Article

Is Non-associative Physics & Language Possible only in Many-sheeted Spacetime?

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Abstract

Language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D imbedding space $M^4 \times CP_2$ is one of the key conjectures of TGD. But what is the situation in many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes- that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA. The mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity.

1 Introduction

In Thinking Allowed Original (see `https://www.facebook.com/groups/thinkallowed/`) there was very interesting link added by Ulla about the possibility of non-associative quantum mechanics (see `http://phys.org/news/2015-12-physicists-unusual-quantum-mechanics.html#jCp`).

Also I have been forced to consider this possibility.

1. The 8-D imbedding space of TGD has octonionic tangent space structure and octonions are non-associative. Octonionic quantum theory however has serious mathematical difficulties since the operators of Hilbert space are by definition associative. The representation of say octonionic multiplication table by matrices is possible but is not faithful since it misses the associativity. More concretely, so called associators associated with triplets of representation matrices vanish. One should somehow transcend the standard quantum theory if one wants non-associative physics.

2. Associativity seems to be fundamental in quantum theory as we understand it recently. Associativity is a fundamental and highly non-trivial constraint on the correlation functions of conformal field theories. It could be however broken in weak sense: as a matter of fact, Drinfeld's associator emerges in conformal field theory context. In TGD framework classical physics is an exact part of quantum theory so that quantum classical correspondence suggests that associativity could play a

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highly non-trivial role in classical TGD. The conjecture is that associativity requirement fixes the dynamics of space-time sheets - preferred extremals of Kähler action - more or less uniquely. One can endow the tangent space of 8-D imbedding $H = M^4 \times CP_2$ space at given point with octonionic structure: the 8 tangent vectors of the tangent space basis obey octonionic multiplication table.

Space-time realized as n-D surface in 8-D H must be either associative or co-associative: this depending on whether the tangent space basis or normal space basis is associative. The maximal dimension of space-time surface is predicted to be the observed dimension $D = 4$ and tangent space or normal space allows a quaternionic basis.

3. There are also other conjectures [**?**]twistorstory about what the preferred extremals of Kähler action defining space-time surfaces are.

   (a) A very general conjecture states that strong form of holography allows to determine space-time surfaces from the knowledge of partonic 2-surfaces and 2-D string world sheets.

   (b) Second conjecture involves quaternion analyticity and generalization of complex structure to quaternionic structure involving generalization of Cauchy-Riemann conditions.

   (c) $M^8 - M^4 \times CP_2$ duality stating that space-time surfaces can be regarded as surfaces in either M^8 or $M^4 \times CP_2$ is a further conjecture.

   (d) Twistorial considerations select $M^4 \times CP_2$ as a completely unique choice since M^4 and CP_2 are the only spaces allowing twistor space with Kähler structure. The conjecture is that preferred extremals can be identified as base spaces of 6-D sub-manifolds of the product $CP_3 \times SU(3)/U(1) \times U(1)$ of twistor spaces associated with M^4 and CP_2 having the property that it makes sense to speak about induced twistor structure.

   The "super(optimistic)" conjecture is that all these conjectures are equivalent.

The motivation for what follows emerged from the observation that language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are however associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D imbedding space $M^4 \times CP_2$ is one of the key conjectures of TGD.

But what about many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes- that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA.

Fortunately the mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity. With good luck this breaking of associativity is all that is needed. With not so good luck this breaking of associativity takes place already at the level of single space-time sheets and something else is needed in many-sheeted space-time.

2 Is non-associative physics possible in many-sheeted space-time?

The key question in the sequel is whether non-associative physics could emerge in TGD via *many-sheeted* space-time as an outcome of many-sheetedness and therefore distinguishing TGD from GRT and various QFTs.

2.1 What does non-associativity mean?

To answer this question one must first understand what non-associativity could mean.

1. In non-associative situation brackets matter. $A(BC)$ is different from $(AB)C$. Here AB need not be restricted to a product or sum: it can be anything depending on A and B. From schooldays or at least from the first year calculus course one recalls the algorithm: when calculating the expression involving brackets one first finds the innermost brackets and calculates what is inside them, then proceed to the next innermost brackets, etc... In computer programs the realization of the command sequences involving brackets is called parsing and compilers perform it. Parsing involves decomposition of program to modules calling modules calling.... Quite generally, the analysis of linguistic expressions involves parsing. Bells start to ring as one realizes that parsings form a hierarchy as also do the space-time sheets!

2. More concretely, there is hierarchy of brackets and there is also a hierarchy of space-time sheets labelled by p-adic primes and perhaps also by Planck constants $h_{eff} = n \times h$. B and C inside brackets form (BC), something analogous to a bound state or chemical compound. In TGD this something could correspond to a "glueing" space-time sheets B and C at the same larger space-time sheet. More concretely, (BC) could correspond to braided pair of flux tubes B and C inside larger flux tube, whose presence is expressed as brackets (..). As one forms $A(BC)$ one puts flux tube A and flux tube (BC) containing braided flux tubes B and C inside larger flux tube. For $(AB)C$ flux one puts tube (AB) containing braided flux tubes A and B and tube C inside larger flux tube. The outcomes are obviously different.

3. Non-associativity in this sense would be a key signature of many-sheeted space-time. It could show itself in say molecular chemistry, where putting on same sheet could mean formation of chemical compound AB from A and B. Another highly interesting possibility is hierarchy of braids formed from flux tubes: braids can form braids, which in turn can form braids,... Flux tubes inside flux tubes inside... Maybe this more refined breaking of associativity could underly the possible non-associativity of biochemistry: biomolecules looking exactly the same would differ in subtle manner.

4. What about quantum theory level? Non-associativity at the level of quantum theory could correspond to the breaking of associativity for the correlation functions of n fields if the fields are not associated with the same space-time sheet but to space-time sheets labelled by different p-adic primes. At QFT limit of TGD giving standard model and GRT the sheets are lumped together to single piece of Minkowski space and all physical effects making possible non-associativity in the proposed sense are lost. Language would be thus possible only in TGD Universe!

2.2 Language and many-sheeted physics?

Non-associativity is an essentially linguistic phenomenon and relates therefore to cognition. p-Adic physics labelled by p-adic primes fusing with real physics to form adelic physics are identified as the physics of cognition in TGD framework.

1. Could many-sheeted space-time of TGD provides the geometric realization of language like structures? Could sentences and more complex structures have many-sheeted space-time structures as geometrical correlates? p-Adic physics as physics of cognition would suggest that p-adic primes

label the sheets in the parsing hierarchy. Could bio-chemistry with the hierarchy of magnetic flux tubes added, realize the parsing hierarchies?

