2} - \frac{\rho K_3}{2CZt} + \alpha) \end{array} \right] \tag{5.6}$$

6. Solution of Equations (2.5) and (2.6)

Equation (2.5) is $L_i = L^j_{[ij]} = 0$ i.e., $L_i = L^j_{[ij]} = \Gamma^j_{[ij]} - \gamma^j_{[ij]} = q^j_{ij} - S^j_{ij} = 0$

Substituting the components of Γ^s_{ij} from equation (4.6) and those of L^s_{ij} from (4.13) into the equation (2.5) we find that when i=1 and 2 , we get

$$K_3 + ZK_4 = 0 \tag{6.1}$$

and when i=2 and 3, we get

$$\sigma\psi - \rho\mu = 0 \tag{6.2}$$

Using the conditions (6.1) and (6.2) , and the equations (5.4)we get the solutions of the field equations(2.6) as:

$$-2K_{1,1} - \frac{K_1^2}{4} - (\frac{K_3 A}{2CZ^2})_{,3} + (\frac{K_4 A}{2C})_{,4} + \frac{K_1 D\psi}{2m} + \frac{D^2\psi^2}{4m^2} - \frac{AB\psi^2}{2m^2} + \frac{A^2\mu^2}{4m^2} = 0$$

$$-2K_{2,2} + \frac{\mu^2}{m} + \frac{1}{4}K_2^2 + (\frac{-K_3 B}{2CZ^2})_{,3} + (\frac{K_4 B}{2C})_{,4} + \frac{1}{4m^2}(B^2\psi^2 - 2AB\mu^2 - \mu^2 D^2) + \frac{\sigma}{2C^2}(\frac{K_3 K_4}{Zt^2}) = 0$$

$$\left[Z^2(a^2 + 2bd + e^2 - \frac{(\bar{a}+\bar{e})}{z}) + \frac{(a+e)Z}{z}(\frac{Z\bar{C}}{C}+1)\right] - 2K_{3,3} - 2K_3(\frac{1}{z} + \frac{\bar{C}Z}{2Cz}) + (\frac{1}{2}K_{3,3} + \frac{Z^2}{2}K_{4,4} + \frac{ZK_4}{z}) = 0$$

$$\left[(a^2 + 2bd + e^2 + \frac{(a+e)\bar{C}}{Cz} - \frac{(\bar{a}+\bar{e})}{z})\right] + \left[(\frac{1}{2Z^2}(K_{3,3} + \frac{2K_3}{z}) + \frac{1}{2}K_{4,4}\right] + (\frac{\mu C}{2m}K_1 - \frac{\psi C}{2m}K_2 - \frac{K_3^2}{2Z^2} + \frac{\bar{C}}{Cz}K_4) = 0$$

$$-(K_{1,2} + K_{2,1}) + \frac{1}{m}(A\mu K_2 + B\psi K_1) + (\frac{-K_3 D}{2CZ^2})_{,3} + (\frac{K_4 D}{2C})_{,4}$$

$$+\frac{1}{4m^2}(2K_1 B\psi + BD\psi^2 - K_2 A\mu m + AD\mu^2 - 2AB\mu\psi) + (\frac{K_3}{2} - \frac{\rho\mu}{2zm})(\frac{-K_3 D}{2CZ^2} - \frac{\bar{D}}{2Ct}) + (\frac{K_4}{2} + \frac{\rho\mu}{2mt})(\frac{K_4 D}{2C} - \frac{\bar{D}}{2Cz}) + 0$$

(6.3)

Thus the equations (6.1) and (6.2) are the necessary conditions in such a way that the gauge invariant generalized second field (2.6) are satisfied in plane symmetry in the sense of Taub.

Lemma: If $K_3 + ZK_4 = 0$ and $\sigma\psi - \rho\mu = 0$,then the necessary condition for (g_{ij}) given by (3.4) to be the solution of Buchdahl's field equation in Bondi space-time is that

$$\left[(a^2 + 2bd + e^2) + \frac{(a+e)\bar{C}}{Cz} - \frac{(\bar{a}+\bar{e})}{z}\right] = 0 \tag{6.4}$$

and

$$Z^2\left[(a^2 + 2bd + e^2) + \frac{(a+e)\bar{C}}{Cz} - \frac{(\bar{a}+\bar{e})}{z}\right] + \frac{(a+e)Z}{z} = 0 \tag{6.5}$$

There are several possibilities under which the solution of equation (2.6) may be considered. However in this paper we consider only the case given by equations (6.1) and (6.2) and write our conclusion in the form of the above Lemma.

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