Article

Physical Aspects of Bianchi Type Space-time with Quadratic Equation of State in *f*(*R*) Gravity

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Abstract

In this viewpoint, we examine Bianchi type-I space-time with quadratic equation of state in the metric version of $f(R)$ gravity. The exact solutions of the field equations are obtained by applying volumetric power law and exponential law of expansion. The physical and kinematical parameters of the models have been discussed using some physical quantities. Also, the function of the Ricci scalar is evaluated for each model.

Keywords: Bianchi Type-I, space-time, quadratic equation, f(R) gravity.

1. Introduction

The prediction about expansion of the universe is flat, past decelerating and present accelerating. The proposals that have been put forward to explain this observed phenomenon can basically be classified into two categories. One can measure due to mysterious energy with negative pressure dubbed as Dark Energy (DE) [1] and second is a classical generalization of General Relativity (GR). There are various generalization of GR namely $f(G), f(R,G), f(R)$, $f(T)$and $f(R,T)$ gravity. In $f(R,T)$ gravity [2] the Gravitational Lagrangian is given by an arbitrary function of the Ricci scalar (R) and trace of the stress energy tensor (T). Several authors have investigated the aspect of cosmological models in this gravity [3-7]. Einstein [8] has presented another form of gravity called Teleparallel Gravity, namely $f(T)$ gravity to explain the current accelerating expansion without introducing dark energy, which allows one to say gravity is not due to curvature, but due to torsion. In this theory some authors [9-12] have discussed several features of cosmological models. In a classical generalization of GR one replaces the Ricci scalar R in the Einstein-Hilbert action by an arbitrary function of R belongs to the well-known $f(R)$ modified gravity.

Considering a viable $f(R)$ gravity models Nojiri and Odintsov [13] shows that the unification of early-time inflation and late-time acceleration. Deriving the exact solution from a power law $f(R)$ cosmological model Capozziello et al. [14] achieve dust matter and DE phase. Using the same theory Azadi et al. [15] studied vacuum solution in cylindrically symmetric space time. Bianchi type-III cosmological model with bulk viscosity in $f(R)$ gravity investigated by katore and Shaikh [16]. Sharif and Yousaf [17] studied the impact of DE and dark matter models on the dynamical evolution of collapsing self-gravitating systems in this gravity.

Quadratic equation of state is needed to explore in cosmological models for the general relativistic dynamics. The general form of the quadratic equation of state is given by

$$p = p_0 + \alpha\rho + \beta\rho^2 ,$$

where p_0, α and β are the parameters.

Ananda and Bruni [18] discussed the anisotropic homogeneous and inhomogeneous cosmological models in GR with the equation of state of the form

$$p = \alpha\rho + \frac{\rho^2}{\rho_c} .$$

Along with many authors [19-24] have investigate quadratic equation of state for different context using cosmological models. Recently, Singh and Bishi [25] investigate Bianchi type-I cosmological model in $f(R,T)$ modified gravity for two different classes of $f(R,T)$ in presence of cosmological constant and quadratic equation of state based on the expansion law.

In our present study, we have consider the quadratic equation of state of the form

$$p = \alpha\rho^2 - \rho . \tag{1.1}$$

Incited by the above investigation, we investigate some features of Bianchi type-I space-time with quadratic equation of state in the frame work of $f(R)$ gravityby applying volumetric power law and exponential law of expansion. This paper is organized as follows: In section 2, we define some basic and field equation of $f(R)$ gravity. Metric and energy momentum tensor are given in section 3. Section 4, 5 and 6, contains field equations, their solutions using volumetric power and exponential law along with some geometrical and physical quantities. Finally, conclusions are summarized in the last section 7.

2. Some Basics of $f(R)$ Gravity

$f(R)$ theory of gravity is the generalization of GR (just by replacing R by $f(R)$ in the standard Einstein–Hilbert action).

The action for this theory is given by

$$S = \frac{1}{2k^2}\int d^4\sqrt{-g}\, f(R) + \int d^4 x L_m (g_{\mu\nu}, \psi_m) . \tag{2.1}$$

Here $f(R)$ is a general function of the Ricci Scalar, $k^2 = 8\pi G = 1, g$ is the determinant of the metric $g_{\mu\nu}$ and L_m is the metric Lagrangian that depends on $g_{\mu\nu}$ and the matter field ψ_m.

