Article

GHP Formalism & Brans-Edgar Equations

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Abstract

We employ the Geroch-Held-Penrose technique to rewrite the Brans-Edgar relations associated with vacuum metrics of type II.

Keywords: Bianchi identities, GHP formalism, Brans-Edgar equations, Newman-Penrose expressions.

1. Introduction

In this work we use the notation and quantities explained in [1]. The Riemann tensor is equivalent to the known 18 equations of Newman-Penrose (NP) [2, 3], and the Bianchi identities are compatibility conditions for them, which for empty solutions type II are given by:

$$D\psi_2 = 3\rho\psi_2\,, \qquad \delta\psi_2 = 3\tau\psi_2\,, \qquad \bar{\delta}\psi_2 = -3\pi\psi_2\,, \qquad \Delta\psi_2 = -3\mu\psi_2\,,$$

$$D\psi_4 = (\rho - 4\varepsilon)\psi_4 - 3\lambda\psi_2\,, \qquad \delta\psi_4 = (\tau - 4\beta)\psi_4 - 3\nu\psi_2\,, \tag{1}$$

in the canonical tetrad with $\psi_0 = \psi_1 = \psi_3 = 0$ [4]; furthermore $\kappa = \sigma = 0$ by the Goldberg-Sachs theorem [5, 6]. The Brans [7]-Edgar [8] equations appear when we use the commutators of NP [1-3] to establish the corresponding compatibility of the set (1):

$$\bar{\delta}\rho + D\pi - \rho(\alpha + \bar{\beta}) - \pi(\bar{\varepsilon} - \varepsilon) + \mu\bar{\kappa} + \bar{\sigma}\tau = 0, \tag{2}$$

$$\Delta\pi - \bar{\delta}\mu + \rho\nu - \lambda\tau - \mu(\alpha + \bar{\beta} - \bar{\tau}) - \pi(\bar{\gamma} - \gamma - \bar{\mu}) = 0, \tag{3}$$

$$\Delta\rho + D\mu + \mu(\varepsilon + \bar{\varepsilon}) - \rho(\gamma + \bar{\gamma}) + \tau\bar{\tau} - \pi\bar{\pi} = 0, \tag{4}$$

$$\delta\mu + \Delta\tau - \rho\bar{\nu} - \bar{\lambda}\pi + \mu(\bar{\alpha} + \beta) + \tau(\bar{\gamma} - \gamma) = 0. \tag{5}$$

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In Sec. 2 we give a summary of the Geroch-Held-Penrose (GHP) formalism [1, 9]; in Sec. 3 this technique is applied to the system (2)-(5), thus finding that only it is necessary to write three equations type GHP because (5) will be the primed version of (2). This is reached due to the explicit presence in (2) of terms with κ and σ despite that in this case $\kappa = \sigma = 0$.

2. GHP's technique

In [9] the GHP formalism is developed in a very complete manner, so that here we give a summary of it, which in Sec. 3 is applied to equations (2)-(5). In each event of the space-time we have an ordered null tetrad $(m^r, \overline{m}^r, l^r, n^r)$ type NP:

$$m^a \overline{m}_a = -n^a l_a = 1. \tag{6}$$

Nevertheless in such event we could employ, without violating (6), the tetrad:

$$(\eta\overline{\eta}^{-1}\, m^r, \eta^{-1}\overline{\eta}\,\overline{m}^r, \eta^{-1}\overline{\eta}^{-1}\, l^r, \eta\overline{\eta}\, n^r), \tag{7}$$

where η is an arbitrary complex scalar. This new choice of the tetrad generates changes in several quantities of the NP formalism; in particular, the GHP technique pays special attention to those objects that under (7) suffer the transformation:

$$A \;\longrightarrow\; \eta^p \overline{\eta}^q\, A\,, \tag{8}$$

so we say that A has type $\{p, q\}$ with spin-weight $\frac{1}{2}(p-q)$ and with boost-weight $\frac{1}{2}(p+q)$. For example, in [1, 9] we find that only 8 of the 12 spin coefficients have the change (8) under (7):

$$\begin{gathered}\kappa\colon\{3,1\}, \quad \sigma\colon\{3,-1\}, \quad \varrho\colon\{1,1\}, \quad \tau\colon\{1,-1\},\\ \pi\colon\{-1,1\}, \quad \mu\colon\{-1,-1\}, \quad \lambda\colon\{-3,1\}, \quad \nu\colon\{-3,-1\}.\end{gathered} \tag{9}$$

The operation" $'$ ", which commutes with the complex conjugation process, is important in the GHP formalism and it realizes the interchange of m^a and l^a with $\overline{m}^a$ and n^a, respectively:

$$m^{a\prime} = \overline{m}^a, \qquad \overline{m}^{a\prime} = m^a, \qquad l^{a\prime} = n^a, \qquad n^{a\prime} = l^a\,, \tag{10}$$

therefore:

$$\nu = -\kappa', \quad \tau = -\pi', \quad \alpha = -\beta', \quad \mu = -\rho', \quad \sigma = -\lambda', \quad \gamma = -\varepsilon'. \tag{11}$$

