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Spherical Images and Characterizations of Time-like Curve According to New Version of the Bishop Frame in Minkowski 3-Space

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Abstract

In this work, we introduce a new version of Bishop frame using a common vector field as binormal vector field of a regular timelike curve and call this fame as "Type-2 Bishop frame in E_1^3". Thereafter, by translating type-2 Bishop frame vectors to O the center of Lorentzian sphere of three-dimensional Minkowski space, we introduce new spherical images and call them as type-2 Bishop spherical images in E_1^3. Serret-Frenet apparatus of these new spherical images are obtained in terms of base curves's type-2 Bishop invariants. Additionally, we express some interesting theorems and illustrate an example of our main results.

Keywords: Timelike curve, spherical image, Minkowski space, Bishop frame, general helix, slant helix, Bertrand mate.

1 Introduction

The local theory of space curves are mainly developed by Serret-Frene theorem which expresses the derivative of a geometrically chosen basis of $\mathbb{R}^3$ by the aid of itself is proved. In resent years, in the same and different space, researchers treat same clasical topics in analogy with the curves in the Euclidean, Minkowski, Galilean and complex space, see [12,13,14,15,17,18]. Then it is observed that by the solution of some special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, Mannheim curves, constant breadth curves, slant helix, general helix, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidian space. It is a well-known concept in the local differential geometry of the curves. Such curves are obtained in terms of the Serret-Frenet vector fields [6].

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in the Euclidian space [4,11], in Minkowski space [5,7]. Bishop and Serret-Frenet frames have a common vector field, namely the tangent vector field of the Serret-Frenet frame.

In this work, we made spherical image for timelike curve. In smilar to research has been studied by Yılmaz [16]. Spherical indicatrices have been examined by authers, see [13,14,15,16]. We introduce a new version of the Bishop frame for timelike curves in E_1^3. We call it is "Type-2 Bishop frame" of regular curves. Thereafter, translating new frames vector fields to the center

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of unit sphere, we obtain new spherical images. We call them as "Type-2 Bishop Spherical Image" of regular curves. We also distinguish them by the names, Ω_1, Ω_2 and binormal Bishop spherical images. Besides, we investigate their Serret-Frenet apparatus according to type-2 Bishop invariants. We establish some results of spherical images and illustrate an example of main results.

2 Preliminaries

The Minkowski three dimensional space E_1^3 is a real vector space $\mathbb{R}^3$ endowed with the standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is rectangular coordinate system of E_1^3. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in E_1^3, the Lorentzian cross product of u and v defined by

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$; timelike if $g(v,v) < 0$ and null(lightlike) if $g(v,v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vector δ' are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by

$$\|a\| = \sqrt{|g(a,a)|}.$$

The curve $\delta = \delta(s)$ is called a unit speed curve if velocity vector δ' is unit i.e, $\|\delta'\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v,w) = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\delta = \delta(s)$ in the space E_1^3.

For an arbitrary timelike curve $\delta = \delta(s)$ in E_1^3, the following Serre-Frenet formulea are given in as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1.1}$$

where $< T,T >= -1$, $< B,B >=< N,N >= 1$, $T(s) = \delta'(s)$, $N(s) = \dfrac{T'(s)}{\kappa(s)}$, $B(s) = T(s) \times N(s)$ and first curvature and second curvature $\kappa(s)$, $\tau(s)$ respectively. $\kappa(s) = \|\delta''\|$, $\tau(s) = \dfrac{\det(\delta', \delta'', \delta''')}{\kappa^2}$ [8].

Denoted by $\{T, N_1, N_2\}$ moving Type-1 Bishop frame along to timelike curve $\alpha : I \subset \mathbb{R} \longrightarrow \mathrm{E}_1^3$ in the Minkowski 3-space E_1^3. For an arbitrary timelike curve $\alpha(s)$ in 3-space E_1^3, the following Type-1 Bishop formulea are given by

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} . \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

The relations between κ, τ, θ and k_1, k_2 are given as follow $\kappa(s) = \sqrt{k_1^2 + k_2^2}$, $\theta(s) = \arctan(\dfrac{k_2}{k_1})$, $k_1 \neq 0$. So that $k_1$and k_2 effectively corrospond to cartasian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ due to the differentiation [7].

