Article

A New Version of Bishop Frame and Application to Spherical Images of Spacelike Curve in E_1^3 Minkowski 3-Space

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Abstract

In this work, I introduce a new version of Bishop frame using a common vector field as binormal vector field of a regular curve and call this frame as "Type-2 Bishop frame in E_1^3". Thereafter, by translating type-2 Bishop frame vectors to O the center of Lorentzian sphere of three-dimensional Minkowski space, I introduce new spherical images and call them as type-2 Bishop spherical images in E_1^3. Serret-Frenet apparatus of these new spherical images are obtained in terms of base curves's type-2 Bishop invariants. Additionally, I express some interesting theorems and illustrate one example of our main results.

Keywords: Spacelike curve, spherical image, Minkowski space, Bishop frame,general helix, Bertrand mate.

1 Introduction

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R:Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in Minkowski space, see [1,2,3,7,8]. And recently, this special frame is extended to study of canal and tubular surfaces, we refer to [7].

In this work, using common vector field as the binormal vector of Serret-Frenet frame, I introduce a new version of the Bishop frame in E_1^3.I call it is "Type-2 Bishop frame" of regular curves. Thereafter, translating new frames vector fields to the center of unit sphere, I obtain new spherical images. We call them as "Type-2 Bishop Spherical Image" of regular curves.

2 Preliminaries

The Minkowski three dimensional space E_1^3 is a real vector space $\mathbb{R}^3$ endowed with the standard flat Lorentzian metric given by $<,>_L= -dx_1^2 + dx_2^2 + dx_3^2$ where (x_1, x_2, x_3) is rectangular coordinate system of E_1^3. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in E_1^3, the Lorentzian cross product of u and v defined by

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

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Recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$; timelike if $g(v,v) < 0$ and null(lightlike) if $g(v,v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vector $\delta^{\prime}$ are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|g(a,a)|}$.The curve $\delta = \delta(s)$ is called a unit speed curve if velocity vector $\delta^{\prime}$ is unit i.e, $\|\delta^{\prime}\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v,w) = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\delta = \delta(s)$ in the space E_1^3.

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by $S_1^2(r) = \left\{p = (p_1, p_2, p_3) \in E_1^3 : g(p,p) = r^2\right\}$.

Proposition 2.1.1: Let two regular curves be α and β in E_1^3. $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of α and β, respectively. If the principal normal vectors are linearly dependent, i.e $N = \lambda N^*$ $(\lambda \in \mathbb{R})$, then α and β called Bertrand mates.

Proposition 2.1.2: Let two regular curves be α and β in E_1^3. $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of α and β, respectively. If the tangent vectors of these curves are perpendicular to each other, i.e $< T, T^* >= 0$, then α is involute of β.

Proposition 2.1.3: Let $\varphi = \varphi(s)$ and $\varphi^* = \varphi^*(s)$ be simple closed curves in E_1^3. These curves will be denoted by C. The normal plane at every point P on the curve meets the curve at a single point Q other than P we call the point Q the opposite point of P. We consider this curves having parallel tangents T and T^* opposite directions at opposite points φ and φ^* of the curve, then φ and φ^* curves called constant breadth, see [9].

3 Type-2 Bishop Frame of a Regular Curve in E_1^3

Theorem 3.1.1: Let $\alpha = \alpha(s)$ be spacelike curve with a spacelike principal normal unit speed. If $\{\Omega_1, \Omega_2, B\}$ is adapted frame, then we have

$$\begin{bmatrix} \Omega_1^{\prime} \\ \Omega_2^{\prime} \\ B^{\prime} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.1}$$