2. DNA is a language and might provide a key example about parsing hierarchy. The mystery is that human DNA and DNAs of most simplest creatures do not differ much. Our cousins have almost identical DNA with us. Why do we differ so much? Could the number of parsing levels be the reason- p-adic primes labelling space-time sheets? Could our DNA language be much more structured than that of our cousins. At the level of concrete language the linguistic expressions of our cousin are indeed simple signals rather than extremely complex sentences of old-fashioned German professor forming a single lecture each. Could these parsing hierarchies realize themselves as braiding hierarchies of magnetic flux tubes physically and - more abstractly - as analos of parsing hierarchies for social structures. Indeed, I have proposed that the presence of collective levels of consciousness having the hierarchy of magnetic bodies as a space-time correlates distinguishes us from our cousins so that this explanation is consistent with more quantitative one relying on language.

3. I have also proposed that intronic portion of DNA is crucial for understanding why we differ so much from our cousins [**?**]dnatqc,dnatqccodes. How does this view relate to the above proposal? In the simplest model for DNA as topological quantum computer introns would be connected by flux tubes to the lipids of nuclear and cell membranes. This would make possible topological quantum computations with the braiding of flux tubes defining the topological quantum computer program.

   Ordinary computer programs rely on computer language. Same should be true about quantum computer programs realized as braidings. Now the hierarchical structure of parsings would correspond to that of braidings: one would have braids, braids of braids, etc... This kind of structure is also directly visible as the multiply coiled structure of DNA. The braids beginning from the intronic portion of DNA would form braided flux tubes inside larger braided flux tubes inside.... defining the parsing of the topological quantum computer program. The higher the number of parsing levels, the higher the position in the evolutionary hierarchy. Each braiding would define one particular fundamental program module and taking this kind of braided flux tubes and braiding them would give a program calling these programs as sub-programs.

4. The phonemes of language have no meaning to us (at our level of self hierarchy) but the words formed by phonemes and involving at basic level the braiding of "phoneme flux tubes" would have. Sentences and their substructures would in turn involve braiding of "word flux tubes". Spoken language would correspond to temporal sequences of braidings of flux tubes at various hierarchy levels.

5. The difference between us and our cousins (or other organisms) would not be at the level of visible DNA but at the level of magnetic body. Magnetic bodies would serve as correlates also for social structures and associated collective levels of consciousness. The degree of braiding would define the level in the evolutionary hierarchy. This is of course the basic vision of TGD inspired quantum biology and quantum bio-chemistry in which the double formed by organism and environment is completed to a triple by adding the magnetic body.

2.3 What about the hierarchy of Planck constants?

p-Adic hierarchy is not the only hierarchy in TGD Universe: there is also the hierarchy of Planck constants $h_{eff} = n \times h$ giving rise to a hierarchy of intelligences. What is the relationship between these hierarchies?

1. I have proposed that speech and music are fundamental aspects of conscious intelligence and that DNA realizes what I call bio-harmonies in quite concrete sense [**?**]harmonytheory [**?**]hearing: DNA codons would correspond to 3-chords. DNA would both talk and sing. Both language and music

are highly structured. Could the relation of h_{eff} hierarchy to language be same as the relation of music to speech?

2. Are both musical and linguistic parsing hierarchies present? Are they somehow dual? What does parsing mean for music? How musical heard sounds could give rise to the the analog of braided strands? Depending on the situation we hear music both as separate notes and as chords as separate notes fuse in our mind to a larger unit like phonemes fuse to a word. Could chords played by single instrument correspond to braidings of flux tubes at the same level? Could the duality between linguistic and musical intelligence (analogous to that between function and its Fourier transform) be very concrete and detailed and reflect itself also as the possibility to interpret DNA codons both as three letter words and as 3-chords [**?**]harmonytheory?

3 Braiding hierarchy mathematically

More precise formulation of the braided flux tube hierarchy leads naturally to the notions of braid group and operad that I have considered earlier. They have a close relationship with quantum groups - more precisely, bialgebras and Hopf algebras and their generalizations quasi-bialgebras and quasi-Hopf algebras, which in turn allow to characterize what might be called minimal breaking of associativity in terms of Drinfeld associator. These notions are already familiar from conformal field theories and string theories them so that there are good hopes that no completely new mathematics is not needed.

It must be made clear that I am not a mathematician and the following is just a modest attempt to understand what the problem is. I try to identify the algebraic structure possibly allowing to realize the big vision and gather some results about these structures from Wikipedia: I confess that I do not understand the formulas at the deeper level and my goal is to find their physical interpretation in TGD framework.

3.1 How to represent the hierarchy of braids?

Before going to web to see how modern mathematics could help in the problem, try first to formulate the situation more concretely. One must consider a more detailed representation for braids and for their hierarchy.

Consider first rough physical geometric view about braids of braids represented in terms of flux tubes.

1. Braid strands have two ends: one can label them as "lower" and "upper". Flux tubes can be labelled by p-adic prime p and $h_{eff} = n \times h$. Magnetic flux tubes can carry monopole flux and this could be crucial for the breaking of associativity - at least it is so in the proposed model (see `http://phys.org/news/2015-12-physicists-unusual-quantum-mechanics.html#jCp`). The possibility of apparent magnetic monopoles in TGD framework indeed involves many-sheetedness in an essential manner: monopole flux flows from space-time sheet to another one through wormhole contact. This can be taken as one possible hint about the concrete physics involved.

2. One can get more precise picture by using formulas. One has labelling of flux tubes by primes p and Planck constants h_{eff}: to be short call this label $a, b, c, ...$ Since the values of p and h_{eff} are graded one could also speak of grading. The states for given value of a assignable to braid strands are labelled by the quantum states $A, B, ...$ associated with them and analogous to algebra elements. One must however consider all possible situations so that has operators $A_a, B_a, ...$ analogous to algebra elements of a graded algebra about which Clifford algebras and super-algebras are familiar examples.

3. Consider now the physical interpretation for the breaking of associativity. For ordinary associative algebra one considers $A(BC) = (AB)C$. This condition as such make sense if $A(BC)$ and $(AB)C$ are inside same flux tube and perhaps also that the strands A, B, C are not braids. In the general

case one must must add the labels a, b, c, d and a, b_1, c_1, d_1 and one obtains $((A_dB_d)_c)C_b)_a$ and $(A_{b_1}(B_{d_1}C_{d_1}))_{c_1})_a$. Obviously, these two states need not identical unless one has $a = b = c = d = b_1 = c_1 = d_1$, which is also possible and means that all strands are at the same flux tube labelled by a. The challenge is to combine various almost copies of algebraic structure defined by braidings and labelled by $a, b, ..$ to larger algebraic structure and formulate the breaking of associativity for this structure.

3.2 Braid groups as coverings of permutation groups

Consider next the definition of braid group.