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2 = \left\{ p = (p_1, p_2, p_3) \in E_1^3 : g(p, p) = r^2 \right\}$$

Theorem 2.1.1: Let $\varphi(s)$ be a unit speed timelike curve in E_1^3. Then φ is a slant helix if and only if either one the next two functions

$$\frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$$

is constant everywhere $\tau^2 - \kappa^2$does not vanish.

Theorem 2.1.2: Let $\varphi(s)$ be a timelike curve with curvatures κ and τ. The curve φ is a general helix if and only if $\dfrac{\kappa}{\tau} = cons\tan t$.

3 Type-2 Bishop Frame of a Regular Curve in E_1^3

Theorem 3.1.1: Let $\alpha = \alpha(s)$ be timelike curve with a spacelike principal normal unit speed. If $\{\Omega_1, \Omega_2, B\}$ is adapted frame, then we have defined by "Type-2 Bishop Frame in E_1^3"

$$\begin{bmatrix} \Omega_1' \\ \Omega_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.1}$$

where $g(\Omega_1, \Omega_1) = -1$, $g(\Omega_2, \Omega_2) = g(B, B) = 1$, and $g(\Omega_1, \Omega_2) = g(\Omega_1, B) = g(\Omega_2, B) = 0$. If Ω_1 timelike Ω_2 and B spacelike vectors. Thus we have equation (3.1.1) or shortly $X' = AX$. Morever A is semi skew matrix where ξ_1 first curvature and ξ_2 called second curvature of the curve, there the curvatures are defined by

$$\xi_1 =< \Omega_1', B >, \quad \xi_2 =< \Omega_2', B > . \tag{3.1.2}$$

Theorem 3.1.2: Let $\{T, N, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet ve Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & -\cosh\theta(s) & 0 \\ \cosh\theta(s) & -\sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}$$

where θ is the angle between the vectors N and Ω_1.

Proof: We write the tangent vector according to frame $\{\Omega_1, \Omega_2, B\}$ as

$$T = \sinh\theta(s)\Omega_1 - \cosh\theta(s)\Omega_2 \tag{3.1.3}$$

and differentiate with respect to s

$$T' = \kappa N = \theta'(s)\left[\cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2\right] + \sinh\theta(s)\Omega_1' - \cosh\theta(s)\Omega_2' \tag{3.1.4}$$

Substituting $\Omega_1' = \xi_1 B$ and $\Omega_2' = \xi_2 B$ to equation (3.1.4), we get

$$\kappa N = \theta'(s)\left[\cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2\right] + \left[\sinh\theta(s)\xi_1 - \cosh\theta(s)\xi_2\right]B \tag{3.1.5}$$

From equation (3.1.5) we get $\theta(s) = Arg\tanh\dfrac{\xi_2}{\xi_1}$, $\theta'(s) = \kappa(s)$, $N = \cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2$, and

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & -\cosh\theta(s) & 0 \\ \cosh\theta(s) & -\sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.6}$$

Since there is a solition for θ satisyfing any initial condition, this show that localy relatively parallel normal fields exist. Besides equation (3.1.1) can also written as

$$B' = \tau N = \xi_1\Omega_1 + \xi_2\Omega_2$$

Taking the norm of both sides, we have

$$\tau = \sqrt{\left|\xi_2^2 - \xi_1^2\right|} \tag{3.1.7}$$

$$1 = \sqrt{\left|\left(\frac{\xi_1}{\tau}\right)^2 - \left(\frac{\xi_2}{\tau}\right)^2\right|} \tag{3.1.8}$$

and so by (3.1.5), we may express

$\{\ \xi_1 = \tau(s)\cosh\theta(s), \quad \xi_2 = \tau(s)\sinh\theta(s)$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and τ and $\theta(s) =_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \xi_1, \xi_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$ in E_1^3.

