Article

Classification of Axially Symmetric Space-time According to Ricci Tensor Symmetries

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Abstract

In this paper, we study the classification of axially symmetric space-time according to Ricci symmetries (collineations). These classifications have been studied for the cases when one component of Ricci symmetry vector field **V** is non-zero for the cases 1 to 4, two components of **V** are non-zero in cases 5 to 10 and three components of **V** are non-zero in cases 11 to 14 respectively. It is observed that, when Ricci tensor R_{ab} is non-degenerate, the Ricci symmetry is finite dimensional, and, when Ricci tensor is degenerate, we cannot confirm finite dimensionality of Ricci symmetry. Finally, we discuss the results in detail.

Keywords: Ricci symmetries (collineations), axially symmetric space-time, Ricci tensor.

1. Introduction

In general theory of relativity, the classification of space-time symmetries based on concept of isometries of Killing vector play an important role in solution of Einstein's field equations (EFE's). These equations are set of highly non-linear partial differential equations for ten unknown functions g_{ab}. Due this non–linearity it very difficult to obtain exact solutions unless certain symmetry restrictions are applied on space-time which reduces number of unknown functions. In order to classify space-time these symmetries or collineations are proposed by Katzine, Levine *et al* [1-3].

We assumed four dimensional manifold $\boldsymbol{M}$ with Lorentz metric g of signature (+,-,-,-). It is considered that both manifold $\boldsymbol{M}$ and metric g are smooth (C^{∞}) in nature. According to the classification, a vector field **V** defined on $\boldsymbol{M}$ is called Curvature Collineation (CC) satisfying

$$L_v R^a_{bcd} = 0 \tag{1}$$

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where R^{a}_{bcd} is Riemannian curvature tensor of Riemannian space V_n and L_v denotes Lie derivative operator along vector field **V**. Any Curvature collineation (CC) equation (1) is also a Ricci collineation(RC) defied by

$$L_v R_{ab} = 0 \Leftrightarrow R_{ab,c}\, V^c + R_{ac}\, V^c_{,b} + R_{bc}\, V^c_{,a} = 0 \ (\mathrm{a, b} = 0,1,2,3). \tag{2}$$

But the converse need not to be true. As curvature and Ricci tensors play a significant role in understanding the geometric structure of metrics, energy-momentum tensor enables us to understand the physical structure of space-time. Also, in general theory of relativity, Ricci tensor R_{ab} has been more fundamental in study of kinematics of fluid space-time.These classifications of space-time symmetries arise in exact solution of EFE's

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \tag{3}$$

where R_{ab} is Ricci tensor, $R = g^{ab} R_{ab}$ is a Ricci scalar. Here we have assumed that cosmological constant as zero and κ is gravitational constant.

In recent years, much interest have been shown in the study of the space-time symmetries particularly Ricci collineation that arise in understanding the general theory of relativity. These space-time symmetries have been widely studied by Kramer *et al*[4]. Green *et al*[5] and Numez *et al* [6] and considered an example of RC and family of contracted RC symmetries of Robertson-Walker metric. The complete classification of maximal symmetric transverse spaces has been provided by M. Akbar and CAI Rong Gen [7]. Qadir *et al* [8] have studied cylindrically symmetric static space-time and work out complete classification according to their Ricci collineation. Bhokari *et al* [9] have proposed the classification of spherically symmetric static space-time. In addition to this, Amir *et al* [10] have investigated relationship between Ricci collineation vectors and Killing vectors for these space-time. Also, Hall *et al* [11] have studied Ricci and matter collineation.

In this paper, we have investigated the classification of axially symmetric space-time by using Ricci symmetries. We consider the cases 1 to 4 when one component of vector field ***V*** is non-zero, in cases 5 to10, two components of vector field ***V*** are non-zero and in cases 11 to14, three components are non-zero respectively. We substitute all these values in Ricci symmetries (collineations) equations. These collineations are solved in section 2. It is observed that the Ricci tensor R_{ab} is non-degenerate then the Ricci symmetry is finite dimensional and when the Ricci tensor R_{ab} is degenerate we cannot enough confirm finite dimensionality of Ricci tensor. We have discussed the results in detail in section 3. Lastly, conclusion is given in section 4.