1. The notion of braiding can be algebraized using the notion of braid group B_n of n strands, which is covering of the permutation group S_n. For ordinary permutations generating permutations are exchanges of P_i two neighboring elements in the ordered set $(a_1, ..., a_n)$: $(a_i, a_{i+1}) \rightarrow (a_{i+1}, a_i)$. Obviously one has P_i^2 so that permutation is analogous to reflection. For braid group permutation is replaced to twisting of neighboring braid strand. It looks like permutation if one looks at the ends of strands only. If one looks entire strands, there is no reason to have $P_i^2 = 1$ except possibly for the representation of braid group. For arbitrarily large n that one has $P_i^n \neq 1$. 2-D braid group B_n can be represented as a homotopies of 2-D plane with n punctures identifiable as ends of braid strands defined by their non-intersecting orbits.

2. At the level of quantum description one must allow quantum superpositions of different braidings and must describe the quantum state of braid as wave function in braid group: one has element of group algebra of braid group. To each element of braid group one can assign unitary matrix representing the braiding and this unitary matrix would define a "topological time evolution" defined by braiding transforming the initial state at the lower end of braid to the state at upper end of braid. Hence it seems that braid group algebra is the proper mathematical notion. One has quantum superposition of topological time evolutions: something rather abstract.

3.3 Braid having braids as strands

Many-sheeted space-time makes possible fractal hierarchy of braids. Braid group in above sense would act on flux tubes at the same space-time sheets or space-time of QFT and GRT. Braids can have as strands braids so that there is hierarchy of braiding levels. The hierarchy of coilings of DNA provides a simple example (very simple having not much to do with the hierarchy of braidings for flux tubes).

1. Suppose that one has only two levels in the hierarchy. One has n braid strands/flux tubes altogether and there are k larger flux tubes containing n_i, $i = 1, .., k$ flux tubes so that one has $\sum_{i=1}^{k} n_i = n$. One can imagine a coloring of the braid strands inside given flux tube characterizing it. Only braid strands inside same flux tube - with the same color - can be braided. The full braid group B_n braiding freely all n braid strands is restricted to a subgbroup $B_{n_1} \times .... \times B_{n_2}$. This group can be regarded as subgroup of B_n so that permutations of B_{n_i} have a well-defined outcome, which seems however to be trivial classically. In quantum situation the exchange of the factors B_{n_i} however corresponds to braiding and for non-trivial quantum deformations its action is non-trivial. One has braided commutativity instead of commutativity.

2. Besides this there are braidings for the k braids of braids and this gives braid group B_k acting at upper level of hierarchy. Clearly the higher level braids b_i, $i = 1, ..., k$ and lower level braids b_{ij}, $j = 1, ..., n_i$ form a two-levelled entity. The braid groups B_k and B_{n_i} form an algebraic entity such that B_k acts by permuting the entities. Same holds true for the braid group algebras. This structure generalizes to an entire hierarchy of braid groups and their group algebras.

The hierarchy of braid group algebras seems to closely relate to a very general notion known as operad (see `https://en.wikipedia.org/wiki/Operad_theory`). The key motivation of the operad theory is to model the computational trees resulting from parsing. The action of permutations/braidings on the basic objects is central notion and one indeed has hierarchy of symmetric groups/braid groups such that the symmetric/braid group at $n+1$:th level permutes/braids the objects at n:th level. Now the objects would be braids whose strands are braided. The braids can be strands of higher level braids and these strands can be braided. The action of braidings extends to that on braid group algebras defining candidates for wave functions.

4 General formulation for the breaking of associativity in the case of operads

The formulas characterizing weak form of associativity by Drinfeld and others look rather mysterious without understanding of their origins. This understanding emerges from very simple but general basic arguments. Instead of studying given algebra one transcends to a higher abstraction level and studies - not the results of algebraic expressions - but the very process how the algebraic expression is evaluated and what kind of rules one can pose on it. The rules can be abstracted to what is called algebraic coherence.

The evaluation process - parsing - starts from inner most brackets and proceeds outwards so that eventually all brackets have disappeared and one has the value for the expression. This process can be regarded as a tree which starts from n inputs which are algebra elements, in the recent case they could be braid group algebra elements.

For instance, $(AB)C$ corresponds to an tree in which A, B, C are the branches. As one comes downwards, A and B fuse in the upper node and AB and C in the lower node. One manner to see this is as particle reaction proceeding backwards in time. For $A(BC)$ B and C fuse to BC in the upper node and A and BC at the lower node. Associativity says that the two trees give the same result. "Braided associativity" would say that these trees give results differing by an isomorphism just as braided commutativity says that AB and BA give results differing by isomorphism.

One can formulate this more concretely by denoting algebra decomposition $A\otimes B \in V\otimes V \to AB \in V$ by θ. In associativity condition one has 3 inputs so that 3-linear map $V \otimes V \otimes V \to V$ is in question. $(AB)C$ corresponds to $\theta\circ(\theta, 1)$ applied to $(A\otimes B\otimes C)$. Indeed, $(\theta, 1)$ gives $(AB, C) \in V\otimes V$. Second step $\theta\circ$ applied to this gives $(AB)C$. In the same manner, $A(BC)$ corresponds to $(\theta \circ (1, \theta)$ and associativity condition can be expressed as

$$\theta \circ (\theta, 1) = \theta \circ (1, \theta) \ .$$

An important delicacy should be mentioned. Although operations can be non-associative, the composition of operations is assumed to be associative. One can imagine obtaining $((ab)c)d$ either by $\theta \circ (\theta, 1) \circ (\theta, 1, 1))$ or by $(\theta \circ (\theta, 1)) \circ (\theta, 1, 1))$. The condition that these expressions are identical is completely analogous to the associativity for the composition of functions $f \circ (g \circ h) = (f \circ g) \circ h$ and this axiom looks obvious becomes one is used to *define* $f \circ g$ using this formula (starting from rightmost brackets). One could however imagine starting the evaluation of the composition of operators also from leftmost brackets. This makes sense if the composition can be done without the substitution of the value of argument.

4.1 How associativity could be broken?

How to obtain the breaking of associativity? The first thing is to get some idea about what (weak) breaking of associativity could mean.

4.1.1 Breaking of associativity at the level of algebras

Basic examples about breaking of associativity might help in the attempts to understand how many-sheetedness could induce the breaking of associativity. The intuitive feeling is that the effect is not large and disappears at QFT limit of TGD.

In the case of algebras one has bilinear map $V \otimes V \to V$. Now this map is from $V \otimes V \to V \otimes V$ so that the two situations need not have much common. Despite this one can look the situation in the case of algebras.