4 New Spherical Images of a Regular Curve

Let $\alpha = \alpha(s)$ be a regular timelike curve in E_1^3. If we translate type-2 Bishop frame vectors to the center O of Lorentzian sphere of three-dimensional Minkowski space, we introduce new spherical images in E_1^3.

4.1 Ω_1 Bishop Spherical Image

Definition 4.1.1: Let $\alpha = \alpha(s)$ be a regular timelike curve in E_1^3. If we translate of the first vector field of type-2 Bishop frame to the center O of the unit sphere S_1^2, we obtain a spherical image $\varphi = \varphi(s_\varphi)$. This curve is called Ω_1 Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular timelikr curve $\alpha = \alpha(s)$. We shall investigate relations among type-2 Bishop and Serret-Frenet invariants. First, we differentiate

$$\varphi' = \frac{d\varphi}{ds_\varphi} \cdot \frac{ds_\varphi}{ds} = \xi_1 B.$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Taking the norm both sides the equation above, we have

$$\begin{aligned} T_\varphi &= B \\ \frac{ds_\varphi}{ds} &= \xi_1 \end{aligned} \tag{4.1.1}$$

we differentiate $(4.1.1)_1$ as

$$T'_\varphi = \dot{T}_\varphi \frac{ds_\varphi}{ds} = \xi_1 \Omega_1 + \xi_2 \Omega_2.$$

So, we have

$$\dot{T}_\varphi = \Omega_1 + \frac{\xi_2}{\xi_1} \Omega_2.$$

Since, we have the first curvature and principal normal of φ

$$\begin{aligned} \kappa_\varphi &= \left\| \dot{T}_\varphi \right\| = \sqrt{\left| \left(\tfrac{\xi_2}{\xi_1} \right)^2 - 1 \right|} \\ N_\varphi &= \frac{1}{\kappa_\varphi} (\Omega_1 + \tfrac{\xi_2}{\xi_1} \Omega_2) \end{aligned} \tag{4.1.2}$$

Cross product of $T_\varphi \times N_\varphi$ gives us the binormal vector field of Ω_1 Bishop spherical image of $\alpha = \alpha(s)$

$$B_\varphi = \frac{1}{\kappa_\varphi} (\frac{\xi_2}{\xi_1} \Omega_1 + \Omega_2) \tag{4.1.3}$$

Using the formula of the torsion, we write a relation

$$\tau_\varphi = \frac{(\xi_1)^7 \cdot \left(\frac{\xi_2}{\xi_1} \right)'}{\left| \xi_2^2 - \xi_1^2 \right|} \tag{4.1.4}$$

Considering equations $(4.1.2)_1$ and (4.1.3), we give:

Corollary 4.1.2: Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of the curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\xi_2}{\xi_1}$ =constant, i.e.), then, the Ω_1 Bishop spherical indicatrix $\varphi = \varphi(s_\varphi)$ is a circle in the osculating plane.

Proof: Let$\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular timelike curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant, in terms of equations $(4.1.2)_1$ and (4.1.4), we have κ_φ =constant and $\tau_\varphi = 0$, respectively. Therefore, φ is a circle in the osculating plane.