2. Classification of axially symmetric space-time

We consider the axially symmetric space-time (Karade and Bhattacharya 1993) [12] in the form of line element given by

$$ds^2 = dt^2 - A^2\left[d\chi^2 + f^2(\chi)\, d\phi^2\right] - B^2\, dz^2, \tag{4}$$

where A and B are functions of t alone. The non-zero components of Ricci tensor are

$$R_{00} = 2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B}, \tag{5}$$

$$R_{11} = -A\ddot{A} - \dot{A}^2 - \frac{A\dot{A}\dot{B}}{B} - \frac{f''}{f}, \tag{6}$$

$$R_{22} = f^2 R_{11}, \tag{7}$$

$$R_{33} = -2\frac{\dot{A}\dot{B}B}{A} - B\ddot{B}, \tag{8}$$

where overhead dot and dash denotes differentiation with respect to t and χ respectively.

Ricci Scalar is given by

$$R = 4\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + 2\frac{\dot{A}^2}{A^2} + 4\frac{\dot{A}\dot{B}}{AB} - 2\frac{f''}{A^2 f}. \tag{9}$$

Using Einstein field equations (EFE's) (2), the expression for non-zero components of energy-momentum tensor T_{ab} are

$$T_{00} = \frac{1}{\kappa}\left[\frac{f''}{A^2 f} - \frac{\dot{A}^2}{A^2} - 2\frac{\dot{A}\dot{B}}{AB}\right], \tag{10}$$

$$T_{11} = \frac{1}{\kappa}\left[A\ddot{A} + \frac{A\dot{A}\dot{B}}{B} + \frac{A^2\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}A}{B}\right], \tag{11}$$

$$T_{22} = f^2 T_{11}, \tag{12}$$

$$T_{33} = \frac{1}{\kappa}\left[2\frac{B^2\ddot{A}}{A} + \frac{\dot{A}^2 B^2}{A^2} + 2\frac{\dot{A}\dot{B}B}{A} - \frac{B^2 f''}{A^2 f}\right]. \tag{13}$$

Using equation (1), we have the following RC equations

$$R_{00,0}\,\mathrm{V}^0 + 2R_{00}\,\mathrm{V}^0{}_{,0} = 0, \tag{14}$$

$$R_{00}\,\mathrm{V}^0{}_{,1} + R_{11}\,\mathrm{V}^1{}_{,0} = 0, \tag{15}$$

$$R_{00}\,\mathrm{V}^0{}_{,2} + f^2 R_{11}\,\mathrm{V}^2{}_{,0} = 0, \tag{16}$$

$$R_{00}\,\mathrm{V}^0{}_{,3} + R_{33}\,\mathrm{V}^3{}_{,0} = 0, \tag{17}$$

$$R_{11,0}\,V^0 + R_{11,1}\,V^1 + 2\,R_{11}\,V^1_{\ ,1} = 0, \tag{18}$$

$$R_{11}\,V^1_{\ ,2} + f^2 R_{11}\,V^2_{\ ,1} = 0, \tag{19}$$

$$R_{11}\,V^1_{\ ,3} + R_{33}\,V^3_{\ ,1} = 0, \tag{20}$$

$$f^2 R_{11}\,V^2_{\ ,3} + R_{33}\,V^3_{\ ,2} = 0, \tag{21}$$

$$R_{33,0}\,V^0 + 2\,R_{33}\,V^3_{\ ,3} = 0, \tag{22}$$

$$R_{11,0}\;V^0 + \frac{\left(f^2 R_{11}\right)_{,1}}{f^2}\,V^1 + 2R_{11}\;V^2_{\ ,2} = 0, \tag{23}$$

where comma denotes the partial derivatives and indices 0,1, 2, 3, 4 corresponds to variables t, χ, ϕ and z respectively.

