Article

Transformation of Dirac Spinor under Boosts & 3-Rotations

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Abstract

We exhibit the transformation rule for the 4-spinor of Dirac under 3-rotations and boosts.

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1. Introduction

In the Dirac equation for spin-1/2 particles [1-3] $[(x^\mu) = (t, x, y, z), \hbar = c = 1]$:

$$\left(i\gamma^\mu \partial_\mu - m_0\right)\psi = 0, \qquad i = \sqrt{-1}, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1}$$

ψ is a 4-spinor with the γ^μ matrices verifying the anticommutator [4-6]:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_{4x4}\,, \qquad (g^{\mu\nu}) = Diag(1, -1, -1, -1). \tag{2}$$

Here we consider the options:

a) Dirac-Pauli (or standard) representation [7].

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \ j = 1,2,3, \quad \gamma^5 \equiv i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3}$$

with the Cayley [8]-Sylvester [9]-Pauli [10] matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

b) Weyl (or chiral) representation [3, 11-13].

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$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ j = 1, 2, 3, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

to study the transformation law of ψ under the homogeneous Lorentz group [14-16]:

$$\tilde{x}^\mu = L^\mu{}_\nu \, x^\nu , \tag{6}$$

which implies the existence [4, 17] of a non-singular matrix S such that:

$$L_{\mu\alpha} \, S \, \gamma^\alpha = \gamma_\mu \, S, \tag{7}$$

and we obtain the relativistic invariance of (1) if the Dirac spinor obeys the transformation rule:

$$\tilde{\psi} = S \, \psi \, . \tag{8}$$

In this work we deduce the structure of S for boosts and 3-rotations, working with the representations (3) and (5).

2. Construction of *S*

First we consider infinitesimal Lorentz transformations, besides L is an orthogonal matrix ($L^\mu{}_\alpha L^{\nu\alpha} = g^{\mu\nu}$) [14], then it differs infinitesimally from the unit matrix by a skew-symmetric matrix [18]:

$$L_{\mu\alpha} = g_{\mu\alpha} + \varepsilon \, F_{\mu\alpha} \, , \qquad F_{\beta\nu} = -F_{\nu\beta} \, , \qquad S = I + \varepsilon \, Q \, , \quad \varepsilon \ll 1, \tag{9}$$

and we must determine Q with the constraint required by (7) via the commutator:

$$[\gamma_\mu, Q] = F_{\mu\beta}\gamma^\beta = \frac{1}{2} F_{\alpha\beta}\left(\delta_\mu^\alpha \, \gamma^\beta - \delta_\mu^\beta \, \gamma^\alpha\right) = \frac{1}{4} F_{\alpha\beta}[\gamma_\mu, \gamma^\alpha\gamma^\beta] = \left[\gamma_\mu, \frac{1}{4} F_{\alpha\beta} \, \gamma^\alpha\gamma^\beta\right],$$

that is, $Q = \frac{1}{4} F_{\alpha\beta} \, \gamma^\alpha\gamma^\beta$, hence for a finite Lorentz transformation [we take $\varepsilon = \frac{1}{N}$]:

$$S = lim_{\, N\to\infty} \, (I + \tfrac{\varepsilon}{4} \, F_{\mu\nu} \, \gamma^\mu\gamma^\nu)^N = \exp\left(\tfrac{1}{4} F_{\mu\nu} \, \gamma^\mu\gamma^\nu\right). \tag{10}$$

Therefore, given L we have $F_{\mu\nu}$, then (8) and (10) allow to construct the new 4-spinor.

3. Rotations in three dimensions

For rotations around of the axes X, Y, Z, the matrix $(F_{\mu\nu})$ is given by [3, 19]:

$$-iJ_1\theta_1 = \theta_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$-iJ_2\theta_2 = \theta_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, -iJ_3\theta_3 = \theta_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

respectively, then in the Dirac-Pauli and Weyl representations the matrix (10) takes the form:

$$S = \begin{pmatrix} \exp(\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}) & 0 \\ 0 & \exp(\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}) \end{pmatrix}. \tag{12}$$

4. Boosts

In this case, for boosts in the directions X, Y, Z, the matrix $(F^{\mu}{}_{\nu})$ has the structure [3, 12]:

$$-iK_1\phi_1 = \phi_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad -iK_2\phi_2 = \phi_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-iK_3\phi_3 = \phi_3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

respectively, where $tanh\ \phi_k = v_k$.

We note that the full matrix of Lorentz transformations of rotations and boosts adopts the expression:

$$(L^{\mu}{}_{\nu}) = \exp\left(i\vec{J}\cdot\vec{\theta} + i\vec{K}\cdot\vec{\phi}\right) = exp \begin{pmatrix} 0 & -\phi_1 & -\phi_2 & -\phi_3 \\ -\phi_1 & 0 & \theta_3 & -\theta_2 \\ -\phi_2 & -\theta_3 & 0 & \theta_1 \\ -\phi_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}, \tag{14}$$

where only one rotation or one boost angle can be applied at any one time [12]. In (14) we see six parameters for the homogeneous Lorentz group.