The corresponding field equations are found by varying the action with respect to the metric $g_{\mu\nu}$

$$F(R)\ R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = T^M_{\mu\nu}, \qquad (2.2)$$

where $\Box \equiv \nabla^\mu \nabla_\mu$, $F(R) \equiv \dfrac{df(R)}{dR}$ (2.3)

∇_μ is the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m.

3. Metric and Energy Momentum Tensor

We consider a spatially homogeneous and anisotropic Bianchi type-I space-time of the form

$$ds^2 = dt^2 - A^2(t) dx^2 - B^2(t)\left(dy^2 + dz^2\right), \qquad (3.1)$$

where the metric potentials A and B are the functions of time *t*.

Some geometrical parameters related with the metric potential for the space-time (3.1) are defined as

The average scale factor (a) and the volume (V) are

$$a = \sqrt[3]{AB^2}, \qquad V = a^3 = AB^2, \qquad (3.2)$$

The Hubble parameter (H) is given by

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \qquad (3.3)$$

The expansion scalar (θ), Anisotropy parameter (Δ) and shear scalar (σ) are defined as

$$\theta = \frac{A_4}{A} + 2\frac{B_4}{B}, \qquad (3.4)$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i^2}{H}\right)^2, \qquad (3.5)$$

$$\sigma^2 = \frac{3}{2}\Delta H^2. \qquad (3.6)$$

Let us consider that the matter content is a perfect fluid such that the energy momentum tensor $T_{\mu\nu}$ is taken as

$$T_{\mu\nu} = \left(p + \rho\right) u_\nu u_\mu - p g_{\mu\nu}, \qquad (3.7)$$

where p and ρ be the pressure and energy density of the fluid respectively, u^{ν} is the four-velocity vector of the fluid satisfying $u^{\nu}=(0,0,0,1)$ and $u^{\nu}u_{\nu}=1$.

Here, we have assumed an equation of state (EoS) in the general form $p=p(\rho)$ for the matter distribution [18]

$$p=\varepsilon\rho^2-\rho \tag{3.8}$$

where ε is the constant and strictly $\varepsilon\neq 0$.

4. Field equations and their solutions

In the presence of perfect fluid source given in equation (3.7), the field equations (2.2) corresponding to the metric (3.1) lead to the following set of linearly independent differential equations

$$\left(\frac{\ddot{A}}{A}+2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)F-\frac{1}{2}f(R)-2\left(\frac{\dot{B}}{B}\right)\dot{F}-\ddot{F}=\left(\rho-\varepsilon\rho^2\right), \tag{4.1}$$

$$\left(\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{B}^2}{B^2}\right)F-\frac{1}{2}f(R)-\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\dot{F}-\ddot{F}=\left(\rho-\varepsilon\rho^2\right), \tag{4.2}$$

$$\left(\frac{\ddot{A}}{A}+2\frac{\ddot{B}}{B}\right)F-\frac{1}{2}f(R)-\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\dot{F}=\left(\rho\right). \tag{4.3}$$

Here the overhead dot denotes differentiation with respect to *t*.

Making use of equation (4.1) and (4.3) along with (1.1), we have got

$$\rho^2=\frac{1}{\alpha}\left\{\left(2\frac{\ddot{B}}{B}-2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)F-\frac{\dot{A}}{A}\dot{F}+\ddot{F}\right\}. \tag{4.4}$$

Using equations (4.1) and (4.2), we get

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)+\left(\frac{\dot{A}^2}{A^2}-\frac{\dot{B}^2}{B^2}\right)+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A}+\frac{\dot{F}}{F}\right)=0, \tag{4.5}$$

whichyield

$$\frac{A}{B}=\exp\int\frac{c}{AB^2F}dt. \tag{4.6}$$

We work outabove equation (4.6) using power law relation between F and a, established by Kotub Uddin et al. [26]; Sharif and Shamir [27, 28] in the $f(R)$ gravity which shows that

$$F \propto a^m, \tag{4.7}$$

where m is an arbitrary constant.