Case 1: $\mathbf{V} = \left(V^0, 0, 0, 0\right)$, where $V^0 \neq 0$

Using Ricci symmetries equations (14)-(23) and substituting $V^1 = V^2 = V^3 = 0$ into these equations, we obtain following constraints equations

$$R_{00,0}\,V^0 + 2\,R_{00}\,V^0_{\ ,0} = 0, \tag{24}$$

$$R_{00}\;V^0_{\ ,a} = 0, \quad \text{where } a = 1,2,3\,, \tag{25}$$

$$R_{11,0}\;V^0 = 0, \tag{26}$$

$$R_{33,0}\;V^0 = 0. \tag{27}$$

From (26) and (27), we get, $R_{11} = \alpha$, $R_{33} = \gamma$, where α and γ are constants.

This case has three sub-cases as listed below

(a) $R_{00,0} = 0$ gives $R_{00} = \beta \neq 0$. Using constraints equations (24) and (25) we have $V^0 = const.$

(b) $R_{00} = 0$. In this sub-case the vector components v^0 is an unconstrained function of variables $t, \chi, \phi, z.$

(c) $R_{00,0} \neq 0$ and $R_{00} \neq 0$. Using (24) and (25), we get $V^0 = \dfrac{c_1}{\sqrt{R_{00}}}$ where c_1 is constant.

Case 2: $\mathbf{V} = \left(0, V^1, 0, 0\right)$, where $V^1 \neq 0$

Using RC equations (14)-(23) and substituting $V^0 = V^2 = V^3 = 0,$ we get

$$R_{11}\ V^1{}_{,a}=0,\quad a=0,2,3 \tag{28}$$

$$R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}=0, \tag{29}$$

$$\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ V^1=0. \tag{30}$$

Here there are two possibilities

a) $R_{11}=0$ then (28)-(30) gives V^1 is an unconstrained function of variables t,χ,ϕ,z.

b) $R_{11}\neq 0$ then (29) gives $V^1=\frac{c_1}{\sqrt{R_{11}}}$.

Case 3: $\mathbf{V}=\left(0,0,V^2,0\right)$, where $V^2\neq 0$

In this case, equations (14)-(23) becomes

$$R_{11}\ V^2{}_{,2}=0, \tag{31}$$

$$\left(f^2R_{11}\right)V^2{}_{,a}=0,\quad \text{where } a=0,1,3, \tag{32}$$

which yields two sub-cases a) $R_{11}=0$ and b) $R_{11}\neq 0$.

For sub-case (a) we see that v^2 is an unconstrained function of variables t,χ,ϕ,z and for sub-case (b), $V^2=const.$

Case 4: $\mathbf{V}=\left(0,0,0,V^3\right)$, where $V^3\neq 0$

Therefore, equations (14)-(23) becomes

$$R_{33}\ V^3{}_{,a}=0,\quad \text{where } a=0,1,2,3.$$

In this case, if $R_{33}=0$ then V^3 is an unconstrained function of variables and if $R_{33}\neq 0$ then $V^3=const.$

Case 5: $\mathbf{V}=\left(V^0,V^1,0,0\right)$, where $V^0\neq 0$ and $V^1\neq 0$

Substituting $V^2=0$ and $V^3=0$ into Ricci symmetries equations (14)-(23), we find

$$R_{00,0}\ V^0+2R_{00}\ v^0{}_{,0}=0, \tag{33}$$

$$R_{00}\ V^0{}_{,1}+R_{11}\ V^1{}_{,0}=0, \tag{34}$$

$$R_{00}\ V^0{}_{,2}=0, \tag{35}$$

$$R_{00}\ V^0{}_{,3}=0, \tag{36}$$

$$R_{11,0}\ V^0+R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}=0, \tag{37}$$

$$R_{11}\ V^1{}_{,2}=0, \tag{38}$$

$$R_{11}\ V^1{}_{,3}=0, \tag{39}$$

$$R_{11,0}\ V^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ V^1=0, \tag{40}$$

$$R_{33,0}\ V^0=0. \tag{41}$$

Equation (41) implies that $R_{33,0}=0\, i.e.\, R_{33}=\gamma$ where γ is constant, we get following sub-cases

a) $R_{00}=0,\quad R_{11}=0,$

b) $R_{00}=0,\quad R_{11}\neq 0,$

c) $R_{00}\neq 0,\quad R_{11}=0,$

d) $R_{00}\neq 0,\quad R_{11}\neq 0$.

