Article

Lanczos Generators for the Weyl Tensor in Kerr Geometry

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Abstract

We show two Lanczos potentials for the conformal tensor in Kerr spacetime.

Keywords: Rotating black hole, Lanczos generator, Weyl tensor.

1. Introduction

We know that in an arbitrary R_4 the Lanczos potential K_{abc} [1] generates the Weyl tensor via the expression [2-4]:

$$C_{abcd} = K_{abc;d} - K_{abd;c} + K_{cda;b} - K_{cdb;a} + \frac{1}{2}[g_{ad}(K_{bc} + K_{cb}) + g_{bc}(K_{ad} + K_{da}) - g_{ac}(K_{bd} + K_{db}) - g_{bd}(K_{ac} + K_{ca})] + \frac{2}{3}(g_{ad}g_{bc} - g_{ac}g_{bd})K^{qp}{}_{p;q} \,, \quad (1)$$

where

$$K_{ab} = K_a{}^c{}_{b;c} - K_a{}^c{}_{c;b} \,, \quad (2)$$

with the properties:

$$K_{abc} = -K_{bac} \,, \qquad K_{abc} + K_{bca} + K_{cab} = 0 \,. \quad (3)$$

In this work we exhibit two Lanczos generators for the corresponding conformal tensor in Kerr spacetime [5-7].

2. Rotating black hole

The Kerr metric, in Boyer-Lindquist's coordinates [8], is given by [6, 7]:

$$ds^2 = \frac{\Sigma}{C}\, dr^2 + \Sigma\, d\theta^2 - \frac{4amr\, sin^2\theta}{\Sigma} dt\, d\phi + sin^2\theta \left(r^2 + a^2 + \frac{2a^2mr\, sin^2\theta}{\Sigma}\right) d\phi^2 - \left(1 - \frac{2mr}{\Sigma}\right) dt^2, \quad (4)$$

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such that

$$A = r + ia\cos\theta, \qquad \Sigma = A\bar{A} = r^2 + a^2\cos^2\theta, \qquad C = r^2 - 2mr + a^2, \qquad i = \sqrt{-1}\,. \tag{5}$$

The parameters m and a are associated with the mass and the angular momentum of the black hole, respectively. Then it is possible to construct the null vectors:

$$(l^b) = \frac{1}{\sqrt{2\Sigma C}}(-C, 0, a, r^2 + a^2), \quad (n^b) = \frac{1}{\sqrt{2\Sigma C}}(C, 0, a, r^2 + a^2),$$

$$(m^b) = \frac{1}{\sqrt{2\Sigma}}\left(0, -i, \frac{1}{\sin\theta}, a\sin\theta\right), \tag{6}$$

in according with the Newman-Penrose formalism [6, 7, 9, 10].

Thus the Kerr geometry accepts the following Lanczos spintensor:

$$K_{abc} = Q_{ca;b} - Q_{cb;a}\,, \tag{7}$$

verifying (3) with the symmetric tensor:

$$Q_{ab} = \frac{1}{4}(l_a l_b + n_a n_b - m_a m_b - \bar{m}_a \bar{m}_b), \tag{8}$$

or in terms of the metric:

$$(Q_{bc}) = \frac{1}{4}\begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & -g_{33} & -g_{34} \\ 0 & 0 & -g_{34} & -g_{44} \end{pmatrix}. \tag{9}$$

It is easy to prove the relations:

$$K_a{}^b{}_b = B_{;a}\,, \qquad B = \frac{1}{4}Ln\,(C\sin^2\theta), \qquad (K^{ab}{}_b)_{;a} = \Box B = 0, \tag{10}$$

$$K_{ab}{}^c{}_{;c} = 0 \qquad \text{Lanczos differential gauge,} \tag{11}$$

then from (1) and (2):

$$C_{abcd} = K_{abc;d} - K_{abd;c} + K_{cda;b} - K_{cdb;a} + g_{ad}K_{bc} + g_{bc}K_{ad} - g_{ac}K_{bd} - g_{bd}K_{ac}\,, \tag{12}$$

such that

$$K_{bc} = K_{cb} = K_b{}^a{}_{c;a} - B_{;bc}\,. \tag{13}$$

For the same Weyl tensor we can construct the Lanczos potential $\widetilde{K}_{abc}$ connected to K_{abc} via the gauge transformation [3, 4, 11]:

$$\widetilde{K}_{abc} = K_{abc} + g_{ca}Q_b - g_{cb}Q_a \,, \qquad (14)$$

with an arbitrary vector Q_c. If in this Kerr solution we select $Q_a = \frac{1}{3} B_{;a}$, we find that (14) satisfies the symmetries (3), (11) and also:

$$\widetilde{K}_a{}^b{}_b = 0 \qquad \text{Lanczos algebraic gauge,} \qquad (15)$$
$$\Box \widetilde{K}_{abc} = 0 \qquad \text{Lanczos-Illge [12] wave equation.}$$

The generators (7) and (14) are compatible with the Lanczos potentials obtained in [13-17], and it must be interesting to study if these spintensors have relationship with the angular momentum of this black hole.

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