Equation (4.7) leads to

$$F = ba^m, \tag{4.8}$$

where b is the proportionality constant.

Using the equation (3.2), equation (4.8) turns out to be

$$F = b\left(AB^2\right)^{m/3}. \tag{4.9}$$

Making use of equation (4.6) and (4.9), we have

$$\frac{A}{B} = \exp\left\{\frac{c}{b}\int \frac{1}{(AB^2)^{\frac{m+3}{3}}}\,dt\right\}. \tag{4.10}$$

As the system of field equations is consistent having three equations with six unknown. Hence, one can introduce more conditions either by an assumptions corresponding to some physical situations or an arbitrary mathematical supposition. The solutions to the field equations are generated by using two different forms of volumetric expansion laws i) power law and ii) exponential law [29, 30]

$$V = c_1 t^{3k}, \tag{4.11}$$

$$V = c_2 e^{3nt}, \tag{4.12}$$

where c_1, c_2 k and n are constants.

The model with exponential expansion exhibit accelerating volumetric expansion whereas the model of power law gives constant value of deceleration parameter. If $k \geq 1$, the model represents accelerated expansion, if $k < 1$the model exhibit a decelerating volumetric expansion and for $k = 1$, the average scale factor has a linear growth with constant velocity and the model is in inflationary phase.

5. Power Law Model

Using equations (3.2) and (4.10) for the power law volumetric expansion (4.11), we obtain

$$A = c_1^{1/3} t^k \exp\left\{\frac{-2c}{3b(km+3k-1)c_1^{m+3/3}} t^{1-3k-km}\right\}, \tag{5.1}$$

$$B = c_1^{1/3} t^k \exp\left\{ \frac{c}{3b(km+3k-1)c_1^{\frac{m+3}{3}}} t^{1-3k-km} \right\}. \tag{5.2}$$

From the above equations (5.1) and (5.2) it is observed that both metric potentials are the product of exponential and power term. At an initial stage, both the metric potentials are vanishes as $t \to 0,\ A, B \to 0$. Hence, initially the model has singularity and at large time i.e. $t \to \infty,\ A, B \to \infty$.This result is similar to that of the result obtained by Singh and Beesham [31].

Using equation (5.1) and (5.2), the Ricci scalar of the model is found to be

$$R = \frac{6k(k-1)}{t^2} + \frac{12c^2}{9b^2 c_1^{2(m+3)/3} t^{6k+2mk}}. \tag{5.3}$$

Using equation (2.3) and (5.3), the function of the Ricci scalar of the model is obtained as

$$f(R) = \frac{12bk(1-k)c_1^{m/3}}{(km-2)t^{2-km}} + \frac{24bc^2(3k+mk)}{9b^2(6k+mk)c_1^{(m+6)/3} t^{6k+mk}}. \tag{5.4}$$

Equation (5.4) represents the function of the Ricci scalar of the model, it is clear that the function of the Ricci scalar is positive and decreasing function of time. This is analogous to the function obtained by Shamir [32].

Using equations (5.1) and (5.2) in equation (3.1), the space-time becomes

$$\begin{aligned} ds^2 = dt^2 - c_1^{2/3} t^{2k} \exp\left\{ \frac{-4c}{3b(km+3k-1)c_1^{\frac{m+3}{3}}} t^{1-3k-km} \right\} dx^2 \\ - c_1^{2/3} t^{2k} \exp\left\{ \frac{2c}{3b(km+3k-1)c_1^{\frac{m+3}{3}}} t^{1-3k-km} \right\} \left(dy^2 + dz^2\right) \end{aligned}, \tag{5.5}$$

The model (5.5) with $A(t)$ and $B(t)$ given by the equations (5.1) and (5.2) represents an exact accelerating Bianchi type-I (LRS) space-time. At the initial time $t = 0$ when the model start to expand the directional scale factors $A(t)$ and $B(t)$ are vanishes hence it represent a singular model.