For sub-case (a) V^0 and V^1 are an unconstrained function of variables t,χ,ϕ,z and in sub-case (b) $V^1=c_1 f$ and $V^0=V^0\left(t,\chi,\phi,z\right)$.

For sub-case (c), we have, $V^0=\dfrac{c_1}{\sqrt{R_{00}}}$, $V^1=V^1\left(t,\chi,\phi,z\right)$.

For sub-case (d), we obtain the following constraints equations

$$R_{00}\ V^0{}_{,a}=0\ , \tag{42}$$

$$R_{11}\ V^1{}_{,a}=0, \quad \text{where } a=2,3, \tag{43}$$

$$R_{00,0}\ V^0+2R_{00}\ V^0{}_{,0}=0, \tag{44}$$

$$R_{00}\ V^0{}_{,1}+R_{11}\ V^1{}_{,0}=0, \tag{45}$$

$$R_{11,0}\ V^0+R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}=0, \tag{46}$$

$$R_{11,0}\ \mathrm{V}^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ \mathrm{V}^1=0. \tag{47}$$

Integrating equation (42), (43) yields

$\mathrm{V}^0=g\left(t,\chi\right),\mathrm{V}^1=h\left(t,\chi\right),$

where g and h are arbitrary functions of t and χ.

Furthermore, from equation (44), we find that, $g\left(t,\chi\right)=\dfrac{F\left(x\right)}{\sqrt{R_{00}}}$ and using (46) and (47), we get $h\left(t,\chi\right)=G\left(t\right)f$.

Now, from equation (45), $\dfrac{F_{,\chi}}{f}=-\dfrac{R_{11}}{\sqrt{R_{00}}}G_{,t}=\lambda,$

where λ is constant then for first possibility $\lambda=0,$ we get $\mathrm{V}^0=\dfrac{c_1}{\sqrt{R_{00}}}$ and $\mathrm{V}^1=c_2 f$

and for second possibility $\lambda\neq 0,$ we obtain $\mathrm{V}^0=\dfrac{1}{\sqrt{R_{00}}}\left[\lambda H\left(\chi\right)+c_1\right]$ and $\mathrm{V}^1=c_2-\lambda f\displaystyle\int\frac{\sqrt{R_{00}}}{R_{11}}dt$ where $H\left(\chi\right)=\int f\,d\chi$.

Case 6: $\mathbf{V}=\left(\mathrm{V}^0,0,\mathrm{V}^2,0\right)$, where $\mathrm{V}^0\neq 0$ and $\mathrm{V}^2\neq 0$

Substituting these values of components of V into equations (14)-(23) we get

$$R_{00}\ \mathrm{V}^0{}_{,a}=0, \tag{48}$$

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,a}=0,\ \text{where}\ a=1,3, \tag{49}$$

$$R_{33,0}\,\mathrm{V}^0=0, \tag{50}$$

$$R_{00,0}\ \mathrm{V}^0+2R_{00}\ \mathrm{V}^0{}_{,0}=0, \tag{51}$$

$$R_{00}\ \mathrm{V}^0{}_{,2}+\left(f^2R_{11}\right)\mathrm{V}^2{}_{,0}=0, \tag{52}$$

$$R_{11,0}\ \mathrm{V}^0=0, \tag{53}$$

$$R_{11,0}\ \mathrm{V}^0+2R_{11}\ \mathrm{V}^2{}_{,2}=0. \tag{54}$$

Equations (50) and (53) gives $R_{33} = \gamma = const.,\ R_{11} = \alpha = const.$

Now, we have the following sub-cases

$$\text{a) } R_{00} = 0, \quad R_{11} = \alpha = 0,$$

$$\text{b) } R_{00} = 0, \quad R_{11} = \alpha \neq 0,$$

$$\text{c) } R_{00} \neq 0, \quad R_{11} = \alpha = 0,$$

$$\text{d) } R_{00} \neq 0, \quad R_{11} = \alpha \neq 0.$$

For sub-case (a), V^0 and V^2 are an unconstrained function of variables.