Lie-algebras and Jordan algebras represent key examples about non-associative algebras. Associative algebras, Lie-algebras, and Jordan algebras can be unified by weakning the associativity condition $A(BC) = (AB)C$ to a condition obtained by cyclically symmetrizing this condition to get the condition

$$A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$$

plus the condition

$$(A^2B)A = A^2(BA)$$

defining together with commutativity condition $AB = BA$ Jordan algebra (http://www.bjp-bg.com/papers/bjp2014_2_071-076.pdf). Note that Jordan algebra with multiplication $A \cdot B$ is realized in terms of associative algebra product as $A \cdot B = (AB + BA)/2$. A good guess is that the non-associative Malcev algebra formed by imaginary octonions with product $xy - yx$ satisfies these conditions.

Could the analog of the condition $A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$ make sense also for the braiding group algebra assignable to quantum states of braids? The condition would say that cyclic symmetrization by superposing different braiding topologies gives a quantum state, which is in well-defined sense associative. Cyclic symmetry looks attractive because it plays also a key role in twistor Grassmannian approach.

4.1.2 Bi-algebras and Hopf algebras

One must start from bi-algebra $(B, \nabla, \eta, \Delta, \epsilon)$. One has product ∇ and co-product Δ analogous to replication of algebra element: particle physicists has tendency to see it as "time reversal" of product analogous to particle decay as reversal of particle fusion. The key idea is that co-multiplication is algebra homomorphism for multiplication and multiplication algebra homomorphism for co-multiplication. This leads to four commutative diagrams essentially expressing this property (see https://en.wikipedia.org/wiki/Bialgebra).

Instead of giving the general definitions it is easier to consider concrete example of bi-algebra defined by group algebra. Bi-algebra has product $\nabla : H \otimes H \to H$ and co-product $\Delta : H \to H \otimes H$, which intuitively corresponds to inverse or time reversal of product. In the case of group algebra this holds true in very precise sense since one has $\Delta(g) = g \otimes g$: Δ is clearly analogous to replication. Besides this one has map $\epsilon : H \to K$ assigning to the algebra element a scalar and inverse map taking the unit 1 of the field to unit element of H, called also 1 in the following. For group algebra one has $\epsilon(g) = 1$. Bi-algebras are associative and co-associative. Commutativity is however only braided commutativity.

Hopf algebra $(H, \nabla, \eta, \Delta, \epsilon, S)$ is special case of bi-algebra and often loosely called quantum group. The additional building brick is algebra anti-homomorphism $S : H \to H$ known as antipode. S is analogous to mapping element of h to its inverse (it need not exist always). For group algebra one indeed has $S(g) = g^{-1}$. Besides the four commuting diagrams for bi-algebra one has commutative diagrams $\nabla(S, 1)\Delta = \eta\epsilon$ and $\nabla(1, S)\Delta = \eta\epsilon$, where ϵ is co-unit. The right hand side gives a scalar depending on h multiplied by unit element of H. For group algebra this gives unit at both sides. In the general case the situation $\Delta(h) = h \otimes h$ is true for group like element only and one has more complex formula $\Delta(h) = \sum_i a_i \otimes b_i$. One also defines primitive elements as elements satisfying $\Delta(h) = h \otimes 1 + 1 \otimes h$. Also Hopf algebras are associative and co-associative.

4.1.3 Quasi-bialgebras and quasi-Hopf algebras

Quasi-bi-algebras giving as special case quasi-Hopf algebras were discovered by Russian mathematician Drinfeld (for technical definition, which does not say much to non-specialist see `https://en.wikipedia.org/wiki/Quasi-bialgebra` and `https://en.wikipedia.org/wiki/Quasi-Hopf_algebra`). They are non-associative or associative modulo isomoprhism.

Consider first quasi-bi-algebra $(B, \Delta, \epsilon, \Phi, l, r)$. Δ and ϵ are as for bi-algebra. Besides this one has invertible elements Φ (Drinfeld associator) and r, l called right and lef unit constraints. The conditions satisfied are following

- $$(1 \otimes \Delta) \circ \Delta(a) = \Phi[((\Delta \otimes 1) \circ \Delta(a)]\Phi^{-1} \ .$$
  For $\Phi = 1 \otimes 1 \otimes 1$ one obtains associativity.
- $$[(1 \otimes 1 \times \Delta)(\Phi)][(\Delta \otimes 1 \otimes 1)(\Phi)] = (1 \otimes \Phi)[1 \otimes \Delta \otimes 1)(\Phi)(\Phi \otimes 1) \ .$$
- $$(\epsilon \otimes 1)(\Delta(a)) = l^{-1}al \ , \ (1 \otimes \epsilon)(\Delta(a)) = r^{-1}ar \ .$$
- $$1 \otimes \epsilon \otimes 1)(\Phi) = 1 \otimes 1 \ .$$

These mysterious looking conditions express the fact that Drinfeld associator is a bialgebra co-cycle.

Quasi-bialgebra is braided if it has universal R-matrix which is invertible element in $B \otimes B$ such that the following conditions hold true.

$$(\Delta^{op})(a) = R\Delta(a)R^{-1} \ . \tag{4.1}$$

Note that for group algebra with $\Delta g = g \otimes g$ one has $\Delta^{op} = \Delta$ so that R must commute with Δ. Whether this forces R to be trivial is unclear to me. Certainly there are also other homomorphisms. A good candidate for a non-symmetric co-product is $\Delta g = g \times h(g)$ where h is a homomorpism of the braid group. This requires the replacement $S(g) \to S(h^{-1}g)$ in order to obtain unitarity for $\nabla(1, S)\Delta$ loop removing the braiding.

$$(1 \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{213}^{-1} \ . \tag{4.2}$$

$$(\Delta \otimes 1)(R) = \Phi_{321}^{-1} R_{13} \Phi_{213}^{-1} R_{23} \Phi_{123} \ . \tag{4.3}$$

This and second condition imply for trivial R that also Φ is trivial.

For $\Phi = 1 \otimes 1 \otimes 1$ the conditions reduces to those for ordinary braiding. The universal R-matrix satisfies the non-associative version of Yang-Baxter equation

$$R_{12}\Phi_{321}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123} = \Phi_{321}R_{23}(\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12} \ . \tag{4.4}$$

Quasi-Hopf algebra is a special case of quasi-bialgebra. Also now one has product ∇, co-product Δ, antipode S not present in bialgebra, and maps ϵ and η. Besides this one has two special elements α and β of H such that the conditions $\nabla(S, \alpha) \cdot \Delta = \alpha$ and $\nabla(1, \beta S) \cdot \Delta = \alpha$. To my understanding these conditions generalize the conditions $\nabla(S, 1)\Delta = \eta\epsilon$ and $\nabla(1, S)\Delta = \eta\epsilon$.