The Hubble parameter, expansion scalar, deceleration parameter, Anisotropy parameter and Shear scalar are as follows

$$H = \frac{k}{t}. \tag{5.6}$$

$$\theta = \frac{3k}{t}. \tag{5.7}$$

$$q = -1 + \frac{1}{k} . \tag{5.8}$$

$$\Delta = \frac{2}{9} \frac{c^2}{\left(kbc_1^{(m+3)/3} \right)^2} \frac{1}{t^{6k+2mk-2}} . \tag{5.9}$$

$$\sigma^2 = \frac{c^2}{3\left(bc_1^{(m+3)/3} \right)^2} \frac{1}{t^{6k+2mk}} . \tag{5.10}$$

In power law model it is observed that, the average scale factor and spatial volume increases with time t i.e. when $t \to \infty$ spatial volume $V \to \infty$. Thus inflation is possible in Bianchi type-I space-time. The mean Hubble parameter, expansion scalar and shear scalar all are initially infinitely large. Also, we observed that the expansion scalar and shear scalar are decreases for $k > 0$ with respect to the time. The value of the anisotropy parameter shows that as time tends to infinity, the anisotropy parameter tends to zero i.e. the universe tends to isotropy and at large time the shear becomes insignificant.

With the help of equation (4.1), (4.3) and (1.1) the energy density is obtained as

$$\rho^2 = \frac{1}{\alpha} \left\{ \frac{(km^2 - km - m - 2)kbc_1^{m/3}}{t^{2-km}} + \frac{2c^2}{3c_1^{(m+6)/3}} \frac{1}{t^{6k+mk}} \right\} . \tag{5.11}$$

Using equation (5.11) in (1.1), we have the pressure as

$$p = \left\{ \left(\frac{(km^2 - km - m - 2)kbc_1^{m/3}}{t^{2-km}} + \frac{2c^2}{3c_1^{(m+6)/3}} \frac{1}{t^{6k+mk}} \right) - \sqrt{\frac{1}{\alpha} \left(\frac{(km^2 - km - m - 2)kbc_1^{m/3}}{t^{2-km}} + \frac{2c^2}{3c_1^{(m+6)/3}} \frac{1}{t^{6k+mk}} \right)} \right\} . \tag{5.12}$$

In power law model, it is observed that the energy density is a function of time t and always decrease positively with the expansion. At the initial stage $t \to 0$ the universe has infinitaly large energy density $\rho \to \infty$ but with the expansion of the universe it declines and at large $t \to \infty$ it is null $\rho \to 0$.

6. Exponential Expansion Model

Using equations (3.2) and (4.10) for exponential volumetric expansion (4.12), we obtain

$$A = c_2^{1/3} \exp\left\{ nt + \frac{-2c}{3nb(m+3)c_2^{(m+3)/3}} \exp\{-n(m+3)t\} \right\} , \tag{6.1}$$

$$B = c_2^{1/3} \exp\left\{ nt + \frac{c}{3nb(m+3)c_2^{(m+3)/3}} \exp\{-n(m+3)t\} \right\}. \tag{6.2}$$

From the above equations (6.1) and (6.2) it is observed that at an initial stage near $t = 0$, both the metric potentials admits constant values and they evolve with time without any type of singularity and finally diverge to infinity which resembles with Singh and Beesham [31].

Using equation (6.1) and (6.2), the Ricci scalar of the model is found to be

$$R = 12n^2 + \frac{6c^2}{9b^2 c_2^{2(m+3)/3}} \exp\{-2n(m+3)t\}. \tag{6.3}$$

Using equation (2.3) and (6.3), the function of the Ricci scalar of the model is obtained as

$$f(R) = \frac{12bk(1-k)c_1^{m/3}}{(km-2)t^{2-km}} + \frac{4c^2}{9b(m+6)c_2^{(m+6)/3}} \exp\{-n(m+6)t\}. \tag{6.4}$$

Equation (6.4) represents the function of the Ricci scalar of the model, it is clear that the function of the Ricci scalar is positive and decreasing function of time. This behavior is analogous to the result obtained by Shamir [32].