Proof: Let investigate "Type-2 Bishop Frame in E_1^3" relation with Serret-Frenet frame, where $g(\Omega_1, \Omega_1) = g(\Omega_2, \Omega_2) = 1$, $g(B, B) = -1$, and $g(\Omega_1, \Omega_2) = g(\Omega_1, B) = g(\Omega_2, B) = 0$. If Ω_1, Ω_2 are spacelike vectors but B timelike vector then we can write

$$\begin{aligned} \Omega_1^{\prime} &= a_{11}\Omega_1 + a_{12}\Omega_2 + a_{13}B \\ \Omega_2^{\prime} &= a_{21}\Omega_1 + a_{22}\Omega_2 + a_{23}B \\ B^{\prime} &= a_{31}\Omega_1 + a_{32}\Omega_2 + a_{33}B \end{aligned} \tag{3.1.2}$$

If we take inner product of equations (3.1.2) according to $\{\Omega_1, \Omega_2, B\}$respectively, we find $a_{11} = 0$, $a_{12} =< \Omega_1^{\prime}, \Omega_2 >$, $a_{13} = - < \Omega_1^{\prime}, B >$, $a_{21} =< \Omega_2^{\prime}, \Omega_1 >$, $a_{22} = 0$, $a_{23} = - < \Omega_2^{\prime}, B >$, $a_{31} =< \Omega_1, B^{\prime} >= -a_{13}$, $a_{32} =< \Omega_2, B^{\prime} >= a_{23}$, $a_{33} = 0$. From above equations the Bishop frame has

$$\begin{bmatrix} \Omega_1^{\prime} \\ \Omega_2^{\prime} \\ B^{\prime} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & a_{23} & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}$$

Considering the obtained frame, $a_{12} = 0$, $a_{13} = \xi_1$, $a_{23} = -\xi_2$. We have type-2 Bishop frame in E_1^3.

$$\begin{bmatrix} \Omega_1' \\ \Omega_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.3}$$

Thus we have equation (3.1.3) or shortly $X' = AX$. Morever A is semi skew matrix where ξ_1 first curvature and ξ_2 called second curvature of the curve, there the curvatures are defined by

$$\xi_1 = - < \Omega_1', B >, \quad \xi_2 =< \Omega_2', B > .$$

Theorem 3.1.2: Let $\{T, N, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet ve Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & \cosh\theta(s) & 0 \\ \cosh\theta(s) & \sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.4}$$

where θ is the angle between the vectors N and Ω_1.

Proof: We write the tangent vector according to frame $\{\Omega_1, \Omega_2, B\}$ as

$$T = \sinh\theta(s)\Omega_1 + \cosh\theta(s)\Omega_2$$

and differentiate with respect to s

$$\begin{aligned} T' = \kappa N = \; & \theta'(s)\left[\cosh\theta(s)\Omega_1 + \sinh\theta(s)\Omega_2\right] + \\ & \sinh\theta(s)\Omega_1' + \cosh\theta(s)\Omega_2' \end{aligned} \tag{3.1.3}$$

Substituting $\Omega_1' = \xi_1 B$ and $\Omega_2' = -\xi_2 B$ to equation (3.3), we get

$$\begin{aligned} \kappa N = \; & \theta'(s)\left[\cosh\theta(s)\Omega_1 + \sinh\theta(s)\Omega_2\right] \\ & + \sinh\theta(s)\Omega_1 - \cosh\theta(s)\Omega_2 \end{aligned}$$

From equation (3.1.4) we get $\theta(s) = Arg\tanh\frac{\xi_2}{\xi_1}$, $\theta'(s) = \kappa(s)$, $N = \cosh\theta(s)\Omega_1 + \sinh\theta(s)\Omega_2$, and

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & \cosh\theta(s) & 0 \\ \cosh\theta(s) & \sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3.1.4}$$

Since there is a solition for θ satisyfing any initial condition, this show that localy relatively parallel normal fields exist. Besides equation (3.1.2) can also written as

$$B' = \tau N = -\xi_1\Omega_1 + \xi_2\Omega_2$$

Taking the norm of both sides, we have

$$\tau = \sqrt{\left|\xi_2^2 - \xi_1^2\right|} \tag{3.1.2}$$

$$1 = \sqrt{\left| \left(\frac{\xi_1}{\tau}\right)^2 - \left(\frac{\xi_2}{\tau}\right)^2 \right|} \tag{3.1.5}$$

and so by (3.1.5), we may express

$\{ \ \xi_1 = \tau(s)\cosh\theta(s), \qquad \xi_2 = \tau(s)\sinh\theta(s)$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and τ and $\theta(s) =_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \xi_1, \xi_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$ in E_1^3.