Theorem 4.1.3: Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular curve $\alpha = \alpha(s)$. There exists a relation among Serret-Frenet invariants $\varphi(s_\varphi)$ and type-2 Bishop curvatures of $\alpha = \alpha(s)$ as follow

$$\frac{\xi_2}{\xi_1} =_0^{s_\varphi} \frac{(\kappa_\varphi)^2 \tau_\varphi}{(\xi_1)^4} ds_\varphi \tag{4.1.5}$$

Proof: Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular timelike curve $\alpha = \alpha(s)$. Then equations $(4.1.1)_2$ and (4.1.4) hold. Using $(4.1.1)_2$ in (4.1.4) and by the chain rule, we have

$$\tau_\varphi = \frac{(\xi_1)^7 \dfrac{d}{ds_\varphi} \cdot \left(\frac{\xi_2}{\xi_1}\right) \dfrac{ds_\varphi}{ds}}{\left|\xi_2^2 - \xi_1^2\right|}. \tag{4.1.6}$$

Substituting $(4.1.2)_1$ to (4.1.6) and integrating both sides, we have (4.1.5) as desired.

In the light of theorem 2.1.1 and 2.1.2, we express the following theorems without proofs.

Theorem 4.1.4: Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If φ is a general helix, then, type-2 Bishop curvatures of $\alpha = \alpha(s)$ a satisfy

$$(\xi_1)^5 \cdot \left(\frac{\xi_2}{\xi_1}\right)' = \text{constant}.$$

Theorem 4.1.5: Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If φ is a slant helix, then, type-2 Bishop curvatures of $\alpha = \alpha(s)$ a satisfy

$$\left[\frac{(\xi_1)^8 \left(\frac{\xi_2}{\xi_1}\right)'}{\left(\xi_2^2 - \xi_1^2\right)^{\frac{3}{2}}}\right]' \frac{\left(\xi_1^2 - \xi_2^2\right)^4 \xi_1}{\left[\left(\xi_2^2 - \xi_1^2\right)^3 \text{-} (\xi_1)^{16} \left[\left(\frac{\xi_2}{\xi_1}\right)'\right]^2\right]^{\frac{3}{2}}} = \text{cons.}$$

Remark 4.1.6: Considering $\theta_\varphi =_0^{s_\varphi} \kappa_\varphi ds_\varphi$ and using the transformation matrix, one can obtain the type-2 Bishop trihedral $\{\Omega_{1\varphi}, \Omega_{2\varphi}, B_\varphi\}$ of the curve $\varphi = \varphi(s_\varphi)$.

4.2 Ω_2 Bishop Spherical Image

Definition 4.2.1: Let $\alpha = \alpha(s)$ be a regular curve in E_1^3.If we translate of the second vector field of type-2 Bishop frame to the center of the unit sphere S_1^2, we obtain a spherical image $\beta = \beta(s_\beta)$. This curve is called Ω_2 Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of the timelike regular curve $\alpha = \alpha(s)$.We can write that

$$\beta' = \frac{d\beta}{ds_\beta} \cdot \frac{ds_\beta}{ds} = \xi_2 B.$$

Similar to Ω_2 Bishop spherical image, one can have

$$\begin{aligned} T_\beta &= B \\ \frac{ds_\beta}{ds} &= \xi_2 \end{aligned} \tag{4.2.1}$$

So, by differentiating of the formula $(4.2.1)_1$, we get

$$T_\beta^{'} = \dot{T}_\beta \frac{ds_\beta}{ds} = \xi_1 \Omega_1 + \xi_2 \Omega_2.$$

or in other words

$$\dot{T}_\beta = \frac{\xi_1}{\xi_2}\Omega_1 + \Omega_2.$$

Since, we express

$$\begin{aligned} \kappa_\beta &= \left\| \dot{T}_\beta \right\| = \sqrt{\left| 1 - \left(\tfrac{\xi_1}{\xi_2}\right)^2 \right|} \\ N_\beta &= \frac{1}{\kappa_\beta}\left(\tfrac{\xi_1}{\xi_2}\Omega_1 + \Omega_2\right) \end{aligned} \tag{4.2.2}$$

Cross product of $T_\varphi \times N_\varphi$ gives us

$$B_\beta = \frac{1}{\kappa_\beta}\left(\Omega_1 + \frac{\xi_1}{\xi_2}\Omega_2\right) \tag{4.2.3}$$

