Article

Weyl's Theory under Chiral Approach

H. Torres-Silva[1], J. López-Bonilla[2*] & R. López-Vázquez[2]

[1]Escuela de Ingeniería Eléctrica y Electrónica, Universidad de Tarapacá, Arica, Casilla 6-D, Chile
[2]ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Lindavista 07738, México DF

Abstract

Weyl's theory is studied under chiral approach when the electromagnetic 4-potential is considered as part of self-dual solutions to the Maxwell equations known as "instantons". We discuss some cosmological situations within the chiral approach.

Keywords: General relativity theory, Maxwell equations, Weyl theory, instantons.

1. Introduction

Hermann Weyl (1885–1955) investigated many of the theory's applications in physics and cosmology of the Einstein's general theory of relativity. He generalized the theory's Riemannian basis for a unification of the gravitational and electromagnetic fields for normal Maxwell equations with 4-potential A_μ. Weyl's work is important because he introduced gauge invariance and applied it to field of quantum theory; gauge invariance is now recognized as one of the milestones of quantum physics. Here we will simply sketch Weyl's derivation of his field equations, which result from a variation of Weyl gauge invariant in action Lagrangian. We present aspects of Weyl theory under chiral approach as cosmological effects.

In this four dimensional theory repeated indices are summed, we denote partial derivatives as $F^{\mu\nu}_{,\lambda} = \partial F^{\mu\nu} / \partial x^\lambda = \partial_\lambda F^{\mu\nu}$, etc., and covariant differentiation is expressed by ; . In Riemannian geometry the affine connection ${}^{LC}\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol, (Levi-Civita connection), that is,

$$ {}^{LC}\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}) ; $$

while in Weyl's geometry it is

$$ {}^{W}\Gamma^\alpha_{\mu\nu} = {}^{LC}\Gamma^\alpha_{\mu\nu} - \delta^\alpha_\mu A_\nu - \delta^\alpha_\nu A_\mu + g_{\mu\nu} g^{\alpha\beta} A_\beta , $$

where A_μ is a vector field that Weyl identified with the electromagnetic 4-potential [1-7].

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, México DF
E-mail: jlopezb@ipn.mx

Einstein objected to the fact that the R^2 term is of fourth order in the metric tensor (see eq. (1) below), but we will find that this objection can be averted in the chiral approach and because it is important to study cosmological phenomenon of the accelerated expansion of the Universe which is consistent with recent observational data [8]. One new feature connected with Weyl's theory are the actual gravitation theories $F(R)$ proposed recently by several authors [9, 10] because also it is interesting to explore its properties with $F(R)=R^2$ which is an invariant action in Weyl geometry.

2. The Weyl action Lagrangian and variational principle

The simplest invariant action in Weyl's geometry utilizes the square of the Ricci scalar:

$$I=\int\sqrt{-g}(R^2+F_{\mu\nu}F^{\mu\nu})d^4x \ , \tag{1}$$

here $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$, where A_μ is the 4-potential. The Weyl action is composed of the scalar R^2 and the source-free electromagnetic density $F_{\mu\nu}F^{\mu\nu}$. In Weyl's theory the covariant derivative of the metric tensor is not zero: $g_{\mu\nu;\alpha}=-2g_{\mu\nu}A_\alpha$, while $\sqrt{-g}_{;\alpha}=4\sqrt{-g}A_\alpha$. The variation of Weyl action with respect to $g_{\mu\nu}$ is simplified by utilizing the Palatini method [3]:

$$\delta I=\int\sqrt{-g}({}^{W}G_{\mu\nu}\delta g^{\mu\nu}+{}^{W}J^{\mu}\delta A_\mu)d^4x \ , \tag{2}$$

where:

$$\begin{aligned}{}^{W}G_{\mu\nu}&=-2(g^{\alpha\beta}F_{\mu\alpha}F_{\beta\nu}+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta})+2R(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)+16RA_\mu A_\nu+8R_{;\mu}A_\nu\\&-g_{\mu\nu}g^{\alpha\beta}R_{,\alpha;\beta}-6g_{\mu\nu}g^{\alpha\beta}R_{,\alpha}A_\beta-2g_{\mu\nu}g^{\alpha\beta}RA_{\alpha;\beta}-8g_{\mu\nu}g^{\alpha\beta}RA_\alpha A_\beta\end{aligned} \tag{3}$$

and:

$${}^{W}J^{\mu}=24\sqrt{-g}g^{\mu\nu}(RA_\nu+\frac{1}{2}R_{,\nu})-4\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu} \quad . \tag{4}$$

Setting these quantities to zero we get:

$$\begin{aligned}(g^{\alpha\beta}F_{\mu\alpha}F_{\beta\nu}+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta})&=R(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)+8RA_\mu A_\nu+4R_{;\mu}A_\nu\\&-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{,\alpha;\beta}-3g_{\mu\nu}g^{\alpha\beta}R_{,\alpha}A_\beta-g_{\mu\nu}g^{\alpha\beta}RA_{\alpha;\beta}-4g_{\mu\nu}g^{\alpha\beta}RA_\alpha A_\beta\end{aligned} \tag{5}$$

and:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu}=6\sqrt{-g}g^{\mu\nu}(RA_\nu+\frac{1}{2}R_{,\nu}) \ . \tag{6}$$

The left side of (5) is the familiar energy tensor of the electromagnetic field, while the Weyl-Ricci term is reminiscent of Einstein's gravitational field equations; both quantities are traceless. The remaining terms all involve the 4-potential and its covariant derivative. Eq. (6) is the most notable result of Weyl's theory; it indicates that electromagnetism is an intrinsic part of the Weyl geometry.

The $(\sqrt{-g}F^{\mu\nu})_{,\nu}$ term is the electromagnetic source density, and its divergence must vanish; this results in the condition:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu;\mu} = 6\left[\sqrt{-g}g^{\mu\nu}(RA_\nu + \frac{1}{2}R_{,\nu})\right]_{;\mu} \tag{7}$$

giving:

$$0 = g^{\mu\nu}R_{,\mu;\nu} + 2g^{\mu\nu}RA_{\mu;\nu} + 4g^{\mu\nu}R_{,\mu}A_\nu + 4g^{\mu\nu}RA_{,\mu}A_{,\nu}\,. \tag{7'}$$

By calculating the trace $g^{\mu\nu}\,{}^{W}G_{\mu\nu}$ and setting it to zero, we recover this same condition (which is a consequence of Noether's theorem). We can now use (7') to simplify (5) somewhat; eliminating the $g^{\mu\nu}RA_{\mu;\nu}$ term we get, finally:

$$T_{\mu\nu} = R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + 8RA_\mu A_\nu + 4R_{,\mu}A_\nu - g_{\mu\nu}g^{\alpha\beta}R_{,\alpha}A_\beta - 2g_{\mu\nu}g^{\alpha\beta}RA_\alpha A_\beta \tag{8}$$

3. Weyl's theory under chiral approach: Cosmological aspects

Eq. (8) and its interpretation is open to questions. In view of this, we can ask what happens when the chiral Weyl vector A_μ is set to $(A_\mu)_{chiral} = (A_\mu)_c$ (see appendix A). Under chiral approach with self dual instantons $T_{\mu\nu} = \Theta_{\mu\nu} = 0$ (see eq. (A14) of appendix A), thus we have:

$$R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + (8RA_\mu A_\nu + 4R_{,\mu}A_\nu - g_{\mu\nu}g^{\alpha\beta}R_{,\alpha}A_\beta - 2g_{\mu\nu}g^{\alpha\beta}RA_\alpha A_\beta)_{A_\mu=(A_\mu)_c} = 0\,, \tag{9}$$

when R is constant this expression is reduced to:

$$(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + (8A_\mu A_\nu - 2g_{\mu\nu}A^\beta A_\beta)_{A_\mu=(A_\mu)_c} = 0\,. \tag{10}$$