4 New Spherical Images of a Regular Curve

Let $\alpha = \alpha(s)$ be a regular curve in E_1^3. If we translate type-2 Bishop frame vectors to the center O of Lorentzian sphere of three-dimensional Minkowski space, we introduce new spherical images in E_1^3.

4.1 Ω_1 Bishop Spherical Image

Definition 4.1.1: Let $\alpha = \alpha(s)$ be a regular spacelike curve in E_1^3. If we translate of the first vector field of type-2 Bishop frame to the center O of the unit sphere S_1^2, we obtain a spherical image $\varphi = \varphi(s_\varphi)$. This curve is called Ω_1 Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Let $\varphi = \varphi(s_\varphi)$ be Ω_1 Bishop spherical image of a regular curve $\alpha = \alpha(s)$. We shall investigate relations among type-2 Bishop and Serret-Frenet invariants. First, we differentiate

$$\varphi' = \frac{d\varphi}{ds_\varphi} \cdot \frac{ds_\varphi}{ds} = \xi_1 B.$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Taking the norm both sides the equation above, we have

$$T_\varphi = B, \qquad \frac{ds_\varphi}{ds} = \xi_1 \tag{4.1.1}$$

we differentiate $(4.1.1)_1$ as

$$T_\varphi' = \dot{T}_\varphi \frac{ds_\varphi}{ds} = -(\xi_1\Omega_1 + \xi_2\Omega_2).$$

So, we have

$$\dot{T}_\varphi = -(\Omega_1 + \frac{\xi_2}{\xi_1}\Omega_2).$$

Since, we have the first curvature and principal normal of φ

$$\kappa_\varphi = \left\|\dot{T}_\varphi\right\| = \sqrt{\left|\left(\tfrac{\xi_2}{\xi_1}\right)^2 - 1\right|}, \qquad N_\varphi = \frac{-1}{\kappa_\varphi}(\Omega_1 - \tfrac{\xi_2}{\xi_1}\Omega_2) \tag{4.1.2}$$

Cross product of $T_\varphi \times N_\varphi$ gives us the binormal vector field of Ω_1 Bishop spherical image of $\alpha = \alpha(s)$

$$B_\varphi = \frac{1}{\kappa_\varphi}(-\frac{\xi_2}{\xi_1}\Omega_1 + \Omega_2) \tag{4.1.3}$$

Using the formula of the torsion, we write a relation

$$\tau_\varphi = \frac{(\xi_1)^7 \cdot \left(\frac{\xi_2}{\xi_1}\right)^{'}}{\left|\xi_2^2 - \xi_1^2\right|} \tag{4.1.4}$$

4.2 Ω_2 Bishop Spherical Image

Definition 4.2.1: Let $\alpha = \alpha(s)$ be a regular spacelike curve in E_1^3.If we translate of the second vector field of type-2 Bishop frame to the center of the unit sphere S_1^2, we obtain a spherical image $\beta = \beta(s_\beta)$. This curve is called Ω_2 Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Let $\beta = \beta(s_\beta)$ be Ω_2 Bishop spherical image of the regular curve $\alpha = \alpha(s)$.We can write that

$$\beta^{'} = \frac{d\beta}{ds_\beta} \cdot \frac{ds_\beta}{ds} = -\xi_2 B.$$

Similar to Ω_2 Bishop spherical image, one can have

$$T_\beta = -B, \qquad \frac{ds_\beta}{ds} = \xi_2 \tag{4.2.1}$$

So, by differentiating of the formula $(4.2.1)_1$, we get

$$T_\beta^{'} = \dot{T}_\beta \frac{ds_\beta}{ds} = \xi_1 \Omega_1 + \xi_2 \Omega_2.$$

or in other words

$$\dot{T}_\beta = \frac{\xi_1}{\xi_2}\Omega_1 + \Omega_2.$$

Since, we express

$$\kappa_\beta = \left\|\dot{T}_\beta\right\| = \sqrt{\left|1 - \left(\frac{\xi_1}{\xi_2}\right)^2\right|}, \qquad N_\beta = \frac{1}{\kappa_\beta}\left(\frac{\xi_1}{\xi_2}\Omega_1 + \Omega_2\right) \tag{4.2.2}$$