For sub-case (b), $\mathrm{V}^2 = const.$ and $\mathrm{V}^0 = \mathrm{V}^0\left(t, \chi, \phi, z\right)$.

For sub-case (c), equations (48)-(54) yields $\mathrm{V}^0 = \dfrac{c_1}{\sqrt{R_{00}}}$ and $\mathrm{V}^2 = \mathrm{V}^2\left(t, \chi, \phi, z\right)$.

For sub-case (d), V^0 and V^2 are function of t, ϕ.

Now, from equations (52) and (54), we obtain

$$\frac{f^2 R_{11,0}}{2R_{00}} \mathrm{V}^2{}_{,0} = \mathrm{V}^2{}_{,22}. \tag{55}$$

Therefore, we set, $\mathrm{V}^2 = A\left(t\right)B\left(\phi\right)$

Then equation (55) becomes, $$\frac{f^2 R_{11,0}}{2R_{00}} \frac{A_{,0}}{A} = \frac{B_{,22}}{B} = \lambda. \tag{56}$$

When $\lambda = 0$ then $\mathrm{V}^0 = -\dfrac{2R_{11}}{R_{11,0}} c_1,\ \mathrm{V}^2 = c_1\phi + c_2.$

When $\lambda > 0$ then $\mathrm{V}^0 = -\dfrac{2F\left(t\right)R_{11}\sqrt{\lambda}}{R_{11,0}}\left[c_1 e^{\sqrt{\lambda}\,\phi} - c_2 e^{-\sqrt{\lambda}\,\phi}\right]$

and $\mathrm{V}^2 = F\left(t\right)\left[c_1 e^{\sqrt{\lambda}\,\phi} + c_2 e^{-\sqrt{\lambda}\,\phi}\right].$

When $\lambda < 0$ then $\mathrm{V}^0 = -\dfrac{2F\left(t\right)R_{11}\sqrt{\lambda}}{R_{11,0}}\left[c_1 \cos\left(\sqrt{\lambda}\,\phi\right) - c_2 \sin\left(\sqrt{\lambda}\,\phi\right)\right]$

and $\mathrm{V}^2 = F(t)\left[c_1 \cos\left(\sqrt{\lambda}\,\phi\right) + c_2 \sin\left(\sqrt{\lambda}\,\phi\right)\right]$, where $F(t) = e^{\int 2\lambda R_{00}/f^2 R_{11,0} dt}$.

Case 7: $\mathbf{V} = \left(\mathrm{V}^0, 0, 0, \mathrm{V}^3\right)$, where $\mathrm{V}^0 \neq 0$ and $\mathrm{V}^3 \neq 0$

Ricci symmetries equations (14)-(23) gives

$$R_{00}\ \mathrm{V}^0{}_{,a} = 0, \tag{57}$$

$$R_{33}\ \mathrm{V}^3{}_{,a} = 0, \quad \text{where } a = 1, 2, \tag{58}$$

$$R_{11,0}\ \mathrm{V}^0 = 0, \tag{59}$$

$$R_{00,0}\ \mathrm{V}^0 + 2R_{00}\ \mathrm{V}^0{}_{,0} = 0, \tag{60}$$

$$R_{00,0}\ \mathrm{V}^0{}_{,3} + R_{33}\ \mathrm{V}^3{}_{,0} = 0, \tag{61}$$

$$R_{00,0}\ \mathrm{V}^0 + 2R_{33}\ \mathrm{V}^3{}_{,3} = 0. \tag{62}$$

Equation (59) gives $R_{11} = \alpha$ and V^0 is unconstraint function.