Using equations (5.1) and (5.2) in equation (3.1), the space-time becomes

$$\begin{aligned} ds^2 = dt^2 - c_2^{2/3} \exp\left\{ 2nt + \frac{-4c}{3nb(m+3)c_2^{(m+3)/3}} \exp\{-n(m+3)t\} \right\} dx^2 \\ - c_1^{2/3} \exp\left\{ 2nt + \frac{2c}{3nb(m+3)c_2^{(m+3)/3}} \exp\{-n(m+3)t\} \right\} \left(dy^2 + dz^2\right) \end{aligned}, \tag{6.5}$$

The model (6.5) with $A(t)$ and $B(t)$ given by the equations (6.1) and (6.2) represents an exact accelerating Bianchi type-I (LRS) space-time. At the initial time $t = 0$ when the model start to expand the directional scale factors $A(t)$ and $B(t)$ are constant. Hence, initially the model is free from singularity.

The Hubble parameter, expansion scalar, deceleration parameter, Anisotropy parameter and Shear scalar are as follows

$$H = n \tag{6.6}$$
$$\theta = 3n \tag{6.7}$$
$$q = -1 \tag{6.8}$$

$$\Delta = \frac{2}{9}\frac{c^2}{n^2 b^2 c_2^{\frac{2(m+3)}{3}}}\exp\{-2n(m+3)t\} \tag{6.9}$$

$$\sigma^2 = \frac{c^2}{b^2 c_2^{\frac{2(m+3)}{3}}}\exp\{-2n(m+3)t\} \tag{6.10}$$

In this cosmology, we observe that the spatial volume approaches to positive small value at $t \to 0$ and with the expansion of the time the universe expands exponentially. The results of Hubble's parameter and expansion scalar yield the constant values. The value of the anisotropy parameter shows that as time tends to infinity, the anisotropy parameter tends to zero i.e. the universe tends to isotropy and at large time the shear becomes insignificant.

With the help of equation (4.1), (4.3) and (1.1) the energy density is obtained as

$$\rho^2 = \frac{1}{\alpha}\left\{\frac{6c^2}{9bc_2^{(m+6)/3}}\exp\{-n6(m+1)t\} + mn^2 b(m-1)c_2^{m/3}\exp\{mnt\}\right\}. \tag{6.11}$$

Using equation (5.11) in (1.1), we have the pressure as

$$p = \left\{\frac{\left(\frac{6c^2}{9bc_2^{(m+6)/3}}\exp\{-n6(m+1)t\} + mn^2 b(m-1)c_2^{m/3}\exp\{mnt\}\right)}{-\sqrt{\frac{1}{\alpha}\left(\frac{6c^2}{9bc_2^{(m+6)/3}}\exp\{-n6(m+1)t\} + mn^2 b(m-1)c_2^{m/3}\exp\{mnt\}\right)}}\right\}. \tag{6.12}$$

In eqponential expansion model, it is observed that the energy density is a function of time t and always decrease positively with the expansion. At the initial stage $t \to 0$ the universe has constant energy density but with the expansion of the universe it declines and at large $t \to \infty$ it is null $\rho \to 0$.

7. Conclusions

Bianchi type-I space-time with quadratic equation of state in the metric version of $f(R)$ gravity have been studied in relation to volumetric power law and exponential law expansion. Our derived model exhibited both accelerating as well as decelerating phase, in both phases universe is expanding and expansion of the universe is much faster and then slows down for later time.

In power law model, the universe starts with zero volume, the Hubble parameter and the scalar expansion are the functions of time having relation $H, \theta \propto {}^{1}/_{t}$ and initially $t \to 0$ they attain infinitely large value and decreases with expansion and approaches to zero at large expansion. Our derived power law model exhibited both accelerating as well as decelerating phase.

In an exponential expansion, the model starts with constant volume for $c_2 > 0$ and for $c_2 = 0$ it starts with zero volume and expand exponentially with infinite time. The expansion scalar and the shear scalar are constant and the model shows only accelerating phase.
In models, anisotropy parameter and shear scalar are the time dependant (observed with relation $\Delta,\ \sigma^2 \propto {}^{1}/_{t}$). At infinite time anisotropy parameter tends to zero and shear becomes insignificant. The function of Ricci scalar is also related with expansion i.e.$R \propto {}^{1}/_{t}$.

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