Cross product of $T_\varphi \times N_\varphi$ gives us

$$B_\beta = \frac{1}{\kappa_\beta}\left(\Omega_1 + \frac{\xi_1}{\xi_2}\Omega_2\right) \tag{4.2.3}$$

By the formula of the torsion, we have

$$\tau_\beta = \frac{(\xi_2)^7 \cdot \left(\frac{\xi_1}{\xi_2}\right)^{'}}{\left|\xi_2^2 - \xi_1^2\right|} \tag{4.2.4}$$

4.3 Binormal Bishop Spherical Image

Definition 4.3.1: Let $\alpha = \alpha(s)$ be a regular spacelike curve in E_1^3. If we translate of the third vector field of type-2 Bishop frame to the center O of the unit sphere S_1^2, we obtain a spherical image $\phi = \phi(s_\phi)$. This curve is called Binormal Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.

Let $\phi = \phi(s_\phi)$ be Binormal Bishop spherical image of a regular spacelike curve $\alpha = \alpha(s)$. One can differentiate of ϕ with respect to s :

$$\phi' = \frac{d\phi}{ds_\phi} \cdot \frac{ds_\phi}{ds} = -(\xi_1\Omega_1 + \xi_2\Omega_2).$$

In terms of type-2 Bishop frame vector fields, we have tangent vector of the spherical image as follows

$$T_\phi = \frac{-(\xi_1\Omega_1 + \xi_2\Omega_2)}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}, \qquad \frac{ds_\phi}{ds} = \sqrt{\left|\xi_2^2 - \xi_1^2\right|} \tag{4.3.1}$$

In order to determine first curvature of ϕ, we write

$$\dot{T}_\phi = P'(s)\Omega_1 + Q'(s)\Omega_2 + [P(s)\xi_1 - Q(s)\xi_2]B$$

where $P(s) = \dfrac{\xi_1}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$ and $Q(s) = \dfrac{\xi_2}{\sqrt{\left|\xi_2^2 - \xi 1\right|}}$.

Since, we immediately arrive at

$$\kappa_\phi = \left\|\dot{T}_\phi\right\| = \sqrt{\left|(P'(s))^2 + (Q'(s))^2 - [P(s)\xi_1 - Q(s)\xi_2)]^2\right|} \tag{4.3.2}$$

Therefore, we have the principal normal

$$N_\phi = \frac{-1}{\kappa_\phi}\{P'(s)\Omega_1 + Q'(s)\Omega_2 + [P(s)\xi_1 - Q(s)\xi_2]B\} \tag{4.3.3}$$

By the cross product of $T_\phi \times N_\phi$, we obtain the binormal vector field

$$B_\phi = \frac{1}{\kappa_\phi \cdot \sqrt{\left|\xi_2^2 - \xi_1^2\right|}}\{[Q(s)\xi_2 - P(s)\xi_1]\,\Omega_1 + [P(s)\xi_1 - Q(s)\xi_2]\,\Omega_2 - [Q'(s)\xi_1 + P'(s)\xi_2]\,B\} \tag{4.3.4}$$

where $P(s) = \dfrac{\xi_1}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$ and $Q(s) = \dfrac{\xi_2}{\sqrt{\left|\xi_2^2 - \xi_1^2\right|}}$.