By the formula of the torsion, we have

$$\tau_\beta = \frac{(\xi_2)^7 \cdot \left(\frac{\xi_1}{\xi_2}\right)^{'}}{\left| \xi_2^2 - \xi_1^2 \right|} \tag{4.2.4}$$

In terms of equations $(4.2.2)_1$ and (4.2.4), we may give:

Corollary 4.2.2: Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of a regular timelike curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\xi_1}{\xi_2}$ =constant, i.e.), then, the Ω_2 Bishop spherical indicatrix $\beta = \beta(s_\beta)$ is a circle in the osculating plane.

Theorem 4.2.3: Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of a regular timelike curve $\alpha = \alpha(s)$.Then, there exists a relation among Serret-Frenet invariants of $\beta(s_\beta)$ and type-2 Bishop curvatures of $\alpha = \alpha(s)$ as follow

$$\frac{\xi_1}{\xi_2} =_0^{s_\beta} \frac{(\kappa_\beta)^2 \, \tau_\beta}{(\xi_2)^4} ds_\beta$$

Proof: Similar to proof of theorem 4.1.3, above equation can be obtained by equations $(4.2.1)_2$ and $(4.2.2)_1$ and (4.2.4).

In the light of theorem 2.1.1 and 2.1.2, we also give the following theorems for the curve $\beta = \beta(s_\beta)$

Theorem 4.2.4: Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of a regular curve $\alpha = \alpha(s)$.If β is a general helix, then, type-2 Bishop curvatures of $\alpha = \alpha(s)$ a satisfy

$$\frac{(\xi_2)^8 \cdot \left(\frac{\xi_1}{\xi_2}\right)^{'}}{\left(\xi_2^2 - \xi_{12}^2\right)^{\frac{3}{2}}} = \text{constant.}$$

Theorem 4.2.5: Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of a regular curve $\alpha = \alpha(s)$.If β is a slant helix, then, type-2 Bishop curvatures of $\alpha = \alpha(s)$ a satisfy

$$\left[\frac{(\xi_2)^8 \left(\frac{\xi_1}{\xi_2}\right)^{'}}{\left(\xi_2^2 - \xi_1^2\right)^{\frac{3}{2}}}\right]^{'} \frac{\left(\xi_2^2 - \xi_1^2\right)^4 \xi_2}{\left[\left(\xi_2^2\text{-}\xi_1^2\right)^4 \text{-} (\xi_2)^{16}\left[\left(\frac{\xi_1}{\xi_2}\right)^{'}\right]^2\right]^{\frac{3}{2}}} = \text{const.}$$

Remark 4.2.6: Considering $\theta_\varphi =_0^{s_\beta} \kappa_\beta ds_\beta$ and using the transformation matrix, one can obtain the type-2 Bishop trihedral $\{\Omega_{1\beta}, \Omega_{2\beta}, B_\beta\}$ of the curve $\beta = \beta(s_\beta)$.

4.3 Binormal Bishop Spherical Image

Definition 4.3.1: Let $\alpha = \alpha(s)$ be a regular curve in E_1^3. If we translate of the third vector field of type-2 Bishop frame to the center O of the unit sphere S_1^2, we obtain a spherical image $\phi = \phi(s_\phi)$. This curve is called Binormal Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Here, one question may come to mind about binormal spherical image, since, Serret-Frenet and type-2 Bishop frame in E_1^3 have a common binormal vector field. Images of such binormal images are same as we shall demonstrate in section 6. But, here we are concerned with the Binormal Bishop spherical image's Serret-Frenet apparatus according to type-2 Bishop invariants.

