Article

A Note on a Sum of the Series 1/1.3 + 1/3.5 + 1/5.7 + ... = 1/2

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Abstract

In this note it is shown that the interesting summation 1/1.3 + 1/3.5 + 1/5.7 + ... = 1/2 which was deduced from certain Fourier series, can be obtained very quickly with the help of classical Gauss hypergeometric summation theorem.

Keywords: Fourier series, hypergeometric series, Gauss summation theorem.

1. Introduction

It is well known that the Fourier series and the hypergeometric series play a key role in the theory of Mathematics, Mathematical Physics and Engineering. From a Fourier series of a function defined in a given interval, one can deduce several summations. In this work, it is shown that the summation:

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots = \frac{1}{2}\ , \tag{1}$$

which was deduced via a Fourier series, can be obtained with the help of classical Gauss hypergeometric summation formula.

The Fourier series for the function *f(x)* defined in the interval $\alpha < x < \alpha + 2\pi$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\, a_n \cos nx + b_n \sin nx\,]\ , \qquad a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx\ ,$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\ dx\ , \qquad b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\ dx\ . \tag{2}$$

In 1812, C. F. Gauss [1] introduced his famous infinite series as follows:

$$\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} = 1+\frac{a\,.b}{c}\frac{z}{1!}+\frac{a.(a+1)\,b.(b+1)}{c.(c+1)}\frac{z^2}{2!}+\ldots \tag{3}$$

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where $(a)_n$ denote the Pochhammer symbol (or the shifted or raised factorial, since $(1)_n$ = n!) defined by:

$$(a)_n = a\,(a+1)\,(a+2)\ldots(a+n-1), \quad n \in \mathbb{N}, \quad (a)_0 = 1, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a = \frac{\Gamma(a+1)}{\Gamma(a)},$$

where $\Gamma(.)$ is the well-known Gamma function.

The above series (3) is called Gauss series or the ordinary hypergeometric series. It is usually represented by the symbol ${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix};\ z\right]$, the well known Gauss hypergeometric function. The series given (3) is convergent for all values of z provided that $|z|< 1$ and divergent if $|z| > 1$. When z = 1, the series is convergent if Re(c-a-b) > 0 and divergent if Re(c-a-b) ≤ 0. Also, when z = -1, the series is absolutely convergent when Re(c-a-b) > 0 and is convergent but not absolutely when -1 < Re(c-a-b) ≤ 0 and divergent when Re(c-a-b) < -1.

It is interesting to mention here that, whenever hypergeometric function ${}_2F_1$ reduces to gamma function, the results are very important from the application point of view. Thus the classical summation theorem for the series ${}_2F_1$ such as those of Gauss, Gauss second, Kummer and Bailey play an important role in the theory of hypergeometric series.

Here, we shall mention the classical summation theorem [1] due to Gauss for ${}_2F_1$ which will be used in our present investigation**:**

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix};\ 1\right] = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)}, \qquad \text{provided Re(c-a-b)} > 0. \tag{4}$$

2. Derivation of sum of the series (1) using Fourier series

With the help of the definition of the Fourier series given in (2), we proceed to obtain a Fourier series for the following function:

$$f(x) = \sqrt{(1-\cos x)}\,, \qquad 0 < x < 2\pi \tag{5}$$

then it is not difficult to get that $a_0 = \frac{4\sqrt{2}}{\pi}$, $a_n = \frac{-4\sqrt{2}}{\pi\,(n^2-1)}$ and $b_n = 0$, thus from (2) we arrive at the following Fourier series:

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{\pi\,(n^2-1)} \cos nx \ . \tag{6}$$

Thus, if in (6) we set $x = 0$, we get, after some simplification the desired summation (1). This completes the derivation of the sum of the series (1) using Fourier series.

In the next section, we shall derive the (1) using hypergeometric function approach.

3. Derivation of sum of the series (1) via Gauss summation theorem

For this, let us denote the left- hand side of the series (1) by S, which can be written as

$$S = \sum_{n=0}^{\infty} \frac{1}{(2n+1)\,(2n+3)} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)\,\left(n+\frac{3}{2}\right)} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{\Gamma\,(n+\frac{1}{2})}{\Gamma\,(n+\frac{5}{2})} ,$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{5}{2}\right)_n} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{5}{2}\right)_n}\frac{(1)_n}{n\,!} , \qquad (7)$$

writing the last expression in terms of hypergeometric series, we have $S = \frac{1}{3}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\ 1 \\ \frac{5}{2}\end{matrix};\ 1\right]$. Finally, using classical Gauss summation theorem (4), we get S = $\frac{1}{2}$. This completes the proof of the sum of the series (1).

Several other interesting summations can also be obtained in a similar manner which is under investigation.

References

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