By means of obtained equations, we express the torsion of the Binormal Bishop spherical image

$$\begin{aligned}\tau_\phi = & \frac{1}{\kappa_\phi^2}\{\xi_1[\xi_1\xi_1'\xi_2' + \xi_2\xi_2'^2 + \xi_2'(\xi_1^2+\xi_2^2)' \\ & - (\xi_1^2+\xi_2^2)(\xi_2'' + (\xi_1^2+\xi_2^2)\xi_2)] \\ & + \xi_2[(\xi_1^2+\xi_2^2)(\xi_1'' + (\xi_1^2+\xi_2^2)\xi_1) \\ & - \xi_1\xi_1'^2 - \xi_1'\xi_2\xi_2' - \xi_1'((\xi_1^2+\xi_2^2)]\}\end{aligned} \tag{4.3.5}$$

Consequently, we determined Serret-Frenet invariants of the Binormal Bishop spherical image according to type-2 Bishop invariants in E_1^3.

5 Main Results

Theorem 5.1.1: Let $\alpha = \alpha(s)$ be a regular spacelike curve in 3-dimensional Minkowski space. Both of Ω_1 and Ω_2 spherical image of α are Bertrand mates.

Proof: Let us denote the principal normal vectors of Ω_1 and Ω_2 and binormal spherical images as N_φ, N_β and N_ϕ respectively.

The principal normal vectors are given in $(4.1.2)_2$, $(4.2.2)_2$, (4.3.3)

$$N_\varphi = \frac{1}{\kappa_\varphi}(\Omega_1 - \tfrac{\xi_2}{\xi_1}\Omega_2), \qquad N_\beta = \frac{1}{\kappa_\beta}(-\tfrac{\xi_1}{\xi_2}\Omega_1 + \Omega_2)$$

$$N_\phi = \frac{1}{\kappa_\phi}\{\left(\frac{-P(s)P'(s)}{\xi_1}\right)\Omega_1 - \left(\frac{Q(s)Q'(s)}{\xi_2}\right)\Omega_2 + [Q(s)\xi_2 - P(s)\xi_1]\, B\}$$

where

$$P(s) = \frac{-\xi_1}{\sqrt{|\xi_1^2 - \xi_2^2|}} \quad Q(s) = \frac{\xi_2}{\sqrt{|\xi_1^2 - \xi_2^2|}}$$

$$\kappa_\varphi = \sqrt{\left|1 - \left(\tfrac{\xi_2}{\xi_1}\right)^2\right|} \quad \kappa_\beta = \sqrt{\left|\left(\tfrac{\xi_1}{\xi_2}\right)^2 - 1\right|}$$

$$\kappa_\phi = \sqrt{\left(\frac{P(s)P'(s)}{\xi_1}\right)^2 - \left(\frac{Q(s)Q'(s)}{\xi_2}\right)^2 + [Q(s)\xi_2 - P(s)\xi_1)]^2}$$

By putting curvatures κ_φ and κ_β of ξ_1 and ξ_2 spherical images, we have the principal normal vectors as

$$N_\varphi = \frac{1}{\kappa_\varphi}(\Omega_1 - \tfrac{\xi_2}{\xi_1}\Omega_2), \qquad N_\beta = \frac{1}{\kappa_\beta}(-\tfrac{\xi_1}{\xi_2}\Omega_1 + \Omega_2)$$

It can be seen $N_\varphi = -N_\beta$, so the principal normal vectors of Ω_1 and Ω_2 spherical images are linearly dependent. As a result of this from proposition **2.1.1**, they are Bertrand mates.

Theorem 5.1.2: Let $\alpha = \alpha(s)$ be a regular curve in 3-dimensional Minkowski space. Both of Ω_1, Ω_2 and B spherical image of α. Both of Ω_1 and Ω_2 spherical images of α are spherical involutes for binormal spherical image of α.

Proof: Let us denote the tangent vectors of Ω_1 and Ω_2 spherical images as T_φ, T_β and T_ϕ respectively. These tangent vectors are given in $(4.1.1)_1$,$(4.2.1)_1$ and $(4.3.3)_1$. If the inner products are calculated, we get

$$< T_\varphi, T_\phi >= 0, \qquad < T_\beta, T_\phi >= 0$$

The tangent vectors of Ω_1 and Ω_2 spherical images are perpendicular to tangent vectors of binormal spherical images. So the proof is completed from proposition **2.1.2.**

Theorem 5.1.3: Let $\alpha = \alpha(s)$ be a regular curve in 3-dimensional Minkowski space. Both of Ω_1, Ω_2 and B spherical image of α. Binormal vector of Ω_1 are orthogonal to normal vector Ω_2.