Let $\phi = \phi(s_\phi)$ be Binormal Bishop spherical image of a regular curve $\alpha = \alpha(s)$. One can differentiate of ϕ with respect to s

$$\phi^{'} = \frac{d\phi}{ds_\phi} \cdot \frac{ds_\phi}{ds} = \xi_1\Omega_1 + \xi_2\Omega_2.$$

In terms of type-2 Bishop frame vector fields, we have tangent vector of the spherical image as follows

$$T_\phi = \frac{\xi_1\Omega_1 + \xi_2\Omega_2}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}} \qquad \frac{ds_\phi}{ds} = \sqrt{\left|\xi_2^2 - \xi_1^2\right|} \tag{4.3.1}$$

In order to determine first curvature of ϕ, we write

$$\dot{T}_\phi = P^{'}(s)\Omega_1 + Q^{'}(s)\Omega_2 + (P(s)\xi_1 + Q(s)\xi_2)B$$

where $P(s) = \dfrac{\xi_1}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$ and $Q(s) = \dfrac{\xi_2}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$.

Since, we immediately arrive at

$$\kappa_\phi = \left\|\dot{T}_\phi\right\| = \sqrt{(Q(s))^2 + [P(s)\xi_1 + Q(s)\xi_2)]^2 - (P^{\prime}(s))^2} \tag{4.3.2}$$

Therefore, we have the principal normal

$$N_\phi = \frac{1}{\kappa_\phi}\{P^{\prime}(s)\Omega_1 + Q^{\prime}(s)\Omega_2 + [P^{\prime}(s)\xi_1 - Q(s)\xi_2)]\}B \tag{4.3.3}$$

By the cross product of $T_\phi \times N_\phi$, we obtain the binormal vector field

$$B_\phi = \frac{1}{\kappa_\phi}\{[P(s)Q(s)\xi_1 - Q^2(s)\xi_2]\Omega_1 + \left[P^2(s)\xi_1 - P(s)Q(s)\xi_2\right]\Omega_2 - \left[P^2(s)\left(\frac{Q(s)}{P(s)}\right)^{\prime}\right]B\} \tag{4.3.4}$$

where $P(s) = \dfrac{-\xi_1}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$ and $Q(s) = \dfrac{\xi_2}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$.

By means of obtained equations, we express the torsion of the Binormal Bishop spherical image

$$\tau_\phi = \frac{1}{\kappa_\phi^2}\{\xi_1[3\xi_2^{\prime}(\xi_1\xi_1^{\prime}+\xi_2\xi_2^{\prime})- - (\xi_1^2+\xi_2^2)(\xi_1+\xi_1^3+\xi_1\xi_2^2)] + \xi_2[3\xi_1^{\prime}(\xi_1\xi_1^{\prime}+\xi_2\xi_2^{\prime})) - (\xi_1^2+\xi_2^2)(\xi_1+\xi_1^3+\xi_1\xi_2^2)]\} \tag{4.3.5}$$

Consequently, we determined Serret-Frenet invariants of the Binormal Bishop spherical image according to type-2 Bishop invariants in E_1^3. In terms of equations (4.3.2) and (4.3.5), we have :

Corollary 4.3.2: Let $\phi = \phi(s_\phi)$ be binormal Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\xi_1}{\xi_2}$ =constant, i.e), then, Binormal Bishop spherical image $\phi(s_\phi)$ is a circle in the osculating plane.

Remark 4.3.3: Considering $\phi_\varphi =_0^{s_\phi} \kappa_\phi ds_\phi$ and using the transformation matrix, one can obtain the type-2 Bishop trihedral $\{\Omega_{1\phi}, \Omega_{2\phi}, B_\phi\}$ of the $\phi = \phi(s_\phi)$.

5 Main Results

Theorem 5.1.1: Let $\alpha = \alpha(s)$ be a regular timelike curve in 3-dimensional Minkowski space. Both of Ω_1, Ω_2 and B spherical image of α. Both of Ω_1 and Ω_2 spherical images of α are spherical involutes for binormal spherical image of α.