Associativity holds but only modulo a morphism in the same way as commutativity becomes braided commutativity in the case of quantum groups. The braided commutativity is characterized by R-matrix. The morphism defining "braided associativity" is characterized by the product $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ acting on triple tensor product $V \otimes V \otimes V$ and satisfying certain algebraic conditions. Φ has "inverse" $\Phi^{-1} = \sum_i P_i \otimes Q_i \otimes R_i$ The conditions $(1, \beta S, \alpha)\Phi = 1$ and $(S, \alpha, \beta S)\Phi = 1$. Here the action of S is that of algebra anti-homomorphism rather than algebra multiplication.

Drinfeld associator, which is a non-abelian bi-algebra 3-cocycle satisfying conditions analogous to the condition for weakened associativity holding true for Lie and Jordan algebras. These quasi-Hopf algebras are known in conformal field theory context and appear in Knizhnik-Zamolodchikov equations so that a lot of mathematical knowhow exists. According to Wikipedia, quasi-Hopf algebras are associated with finite-D irreps of quantum affine algebras in terms of F-matrices used to factorize R-matrix. The representations give rise to solutions of Quantum Yang-Baxter equation. The generalization of conformal invariance in TGD framework strongly suggests the relevance of Quasi-Hopf algebras in the realization of non-associativity in TGD framework.

4.1.4 Drinfeld double

Drinfeld double provides a concrete example about breaking of associativity. It can be formulated for finite groups as well as discrete groups. Drinfeld's approach is essentially algebraic: one works at the level of group algebra. In TGD framework the approach is geometric: algebraic constructs should emerge naturally from geometry. Braiding operations should induce algebras.

The basic notions involved are following.

1. One begins from a trivial tensor product of Hopf algebras and modified. In trivial case algebra product is tensor product of products, co-product is tensor product of co-products, antipode is tensor product of antipodes, map ϵ is product of the maps from the factors of the tensor product and delta maps unit element of field K to a product of unit elements. Drinfeld double represents a non-trivial tensor product of Hopf algebras.

2. One application of Drinfeld double construction is tensor product of group algebra and its dual. One can also interpret it as tensor product of braids as non-closed paths and closed braids (knots) as closed paths: in TGD framework this interpretation is suggestive and will be discussed later.

3. Drinfeld double allows breaking of associativity. It can be broken by introducing 3-cocycle (see `http://groupprops.subwiki.org/wiki/3-cocycle_for_a_group_action`) of group cohomology (see `http://groupprops.subwiki.org/wiki/Cochain_complex_for_a_group_action`). In the recent case group cohomology relies on homomorphism of group braid G to abelian group $U(1)$. n-cocycle is a map $G^n \rightarrow U(1)$ satisfying the condition that its derivation vanishes $d_n f = 0$. $d_n \circ d_{n-1} = 0$ holds true identically.

   The explicit definition of n-cocycle is in additive notion for U(1) product (usually multiplicative notation is used is) given by to illustrate that d_n acts like exterior derivative.

$$\begin{aligned}(d_n f)(g_1, g_2, g_n, g_{n+1}) = {} & g_1 f(g_1, ...g_n) - f(g_1 g_2, g_2, ..., g_{n+1}) + f(g_1, g_2 g_3, ..., g_{n+1}) \\ & - ... + (-1)^n f(g_1, g_2 ... g_n g_{n+1}) + (-1)^{n+1} f(g_1, g_2 ... g_n) \ .\end{aligned} \tag{4.5}$$

   This formula is easy to translate to multiplicative notion. The fact that group cohomology is universal concept strongly suggests that 3 co-cycle can be introduced quite generally to break associativity in the sense that different associations differ only by isomorphism.

The construction of quantum double of Hopf algebras is discussed in detail at `https://staff.fnwi.uva.nl/j.v.stokman/quantumdouble1.pdf`. Here however non-associative option is not discussed. In `http://msp.org/agt/2008/8-3/agt-v8-n3-p08-s.pdf` one finds explicit formula for Drinfeld double for the Drinfeld double formed by group algebra and its dual. Just to give some idea what is involved the following gives the formula for the product:

$$(h,y)\circ(g,x)=\frac{\omega(h,g,x)\omega(hgx((hg)^{-1},h,g)}{\omega(h,gx(g)^{-1},h,g)}(hg,x) \ . \tag{4.6}$$

Without background it does not tell much. What is essential however that the starting point is algebraic. The product is non-vanishing only between (g,x) and (h,gxg^{-1}). For gauge group like structure one would have x instead of $g^{-1}xg^{-1}$. ω is 3-cocycle: it it is non-trivial one as associativity modulo isomorphism.

I do not have any detailed understanding of quasi-Hopf algebras but to me they seem to provide a very promising approach in attempts to understand the character of non-associativity associated with the braiding hierarchy. The algebraic construction of Drinfeld double does not seem interesting from TGD point of view but the idea that group cocycle is behind the breaking of associativity is attractive. Also the generalization of construction of Drinfeld double to code what happens in braiding geometrically is attractive. One of the many difficult challenges is to understand the role of the varying parameters p, h_{eff}, q at the level of braid group algebras and their projective representations characterized by quantum phase q.

4.2 Construction of quantum braid algebra in TGD framework

It seems that there is no hope that naive application of existing formulas makes sense. The variety of different variants of quantum algebras is huge and one should have huge mathematical knowledge and understanding in order to find the correct option if it exists at all. Therefore I bravely take the approach of physicists. I try to identify the physical picture and then look whether I can identify the algebraic structure satisfying the axioms of Hopf algebra. In the following I first list various inputs which help to identify constraints on the algebraic structure, which should be simple if it is to be fundamental.

4.2.1 Trying to map out the situation

Usually physicists has enough trouble when dealing with single algebraic structure: say group and its representations. Unfortunately, this does not seem to be possible now. It seems that one must deal with entire collection of algebraic structures defined by braid groups B_n with varying value of n forming a hierarchy in which braid groups act on lower level braid groups.

1. What is clear that the algebraic operation $(A\otimes B)\rightarrow AB$ is somehow related to the braiding of flux tubes or fermionic strings connecting partonic 2-surfaces. One can also consider strings connecting the ends of light-like 3-surfaces so that one has both space-like and time-like braiding. One has flux tubes inside flux tubes.

   The challenge is to identify the natural algebra. It seems best to work with the braiding operations themselves - analogs of linguistic expressions - than the states to which they act. Braiding operations form discrete group, braid group. One must deal with the quantum superpositions of braidings so that one has wave functions in braid group identifiable as elements of discrete group algebra of braid group B_n. One can multiply group algebra elements and include the the group algebra of B_m to that of B_n m a factor of n so that the desired product structure is obtained. The group algebras associated with various braid numbers can be organized to operad.

   The operad formed by the braid group algebras has the desired hierarchical structure, and braid group algebra is one of the basic structures and quantum groups can be assigned with its projective representations.