Proof: Let us denote the binormal vectors of Ω_1 and principal normal vector of Ω_2, B_φ and N_β respectively. From (4.1.3), $(4.2.2)_2$ this vectors are given

$$B_\varphi = \frac{1}{\kappa_\varphi}(-\tfrac{\xi_2}{\xi_1}\Omega_1 + \Omega_2), \qquad N_\beta = \frac{1}{\kappa_\beta}(-\tfrac{\xi_1}{\xi_2}\Omega_1 + \Omega_2)$$

If the Lorentzian inner product of B_φ and N_β are calculated, we get $< B_\varphi, N_\beta >= 0$. It can be seen that, Binormal vector of Ω_1 and normal vector Ω_2 are perpendicular.

Theorem 5.1.4: Let $\alpha = \alpha(s)$ be a regular curve in 3-dimensional Minkowski space. Both of Ω_1 and Ω_2 spherical images curves of α are constant breadth.

Proof: Let us denote the tangent vectors of Ω_1 and Ω_2 spherical images as T_φ, T_β and T_ϕ respectively. These tangent vectors are given in $(4.1.1)_1$,$(4.2.1)_1$ and $(4.3.3)_1$.

$$T_\varphi = -B, \qquad T_\beta = B, \qquad T_\phi = \frac{-\xi_1\Omega_1 + \xi_2\Omega_2}{\sqrt{\left|\xi_1^2 - \xi_2^2\right|}}$$

It can be seen that $T_\varphi = -T_\beta$. From proposition **2.1.3** they are constant breadth.

6 Example

In this section,we illustrate one example of Frenet frame and new spherical images in E_1^3.

Example 6.1.2: Next, let us consider the following unit speed curve $w(s)$ of E_1^3 by $w = w(s) = (s, \sqrt{2}\ln(sech(s)), \sqrt{2}\arctan(\sinh(s)))$. It is rendered in figure 1.

And this curves's curvature functions are expressed as in E_1^3

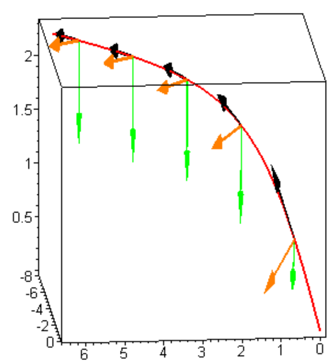
$\{\ \kappa(s) = \sqrt{2}sech(s), \qquad \tau(s) = sech(s).$

The Serret-Frenet frame of the $w = w(s)$ may be written by the aid Mathematical program as follows

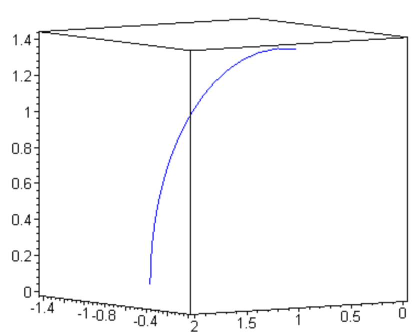
$$\begin{aligned}
&T = (1, \sqrt{2}\tanh(s), \sqrt{2}\sec\mathrm{h}(s)),\\
&N = (0, \mathrm{sech}(s), \text{-}\tanh(s)),\\
&B = (\sqrt{2}, \text{-}\tanh(s), \sec\mathrm{h}(s)),\\
&\theta(s) = \sqrt{2}_0^s \sec h(s)ds{=}\sqrt{2}\arctan(\sinh(s))
\end{aligned}$$

Using transformation matrix equation (3.1.4) we get $w = w(s)$ and tangent, normal,binormal spherical images of unit speed curve with respect to Serret-Frenet frame. respectively Fig 1,2a,2b, 2c.we have type-2 Bishop spherical images of the unit speed curve $w = w(s)$,see figures 3a,3b,3c