Proof: Let us denote the tangent vectors of Ω_1 and Ω_2 spherical images as T_φ, T_β and T_ϕ respectively. These tangent vectors are given in $(4.1.1)_1$,$(4.2.1)_1$ and $(4.3.3)_1$.

$$T_\varphi = B$$
$$T_\beta = B$$

$$T_\phi = \frac{\xi_1\Omega_1 + \xi_2\Omega_2}{\sqrt{|\xi_2^2 - \xi_1^2|}}$$

where $\{\Omega_1, \Omega_2, B\}$ is type-2 Bishop frame and ξ_1 and ξ_2 are Bishop curvatures of α.

If the inner products are calculated, we get

$$< T_\varphi, T_\phi >= 0$$
$$< T_\beta, T_\phi >= 0$$

The tangent vectors of Ω_1 and Ω_2 spherical images is orthogonal to tangent vectors of binormal spherical images. So the proof is completed.

Theorem 5.1.2: Let $\alpha = \alpha(s)$ be a regular timelike curve in 3-dimensional Minkowski space. Both of Ω_1, Ω_2 and B spherical images of α. Binormal vector of Ω_1 are orthogonal to normal vector Ω_2.

Proof: Let us denote the binormal vectors of Ω_1 and principal normal vector of Ω_2, B_φ and N_β respectively. From (4.1.3), $(4.2.2)_2$ this vectors are given

$$B_\varphi = \frac{1}{\kappa_\varphi}(\tfrac{\xi_2}{\xi_1}\Omega_1 + \Omega_2)$$

$$N_\beta = \frac{1}{\kappa_\beta}(\tfrac{\xi_1}{\xi_2}\Omega_1 + \Omega_2)$$

If the Lorentzian inner product of B_φ and N_β are calculated, we get

$$< B_\varphi, N_\beta >= 0$$

It can be seen that, Binormal vector of Ω_1 and normal vector Ω_2 are perpendicular.

6 Example

In this section, first we show how to find a regular curve's type-2 Bishop trihedral and thereafter illustrate one example of new spherical images in E_1^3.

Theorem 6.1.1: Next, let us consider the following unit speed curve $w(s)$ of E_1^3 by

$$w = w(s) = (\sqrt{2}\sinh s,\ \sqrt{2}\cosh s,\ s)$$

It is rendered in figure 1.

And this curves's curvature function is expressed as in E_1^3
$\{\kappa(s) = \sqrt{2}$

The Serret-Frenet frame of the $w = w(s)$ may be written by the aid Mathematical program as follows

$$T = (\sqrt{2}\cosh s,\ \sqrt{2}\sinh s,\ 1)$$
$$N = (\sinh s,\ \cosh s,\ 0)$$
$$B = (-\cosh s,\ -\sinh s,\ -\sqrt{2})$$
$$\theta(s) =_0^s \sqrt{2}ds = \sqrt{2}s$$

Using transformation matrix equation (3.1.6) we get $w = w(s)$ and tangent, normal, binormal spherical images of unit speed curve with respect to Serret-Frenet frame. respectively Fig 1, 2a, 2b, 2c.we have type-2 Bishop spherical images of the unit speed curve $w = w(s)$,see figures 3a, 3b, 3c

$$\Omega_1 = (-\sqrt{2}\sinh\theta\cosh s + \cosh\theta\sinh s,$$
$$-\sqrt{2}\sinh\theta\sinh s + \cosh\theta\cosh s,\ -\sinh\theta)$$
$$\Omega_2 = (-\sqrt{2}\cosh\theta\cosh s + \sinh\theta\sinh s,$$
$$,-\sqrt{2}\cosh\theta\sinh s + \sinh\theta\cosh s,\ -\cosh\theta)$$
$$B = (-\cosh s,\ -\sinh s,\ \sqrt{2})$$

where $\theta = \sqrt{2}s$.