2. For a given flux tube (and perhaps also for the fermionic string(s) assigned with it) one has degrees of freedom due different values of the quantum deformation parameter q for which roots of unity define preferred values in TGD framework. In TGD framework also hierarchy $h_{eff}/h = n$ of Planck constants brings in additional complexity. Also the p-adic prime p is expected to characterize the situation: preferred p-adic primes can be interpreted as so called ramified primes in the adelic vision about quantum TGD [**?**]numbervision unifying real and various p-adic physics to a coherent whole. This brings in new elements. It is still unclear how closely n and $q = exp(i2\pi/m)$ are related and whether one might have $m = n$. Also the relationship of p to n is not well-understood. For instance, could p divide n.

3. Geometrically the association of braid strands means that they belong to the same flux tube. Moving the brackets in expression to transform say $(A(BC))$ to $((AB)C)$ means that strands are transferred from flux tube another one. Hence the breaking of associativity should take place at all hierarchy levels except the lowest one for which flux tube contains single irreducible braid strand - fermion line.

   The general mechanism for a weak breaking of associativity is describable in terms of Drinfeld's associator for quasi-bialgebras and known in some cases explicitly - in particular, shown by Drinfeld to exists when the number field used is rational numbers - is the first guess for the mechanism of the breaking of associativity. Drinfeld's associator is determined completely by group cohomology, which encourages to think that it can be used as such as as a multipler in the definition of product in suitable tensor product algebra. How the Drinfeld's associator depends on the p,n, and q is the basic question.

4. Besides the geometric action of braidings it is important to understand how the braidings act on the fundamental fermions. An attractive idea is that the representation is as holonomies defined by the induced weak gauge potentials as non-integrable phase factors at the boundaries of string world sheets defining fermion lines. The vanishing of electroweak gauge fields at them implies that the non-Abelian part of holonomy is pure gauge as in topological gauge field theories for which the classical solutions have vanishing gauge field. The em part of the induce spinor curvature is however non-vanishing unless one poses the vanishing of electromagnetic field at the boundaries of string world sheets as boundary condition. This seems un-necessary. The outcome would be non-trivial holonomy and restriction to a particular representation of quantum group with quantum phase q coming as root of unity means conditions on the boundaries of string world sheets. Quantum phase would make itself visible also classically as properties of string world sheets which together with partonic 2-surfaces determined space-time surface by strong form of holography. An interesting question relates to the possibility of non-commutative statistics: it should come from the weak part of induced connection which is pure gauge and seems possible as it is possible also in topological QFTs based on Chern-Simons action.

4.2.2 Hints about the details of the braid structure

Concerning the details of the braid structure one has also strong hints.

1. There two are two basic types of braids: I have called them time-like and space-like braids. Time-like (or rather light-like) braids are associated with the 3-D light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes signature from Minkowskian to Euclidian. Braid strands correspond to fermionic lines identifiable as parts of boundaries of string world sheets. Space-like braids are associated with the space-like 3-surfaces at the ends of causal diamond (CD). Also they consist of fermionic lines. These braids could be called fundamental.

   If these braids are associated with magnetic flux tubes carrying monopole flux, the flux tubes are closed. Typically they connect wormhole throats at first space-time sheet, go to the second space-time sheet and return. Hence two-sheeted objects are in question. The braids in question can closed to knots and could correspond to closed loops assigned with the Drinfeld quantum double. The tensor product of the groupoid algebra associated with time-like braids and group algebra associated with space-like braids is highly suggestive as the analog of Drinfeld double.

Also magnetic flux tubes and light-like orbits of partonic 2-surfaces can become braided and one obtains the hierarchies of braids.

2. Since strong world sheets and partonic 2-surfaces have co-dimension 2 as sub-manifolds of space-time surface they can also get braided and knotted and give rise to 2-braids and 2-knots. This is something totally new. The unknotting of ordinary knots would take place via reconnections and the reconnections could correspond to the basic vertices for 2-knots analogous to the crossing of the plane projections of ordinary knot. Reconnections actually correspond to string vertices. A fascinating mathematical challenge is to generalize existing theories so that they apply to 2-braids and 2-knots.

3. Dance metaphor emerged in the model for DNA-lipid membrane system as topological quantum computer [**?**]dnatqc,dnatqccodes. Dancers whose feet are connected to wall by threads define time-like braiding and also space-like braiding through the resulting entanglement of threads. The assumption was that DNA codons or nucleotides are connected by space-like flux tubes to the lipids of lipid layer of cell membrane or nuclear membrane.

   If they carry monopolo flux they make closed loops at the structure formed by two space-time sheets. The lipid layer of cell membrane is 2-dimensional and can be in liquid crystal state. The 2-D liquid flow of lipids induces braiding of both space-like braids if the DNA end is fixed and of time-like braids. This leads to the dance metaphor: the liquid flow is stored at space-time level to the topology of space-time as a space-like braiding of flux tubes induced by it. Space-like braiding would be like written text. Time-like braiding would be like spoken language.

4. If the space-like braids are closed, they form knots and the flow caused at the second end of braid by liquid flow must be compensated at the parallel flux tube by its reversal since braid strands cannot be cut. The isotopy equivalence class of knot remains unchanged since knots get gg^{-1} piece which can be deformed away. Second interpretation is that the braid X transforms to gXg^{-1}. This kind of transformation appears also in Drinfeld construction. This suggests that the purely algebraic tensor product of braid algebra and its dual corresponds in TGD framework semi-direct tensor product of the groupoid of time-like braids and space-like braids associated with closed knots. The semi-direct tensor product would define the fundamental topological interaction between braids.

5. One can also consider sequence of n tensor factors each consisting of time-like and space-like braids. This require a generalization of the product of two tensor factors to $2n$ tensor factors. Dance metaphor suggests that a kind of chain reaction occurs.

4.2.3 What the structure of the algebra could be?

With this background one can try to guess what the structure of the algebra in question is. Certainly the algebra is semi-direct product of above defined braid group algebras. The multiplication rule would have purely geometric interpretation.

1. The multiplication rule inspired by dance metaphor for 2 tensor factors would be

$$(a_1, a_2) \circ (b_1, b_2) = (a_1 a_2 b_1 a_2^{-1}, a_2 b_2) \quad . \tag{4.7}$$

   Here a_1, b_1 correspond label elements of time-like braid groupoid and a_2, b_2 the elements of braid group associated with the space-like braid. This would replace the trivial product rule $(a_1, a_2)(b_1 g_) =$

(a_1b_1, a_2b_2) for the trivial tensor product. The structure is same as for Poincare group as semi-direct product of Lorentz group and translation group: $(\Lambda_1, T_1)(\Lambda_2, T_2) = (\Lambda_1\Lambda_2, T_1 + \Lambda_1(T_2))$.