Hence it is enough the explicit use of $\kappa, \rho, \pi, \lambda, \beta, \varepsilon$ because the another coefficients can be generated from these ones through the operation " $'$ ". It becomes evident that under (10) the NP operators $\delta, \bar{\delta}, \Delta, D$ interchange among them, then it is sufficient the explicit employ of δ and D, because:

$$\bar{\delta} = \delta' \quad \text{and} \quad \Delta = D'. \tag{12}$$

However, when the operators act onto a quantity of type $\{p, q\}$, they do not lead to quantities fulfilling (8) under (7); so for example, if A has the property (8) then in general δA will not satisfy such property. For this reason, in the GHP technique the operators Φ and ∂ are introduced to remedy such situation, and their effect on quantities verifying (8) is defined via the expressions:

$$\Phi A = (D - p\varepsilon - q\bar{\varepsilon})\, A : \quad \{p+1, q+1\}, \quad \partial A = (\delta - p\beta - q\bar{\alpha})\, A : \quad \{p+1, q-1\}. \tag{13}$$

Then by using (11) and (12) is simple to see the action of the corresponding primed operators:

$$\Phi' A = (D' + p\varepsilon' + q\bar{\varepsilon}')\, A = (\Delta - p\gamma - q\bar{\gamma})\, A : \quad \{p-1, q-1\},$$

$$\partial' A = \left(\delta' + p\beta' - q\bar{\beta}\right) A = \left(\bar{\delta} - p\alpha - q\bar{\beta}\right) A : \quad \{p-1, q+1\}, \tag{14}$$

where we have used the fact that under the operation "$\,'\,$", the type $\{p, q\}$ changes to the $\{-p, -q\}$ one; besides, if one realizes that $\bar{A}$ has type $\{q, p\}$, then:

$$\bar{\Phi} = \Phi \qquad \text{and} \qquad \bar{\partial} = \partial'. \tag{15}$$

As a warning up for the next Sec., we exemplify the action of the operators (13) – (15) by means of (9):

$$\partial\mu = (\delta + \beta + \bar{\alpha})\, \mu, \quad \Phi\mu = (D + \varepsilon + \bar{\varepsilon})\, \mu, \quad \partial'\mu = \left(\bar{\delta} + \alpha + \bar{\beta}\right) \mu, \quad \partial'\rho = \left(\bar{\delta} - \alpha - \bar{\beta}\right) \rho,$$
$$\Phi'\pi = (\Delta + \gamma - \bar{\gamma})\, \pi, \quad \Phi\pi = (D + \varepsilon - \bar{\varepsilon})\, \pi, \quad \Phi'\tau = (\Delta - \gamma + \bar{\gamma})\, \tau, \quad \Phi'\rho = (\Delta - \gamma - \bar{\gamma})\, \rho. \tag{16}$$

The above relations are useful when rewriting (2),…,(5) of Brans-Edgar within the GHP formalism.

3. Brans-Edgar equations in their GHP version

The system (2)-(5) represents the integrability conditions such that (1) has solution. If now the corresponding compatibility constraints are calculated [10] for (2),…,(5) then one gets 0 = 0, this means that the 18 NP equations together with the Bianchi identities and the Brans-Edgar equations conform a completely integrable system. If in (2)-(5) we make use of (11),…,(16) it results their GHP version:

$$\partial'\rho + \Phi\pi + \mu\bar{\kappa} + \bar{\sigma}\tau = 0, \tag{17}$$

$$\Phi'\pi + \partial'\rho' - \kappa'\rho + \lambda\pi' + \rho'\pi' - \bar{\rho}'\pi = 0, \tag{18}$$

$$\Phi'\rho - \Phi\rho' + \pi\bar{\pi}' - \pi\bar{\pi} = 0, \tag{19}$$

that is, equations (2)-(4) are equivalent to (17)-(19) in the same order. There is no need to rewrite (5) since this is the primed version of (17); the primed equation of (19) coincides with itself. Finally, applying the operation " ′ " to (18) gives the equation:

$$\delta\rho - D\tau - \rho(\bar{\alpha} + \beta) + \tau(\varepsilon - \bar{\varepsilon}) + \kappa\mu - \sigma\pi + \rho\bar{\pi} + \bar{\rho}\tau = 0,$$

which reduce to 0 = 0 in virtue of the 18 NP equations [1], so that it does not give any new information. The relations (17), (18) and (19) are the GHP version of the Brans-Edgar relations for arbitrary vacuum metrics of type II.

Received Jan. 04, 2016; Accepted Jan. 17, 2016

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