In this approach, $g_{\mu\nu;\alpha} = -2g_{\mu\nu}(A_\alpha)_c = 0$ and $\sqrt{-g}_{;\alpha} = 4\sqrt{-g}(A_\alpha)_c = 0$ so refutation of Einstein is no longer valid (Einstein argued that ds can be associated with the ticking of a clock or the spacings of atomic spectral lines and concluded that if it is not absolutely invariant, many basic physical quantities such as Compton wavelength, electron mass, etc, would vary arbitrarily with time and location) .

Eq. (10) is traceless, Einstein's equation $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R = T_{\mu\nu}$ is divergenceless, whereas Weyl's equation is not. We note that chiral eq. (10) reproduces all of the predictions that general relativity makes via Einstein's equation, and it may be important to F(R) theories, (see references in [10]). The Einstein field equations with $T_{\mu\nu} = 0$ is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \overline{\Lambda} g_{\mu\nu} = 0 \ , \tag{11}$$

where $\overline{\Lambda}$ is Einstein's cosmological constant. Contraction of this expression shows that we can identify the Ricci scalar with this constant via $R = 4\overline{\Lambda}$. But if we insert this result back into (11), we have precisely the chiral Weyl result in (10) (Eqs. 10 and 11 differ only by a traceless term)! We see that chiral Weyl's field equations automatically relate the Ricci scalar to the cosmological constant. This justifies our earlier demand in (9) that R be a non-zero quantity. As $i\mathbf{E} = \pm \mathbf{H}$, we can postulate that the cosmological constant is proportional to the square of the magnetic field H; further studies on this topic is under development.

4. Conclusion

All the usual tests of general relativity—gravitational red shift, radar delay, perihelion shift of Mercury, bending of light with the Einstein theory—also are satisfied using Weyl's approach under chiral approach because we have an additional term due to the non-vanishing of the cosmological constant. At the other end of the cosmological spectrum is the observation that the expansion of the universe appears to be accelerating. This has given rise to the dark energy theory, which also proposes the existence of some kind of repulsive energy field permeating the universe that serves to speed up the expansion, a non-zero cosmological constant can be a solution to these problems.

Appendix A: Self-dual electromagnetic fields (instantons)

Self-dual solutions to the Maxwell equations known as "instantons," have gained recognition among experts in gauge field theory and mathematical physics. Furthermore, the idea of self-duality was found to be of much significance in many problems of algebraic geometry. It is instructive to consider self-dual fields in this simpler context [11-14] is likely to be relevant. We will obtain a field configuration similar to that found in Ref. [12] but follow an alternative route based on the idea of self-duality. There are several reasons for considering self-dual fields in classical electrodynamics: Self-dual solutions are readily calculated and possess trivial energy-momentum properties, and the desired free field configurations are obtainable as superposition of self-dual and anti-self-dual constituents so that the resulting spectral properties may be easily controlled.

To simplify our notation as much as possible, we choose the Gaussian system of units and set the speed of light equal to unity. An electromagnetic field is self-dual/anti-self-dual if [15]:

$$\mathrm{i}\mathbf{E}=\pm\mathbf{H}\ . \tag{A1}$$

How it is possible to obtain equation (A1)? The answer is: if we consider free electromagnetic fields governed by the homogeneous Maxwell equations with the operator $\partial / c\partial t$ transformed to $\partial / c\partial t(1+T\nabla\times)$:

$$\nabla\times\mathbf{E}=-\frac{\partial}{c\partial t}(1+T\nabla\times)\mathbf{H}\ , \tag{A2}$$