We employ (13) into (10) to obtain:

$$S = \begin{pmatrix} \cosh\left(\frac{\phi_k}{2}\right) I & -\sinh(\frac{\phi_k}{2})\sigma_k \\ -\sinh(\frac{\phi_k}{2})\sigma_k & \cosh\left(\frac{\phi_k}{2}\right) I \end{pmatrix} \qquad \text{Dirac-Pauli scheme,} \tag{15}$$

$$= \begin{pmatrix} \exp(-\frac{1}{2}\vec{\sigma}\cdot\vec{\phi}) & 0 \\ 0 & \exp(\frac{1}{2}\vec{\sigma}\cdot\vec{\phi}) \end{pmatrix} \qquad \text{Weyl scheme,} \tag{16}$$

hence, in the representation (5), for rotations and boosts the matrix S acquires the general structure [3, 12]:

$$S = \begin{pmatrix} \exp[\frac{1}{2}\vec{\sigma}\cdot\left(i\vec{\theta}-\vec{\phi}\right)] & 0 \\ 0 & \exp([\frac{1}{2}\vec{\sigma}\cdot\left(i\vec{\theta}+\vec{\phi}\right)] \end{pmatrix}. \tag{17}$$

5. Weyl spinors

We write the Dirac spinor in the form:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{18}$$

where ψ_R and ψ_L are called Weyl spinors [12, 20], then with (8) and (17) it is immediate to deduce their transformation laws (in the chiral scheme) under an arbitrary Lorentz mapping (14) [3]:

$$\tilde{\psi}_R = \exp\left[\frac{1}{2}\vec{\sigma}\cdot\left(i\vec{\theta}-\vec{\phi}\right)\right]\psi_R, \qquad \tilde{\psi}_L = \exp\left[\frac{1}{2}\vec{\sigma}\cdot\left(i\vec{\theta}+\vec{\phi}\right)\right]\psi_L, \tag{19}$$

and they do not preserve parity (they are not invariant with respect to the change $x \to -x$), hence they were assumed to represent neutrinos, which are all left-handed (described by ψ_L) while antineutrinos are all right-handed (described by ψ_R). The Dirac spinor, being composed of both spinors, is fully parity-preserving [12]. The standard representation necessarily mixes the Weyl spinors under Lorentz transformations, so their distinction is not noticeable; ψ_R and ψ_L are dotted and undotted 2-spinors, respectively, that is, they correspond to the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of the Lorentz group [3]. In fact [21]:

$$\psi_R = \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \end{pmatrix}, \qquad \psi_L = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{20}$$

and we can consider (19) with $\theta_1 \neq 0$, therefore:

$$\begin{pmatrix} \tilde{\eta}^1 \\ \tilde{\eta}^2 \end{pmatrix} = U_1 \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} \cos(\frac{\theta_1}{2}) & -i\,sin(\frac{\theta_1}{2}) \\ -i\,sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{pmatrix}, \qquad \det U_1 = 1\,, \tag{21}$$

because [22] $\eta_1 = \epsilon_{A1}\,\eta^A = -\eta^2$ and $\eta_2 = \epsilon_{A2}\,\eta^A = \eta^1$; besides:

$$\begin{pmatrix} \tilde{\xi}^{\dot{1}} & \tilde{\xi}^{\dot{2}} \end{pmatrix} = \begin{pmatrix} \xi^{\dot{1}} & \xi^{\dot{2}} \end{pmatrix} {U_1}^{\dagger}\,. \tag{22}$$

Similarly, from (19) for $\phi_1 \neq 0$:

$$\tilde{\psi}_R{}^T = \psi_R{}^T {U_2}^{\dagger}\,,\ \begin{pmatrix} \tilde{\eta}^1 \\ \tilde{\eta}^2 \end{pmatrix} = U_2 \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}\,,\ U_2 = \begin{pmatrix} \cosh(\frac{\phi_1}{2}) & -\sinh(\frac{\phi_1}{2}) \\ -\sinh(\frac{\phi_1}{2}) & \cosh(\frac{\phi_1}{2}) \end{pmatrix}, \qquad \det U_2 = 1, \tag{23}$$

thus (21), (22) and (23) show [22] the undotted and dotted character of ψ_L and ψ_R, respectively.

The matrix $\exp(\frac{1}{2}\vec{\sigma}\cdot\vec{\phi})$ is not unitary, hence we have above a finite-dimensional and non-unitary representation of the non-compact Lorentz group, however, it has infinite-dimensional unitary representations [3]. If we employ (5) and (18) in the Dirac equation (1) for massless particles, we obtain the Weyl equations [3, 11, 12] $(\partial_0 - \sigma_j\partial_j)\psi_L = 0$ and $(\partial_0 + \sigma_j\partial_j)\psi_R = 0$, that is:

$$(p_0 + \vec{\sigma}\cdot\vec{p})\psi_L = 0\,, \qquad (p_0 - \vec{\sigma}\cdot\vec{p})\psi_R = 0\,, \tag{24}$$

thus $\psi_L(\psi_R)$ are eigenstates of negative (positive) helicity, which tells us how closely aligned the spin of a particle is with its direction of motion.

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