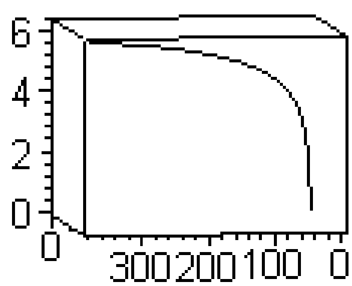
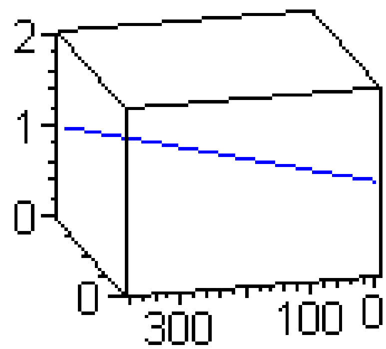
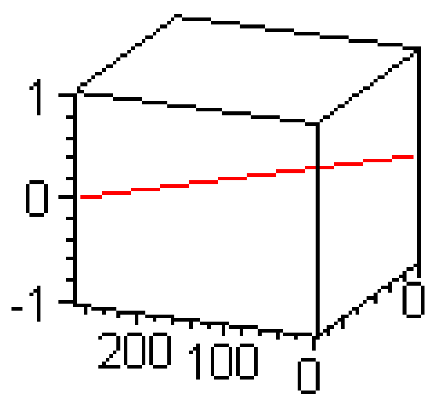


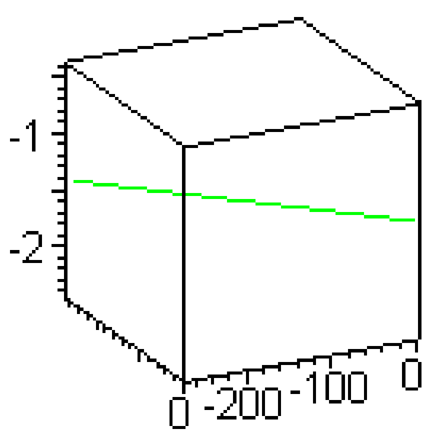
Fig.1



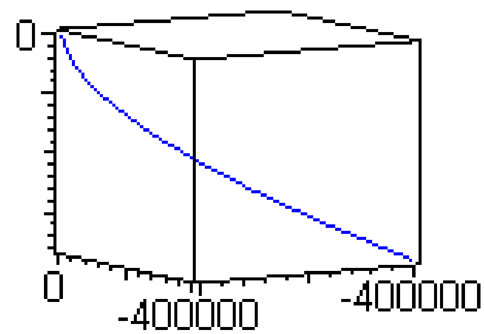
$Fig.2a$



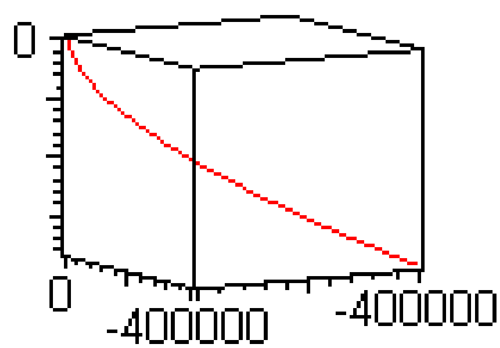
$Fig.2b$



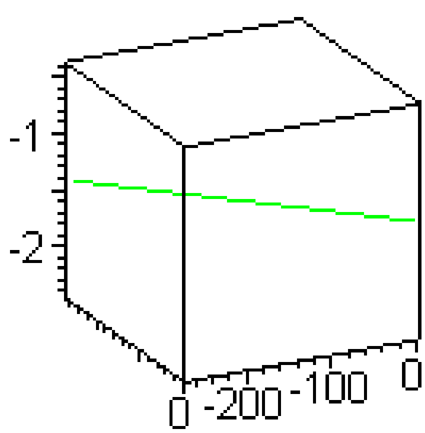
$Fig.2c$



$Fig.3a$



$Fig.3b$



$Fig.3c$

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