$$\nabla\cdot\mathbf{E}=0\,, \tag{A3}$$

$$\nabla\times\mathbf{H}=\frac{\partial}{c\partial t}(1+T\nabla\times)\mathbf{E}\,, \tag{A4}$$

$$\nabla\cdot\mathbf{H}=0\,. \tag{A5}$$

Let some field configuration be self-dual. If this field obeys eqs. (A4) and (A5), it automatically satisfies eqs. (A2) and (A3). Because Maxwell's equations are linear, any superposition of self-dual and anti-self-dual solutions is a further solution. The condition that a field configuration is self-dual is not invariant under the parity transformation $\mathbf{r}\rightarrow-\mathbf{r}$ because of the opposite parity properties of the electric and magnetic field; the mirror-image configuration is anti-self-dual. As will become clear, the physically relevant configurations are represented by a sum of self-dual and anti-self-dual solutions, which is invariant under the parity transformation.

Let us express the electric field intensity $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in terms of scalar and chiral vector potentials V and $\mathbf{A}_c$, then the self-duality condition (1) becomes:

$$\pm(\nabla V+\frac{\partial\mathbf{A}_c}{c\partial t})=-\nabla\times\mathbf{A}_c; \tag{A6}$$

If we fix the gauge $V=0$, then eq. (6) reduces to:

$$\pm\frac{\partial \mathbf{A}_c}{c\partial t} = -\nabla\times\mathbf{A}_c . \tag{A7}$$

Because the self-duality condition (A7) is a linear first-order partial differential equation, it is simpler to solve than the second-order equations that result from Maxwell's eqs. (A2)-(A5). A remarkable property of self-dual configurations is that they carry zero energy and momentum. This property can be verified by applying the self-duality condition (1) to the expressions for the energy density $\varepsilon = (1/8\pi)(\mathbf{E}^2 + \mathbf{H}^2)$ and the Poynting vector $\mathbf{S} = (c/4\pi)(\mathbf{E}\times\mathbf{H})$.

Note that given an antisymmetric field $F_{\mu\nu}$ in Minkowski space, the self-duality condition can be expressed as:

$$*F_{\mu\nu} = \pm iF_{\mu\nu} \quad , \tag{A8}$$

where the Hodge dual field $*F_{\mu\nu}$ is defined by $*F_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Equation (A8) is identical to eq. (1) because $\mathbf{E}$ and $\mathbf{H}$ are expressed in terms of $F_{\mu\nu}$ as $E_i = F_{0i}$ and $H_i = (1/2)\varepsilon_{ijk}F_{jk}$, so eq. (A1) is $F_{0i} = i(1/2)\varepsilon_{ijk}F_{jk}$. If the Bianchi identity:

$$\partial_\mu * F^{\mu\nu} = 0 \tag{A9}$$

is compared with the equations of motion for a free electromagnetic field:

$$\partial_\mu F^{\mu\nu} = 0, \tag{A10}$$

it becomes apparent that if $F_{\mu\nu}$ obeys eqs. (A8) and (A9), then $F_{\mu\nu}$ automatically obeys eq. (A10).

Self-dual configurations possess trivial energy-momentum contents. The stress-energy tensor $\Theta_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\lambda F_{\lambda\mu} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$, can be brought into the form $\Theta_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\lambda F_{\lambda\mu} + *F_\mu{}^\lambda * F_{\lambda\mu}\right)$. The proof is simple, to see, for example, Ref. [13], problem 5.2.8, and thus $\Theta_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\lambda + i*F_\mu{}^\lambda)(+F_{\lambda\nu} + i*F_{\lambda\nu}\right)$; if $*F_{\mu\nu} = \pm iF_{\mu\nu}$ then

$\Theta_{\mu\nu} = 0$. Special result is obtained with $(A_\mu)_{chiral} = (A_\mu)_c$ when $V = 0$ and $\nabla \times \mathbf{A}_c = k\mathbf{A}_c$, $k = \omega / c$ that are used in our paper.

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