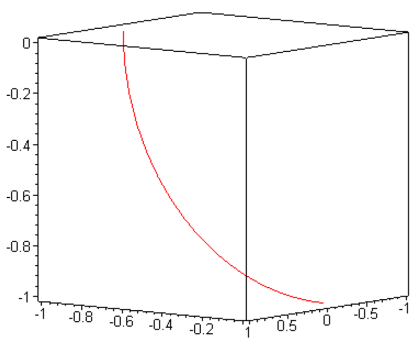
$$\Omega_1 = \frac{1}{\sinh^2\theta + \cosh^2\theta}(-\sinh\theta, -\sqrt{2}\tanh\theta\sec h\theta - \cosh\theta\sec h\theta \\ , -\sqrt{2}\sinh\theta\sec h\theta - \cosh\theta\tanh\theta)$$
$$\Omega_2 = \frac{1}{\sinh^2\theta + \cosh^2\theta}(\cosh\theta, \sqrt{2}\tanh\theta\cosh\theta - \sinh\theta\sec h\theta \\ , \sqrt{2}\cosh\theta\sec h\theta - \sinh\theta\tanh\theta)$$
$$B = (\sqrt{2}, -\tanh(s), \sec h(s))$$



$Fig.1$



$Fig.2a$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-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
1
0.5
0
-0.5
-1

Fig.2b

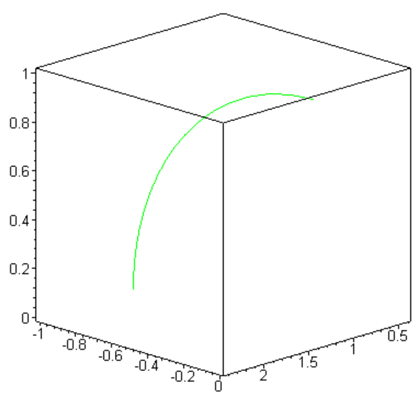
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
2
1.5
1
0.5

Fig.2c

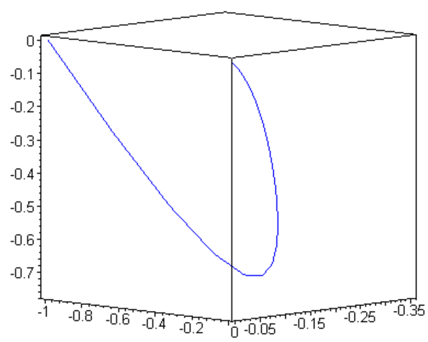
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-1 -0.8 -0.6 -0.4 -0.2 0
-0.05 -0.15 -0.25 -0.35

Fig.3a

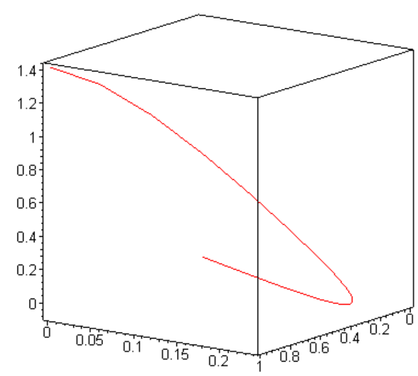
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0 0.05 0.1 0.15 0.2
1 0.8 0.6 0.4 0.2 0

Fig.3b

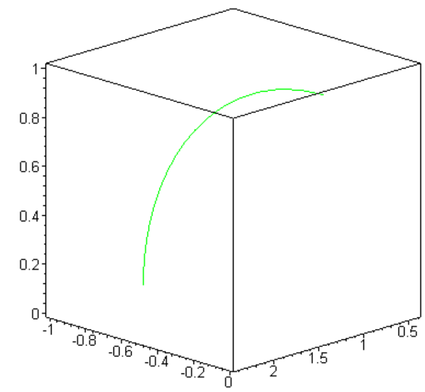

Fig.3c

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