It is easy to check that this product is associative. One can however add exactly the same 3-cocycle factor

$$(h, y) \circ (g, x) = \frac{\omega(h, g, x)\omega(hgx((hg)^{-1}, h, g)}{\omega(h, gx(g)^{-1}, h, g)}(hg, x) \quad . \tag{4.8}$$

Here (h, y) corresponds to (a_1, a_2) and (g, x) to (b_1, b_2). This should give breaking of non-associativity and third group cohomology of braid group B_n would characterize the non-equivalent associators.

2. The product rule generalizes to n factors. This generalization could be relevant for the understanding of braid hierarchy.

$$(a_1, a_2, ...a_n) \circ (b_1, b_2, ...b_n) \equiv (c_1, ..., c_n) \quad , \tag{4.9}$$

where one has

$$\begin{array}{lll} c_n = a_nb_n \quad , & c_{n-1} = a_{n-1}Ad_{a_n}(b_{n-1}) \quad , & c_{n-2} = a_{n-2}Ad_{a_{n-1}a_n}(b_{n-2}) \quad , \\ c_{n-3} = a_{n-3}Ad_{a_{n-2}a_{n-1}a_n}(b_{n-3}) & ,.... & c_1 = a_1Ad_{a_2.....a_n}(b_1) \quad . \\ Ad_x(y) = xyx^{-1} \quad . & & \end{array} \tag{4.10}$$

In this case a good guess for the breaking of associativity is that the associator is defined in terms of n-cocyle in group cohomology.

What is remarkable that this formula guarantees without any further assumptions the condition

$$\begin{array}{c} \nabla_{1\otimes 2}(\Delta_1(a), \Delta_2(b)) = \nabla_1(\Delta_1(a))\nabla_2(\Delta_2(b)) = \sum_{(a)} a_1a_2 \sum_{(b)} b_1b_2 \quad , \\ \Delta_1(a) = \sum_{(a)} a_1 \otimes a_2 \quad , \quad \Delta_2(b) = \sum_{(b)} b_1 \otimes b_2 \end{array} \tag{4.11}$$

as a little calculation shows. For group algebra one has $\Delta(a) = g \otimes g$. $\nabla_{1\otimes 2}$ refers to the product defined above.

3. The formula for $\Delta_{1\otimes 2}$ is also needed. The simplest guess is that it corresponds to replication for both factors. This would mean $\Delta^{op} = \Delta$: non-symmetric form guaranteeing non-trivial braiding is however desirable. A candidate satisfying this condition in $n = 2$ case is asymmetric replication:

$$\begin{array}{c} \Delta_{1\otimes 2}(bab^{-1}, b) \otimes (a, b) \\ \Delta^{op}_{1\otimes 2}(a, b) \otimes (bab^{-1}, b) \quad . \end{array} \tag{4.12}$$

4. In $n = 2$ case the formula for antipode would read as

$$S(a_1, a_2) = (a_2^{-1} a_1^{-1} a_2, a_2^{-1}) \tag{4.13}$$

instead of $S(a_1, a_2) = (a_1^{-1}, a_2^{-1})$. Again the semi-direct structure would be involved. One can check that the formula

$$\nabla_{1\otimes 2}(1, S)\Delta_{1\otimes 2} = 1 \otimes 1 \tag{4.14}$$

holds true.

4.3 Should one quantize complex numbers?

The TGD inspired proposal for the concrete realization of quantum groups might help in attempts to understand the situation. The approach relies on what might be regarded as quantization of complex numbers appearing as matrix elements of ordinary matrices.

1. Quantum matrices are obtained by replacing complex number valued of matrix elements of ordinary matrices with operators. They are are products of hermitian non-negative matrix P analogous to modulus of complex number and unitary matrix S analogous to its phase. One can also consider the condition $[P, S] = iS$ inspired by the idea that radial momentum and phase angle define analog of phase space.

2. The notions of eigenvalue and eigenstate are generalized. Hermitian operator or equivalently the spectrum of its eigenvalues replaces real number. The condition that eigenvalue problem generalizes, demands that the symmetric functions formed from the elements of quantum matrix commute and can be diagonalized simultaneously. The commutativity of symmetric functions holds also for unitary matrices. These conditions is highly non-trivial, and consistent with quantum group conditions if quantum phases are roots of unity. In this framework also Planck constant is replaced by a hermitian operator having $h_{eff} = n \times h$ as its spectrum. Also $q = exp(in2\pi/m)$ generalizes to a unitary operator with these eigenvalues.

3. This leads to a possible concrete representation of quantum group in TGD framework allowing to realize the hierarchy of inclusions of hyperfinite factors obtained by repeatedly replacing the operators appearing as matrix elements with quantum matrices.

4. This procedure can be repeated. One might speak of a fractal quantization. At the first step one obtains what might be called 1-hermitian operators with eigenvalues replaced with hermitian operators. For 1-unitary matrices eigenvalues, which are phases are replaced with unitary operators. At the next step one considers what might be called 2-hermitian and 2-unitary operators. An abstraction hierarchy in which instance (localization to a point as member of class) is replaced with wave function in the class. This hierarchy is analogous to that formed by infinite primes and by the sheets of the many-sheeted space-time. Also braids of braids of ... form this kind of abstraction hierarchy as also the parsing hierarchy for linguistic expressions.

I have proposed that generalized Feynman diagrams or rather - TGD analogs of twistor diagrams - should have interpretation as sequences of arithmetic operators with each vertex representing product or co-product and having interpretation as time reversal of the product operation.

1. The arithmetic operations could be induced by the algebraic operations for Yangian algebra assignable to the super-symplectic algebra. I have also proposed that there TGD allows a very powerful symmetry generalizing the duality symmetry of old-fashioned string models relating s- and t-channel exchanges. This symmetry would state that one can freely move the ends of the propagator lines around the diagrams and that one can remove loops by transforming the loop to tadpole and snipping it away. This symmetry would allow to consider only tree diagrams as shortest representations for computations: this would reduce enormously the calculational complexity. The TGD view about coupling constant evolution allows still to have discrete coupling constant evolution induced by the spectrum of critical values of Kähler coupling strength: an attractive conjecture is that the critical values can be expressed in terms of zeros of Riemann zeta [**?**]fermizeta.

2. One can represent the tree representing a sequence of computations in algebra as an analog of twistor diagram and the proposed symmetry implies associativity since moving the line ends induces motion of brackets. If co-algebra operations are allowed also loops become possible and can be eliminated by this symmetry provided the loop acts as identity transformation. This would suggest strong form of associativity at the level of single sheet and weaker form at the level of many-sheeted space-time. One could however still hope that loops can be cancelled so that one would still have only tree diagrams in the simplest description. One would have however sum over amplitudes with different association structures.

3. Co-product could be associated with the basic vertices of TGD, which correspond to a fusion of light-like parton orbits along their ends having no counterpart in super-string models (tensor product vertex) or the decay of light-like parton orbit analogous to a splitting of closed string (direct sum vertex). For the direct sum vertex one has direct sum (unlike string models): one can say that the particle propagates along two path in the sense of superposition as photons in double slit experiment. For the tensor product vertex $D(g) = \Delta(g) = g \times g$ is the first guess. $D(g) = (1, S)\Delta(g) = g \otimes Sg$ or $D(g) = Sg \otimes g$ or their sum suitably normalized is natural second guess. Unitarity allows only the latter option since $\nabla\Delta$ does not conserve probability for probability amplitudes unlike $\nabla(1, S)\Delta$ although it does so for probability distributions. For the direct sum vertex $\Delta(g) = 1 \otimes g \oplus g \otimes 1$ suitably normalized is the natural first guess.

4. Co-product Δ might allow interpretation as annihilation vertex in particle physics context. Co-product might also allow interpretation in terms of replication - at least at the level of topological dynamics of braiding. The possible application of co-product to the replication occurring biology assumed to be induce by replication of magnetic flux tubes in TGD based vision is highly suggestive idea. Is the identification of co-product as replication consistent with its identification as particle annihilation?

   Second question relates to the antipode S, which is anti-homomorphism and brings in mind time reversal. Could one interpret also S as an operation, which should be included to the braid group algebra in the same way as the inclusion of complex conjugation to the algebra of complex numbers produces quaternions? Could one interpret the identity $\nabla(1 \otimes S)\Delta(g) = \eta\epsilon(g) = 1$ by saying that the annihilation to $g \otimes S(g)$ followed by fusion produces braid wave function concentrated on trivial braiding and destroying the information associated with braiding completely. The fusion would produce non-braided particle rather than destroying particles altogether.

5. The condition that loop involving product and annihilation does not affect braid group wave function would require that it takes g to g. For the standard realization of co-product Δ of group algebra $g \to g \otimes g \to g^2$ so that this is not the case. The condition defining Δ is not easy to modify since one loses homomorphism property of Δ. The repetitions of loops would give sequence of powers g^{2n}. For wave function $\sum D(g)g$ this would give the sequence $\sum D(g)g \to \sum D(g)g^2 \to .... \to \sum D(g)g^{2n}$: since given group element has typically several roots one expects that eventually the wave function becomes concentrated to unity with coefficient $\sum D(g)$! For wave functions one has $\sum D(g) = 0$ if

they are orthogonal to $D(g) = constant$ as is natural to require. Almost all wave functions would approach to zero so that unitary would be lost. For probability distributions the evolution would make sense since the normalization condition would be respected.

Also the irreversible behaviour looks strange from particle physics perspective unless $D(g)$ is concentrated on identity so that braiding is trivial. Topological dissipation might take care that this is the case. For elementary particles partonic 2-surfaces carry in the first approximation only single fermion so that braid group would be trivial. Braiding effects become interesting only for strand number larger than 2. The situations in which partonic surface carries large number of fermion lines would be more interesting. Anyonic systems to which TGD based model assigns large h_{eff} and parton surfaces of nanoscopic size could represent a condensed matter example of this situation.

6. Does the behavior of Δ force to regard generalized Feynman diagrams representing computations with different numbers of self-energy loops non-equivalent and to sum over self-energy loops in the construction of scattering amplitudes? The time evolution implied by topological self energy loops is not unitary which suggest that one must perform the sum. There are hopes that the sum converges since the contributions approaches to $\sum D(g) = 0$. This does not however look elegant and is in conflict with the general vision.

   Particle physics intuition tells that in pair annihilation second line has opposite time direction. Should one therefore identify annihilation $g \to g \otimes S(g)$. Antiparticles would differ from particles by conjugation in braid group. The self energy loop would give trivial braiding with coefficient $\sum D(g)D(g^{-1}) = \sum D(g)D(g)^* = 1$ so that unitarity would be respected and higher self energy loops would be trivial. The conservation of fermion number at fundamental level could also prevent the decays $g \to g \otimes g$.

One could also take biological replication as a guide line.

1. In biological scales replication by $g \to g \otimes g$ vertex might not be prevented by fermion number conservation but probability conservation favors $g \to g \otimes Sg$. Braid replication might be perhaps said to provide replicas of information: whether this conforms with no-cloning theorem remains to be seen. Braid replication followed by fusion means topological dissipation by a loss of braiding and loss of information. Could the fusion of reproduction cells corresponds to product and that replication to co-product possibly involving the action of S one the second line. Fusion followed by replication would lead to a loss of braiding: for $g \to g \otimes g$ perhaps making sense in probabilistic description gradually and for $g \to g \otimes Sg$ instantaneously: a reset for memory? Could these mechanisms serve as basic mechanisms of evolution?

2. There might be also a connection with the p-adic length scale hypothesis. The naive expectation is that $g \to g^2$ in fusion followed by Δ means the increase of the length of braid by factor 2 - kind of ageing? Could the appearance of powers of two for the length of braid relate to the p-adic length scale hypothesis stating that primes p near powers of 2 are of special importance?

To summarize, the proposed framework gives hopes about description of braids of braids of .... Abstraction would mean transition from classical to quantum: from localized state to a de-localized one: from configuration space to the space of complex valued wave functions in configuration space. Now the configuration space would involve different braidings and corresponding evolutions, and various values of p, h_{eff} and q. If this general framework is to be useful it should be able to tell how the braiding matrices depend on p and h_{eff}: note that p and h_{eff} would be fixed only at the highest abstraction level - the largest flux tubes. This indeterminacy could be interpreted in terms of finite measurement resolution and inclusions of HFFs should help to describe the situation. Indeterminacy could also be interpreted in terms of abstraction in a manner similar to the interpretation of negentropically entangled state as a rule for which the state pairs in the superposition represent instances of the rule.

References

[1] Pitkänen M. DNA as Topological Quantum Computer. In *Genes and Memes*. Onlinebook. Available at: http://tgdtheory.fi/public_html/genememe/genememe.html#dnatqc, 2006.

[2] Pitkänen M. Quantum Model for Hearing. In *TGD and EEG*. Onlinebook. Available at: http://tgdtheory.fi/public_html//tgdeeg/tgdeeg/tgdeeg.html#hearing, 2006.

[3] Pitkänen M. The classical part of the twistor story. In *Towards M-Matrix*. Onlinebook. Available at: http://tgdtheory.fi/public_html/tgdquantum/tgdquantum.html#twistorstory, 2006.

[4] Pitkänen M. Three new physics realizations of the genetic code and the role of dark matter in biosystems. In *Genes and Memes*. Onlinebook. Available at: http://tgdtheory.fi/public_html/genememe/genememe.html#dnatqccodes, 2006.

[5] Pitkänen M. Geometric theory of harmony. Available at: http://tgdtheory.fi/public_html/articles/harmonytheory.pdf, 2014.