Now, from (57)-(62) has following sub-cases

a) $R_{00} = 0, \quad R_{33} = 0,$

b) $R_{00} = 0, \quad R_{33} \neq 0,$

c) $R_{00} \neq 0, \quad R_{33} = 0,$

d) $R_{00} \neq 0, \quad R_{33} \neq 0$.

For sub-case (a), each equation has satisfied identically.

For sub-case (b), we have

$$R_{33}\ \mathrm{V}^3{}_{,a} = 0, \quad \text{where } a = 0, 1, 2, \tag{63}$$

$$R_{33,0}\ \mathrm{V}^0 + 2R_{33}\ \mathrm{V}^3{}_{,3} = 0. \tag{64}$$

From equation (63) gives V^3 is function of z alone *i.e.* $\mathrm{V}^3 = F(z)$ and using (64) , we get

$$\mathrm{V}^0 = -\frac{2R_{33}F(z)_{,3}}{R_{33,0}}.$$

For sub-case (c), constraint equations are

$$R_{00}\ \mathrm{V}^0{}_{,b}=0,\quad \text{where } b=1,2,3, \tag{65}$$

$$R_{00,0}\ \mathrm{V}^0+2R_{00}\ \mathrm{V}^0{}_{,0}=0. \tag{66}$$

Equation (65) and (66) implies $\mathrm{V}^0=G(t)$ where $G(t)=\dfrac{c}{\sqrt{R_{00}}}$, $c=const.$ and $\mathrm{V}^3=g(t,\chi,\phi,z)$

Now, for sub-case (d), $\mathrm{V}^0=g(t,z)$ and $\mathrm{V}^3=h(t,z)$.

On differentiating (62) with respect to z, we get, $\mathrm{V}^0{}_{,3}=\dfrac{-2R_{33}}{R_{33,0}}\mathrm{V}^3{}_{,33}$ (67)

and substitute (67) in (61), $\dfrac{R_{33,0}}{2R_{00}}\ \mathrm{V}^3{}_{,0}=\mathrm{V}^3{}_{,33}$. (68)

Assume, $\mathrm{V}^3=A(t)B(z)$

Then (68) yields, $\dfrac{R_{33,0}}{2R_{00}}\ \dfrac{A_{,0}}{A}=\dfrac{B_{,33}}{B}=\lambda.$ (69)

If $\lambda=0$ then $\mathrm{V}^0=\dfrac{-2R_{33}}{R_{33,0}}c_1$ and $\mathrm{V}^3=c_1z+c_2$.

If $\lambda>0$ then $\mathrm{V}^0=-\dfrac{2F(t)R_{33}\sqrt{\lambda}}{R_{33,0}}\left[c_1e^{\sqrt{\lambda}\,z}-c_2e^{-\sqrt{\lambda}\,z}\right]$ and $\mathrm{V}^3=F(t)\left[c_1e^{\sqrt{\lambda}\,z}+c_2e^{-\sqrt{\lambda}\,z}\right]$.

If $\lambda<0$ then $\mathrm{V}^0=\dfrac{2F(t)R_{33}\sqrt{\lambda}}{R_{33,0}}\left[c_1\sin\left(\sqrt{\lambda}\,z\right)-c_2\cos\left(\sqrt{\lambda}\,z\right)\right]$

and $\mathrm{V}^3=F(t)\left[c_1\cos\left(\sqrt{\lambda}\,z\right)+c_2\sin\left(\sqrt{\lambda}\,z\right)\right]$.

where $F(t)$ is an arbitrary functions of t.

Case 8: $\mathbf{V}=\left(0,\mathrm{V}^1,\mathrm{V}^2,0\right)$, where $\mathrm{V}^1\neq0$ and $\mathrm{V}^2\neq0$

On substituting these values in constraint equations (14)-(23), we get

$$R_{11}\ \mathrm{V}^1{}_{,a}=0, \tag{70}$$

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,a}=0,\quad \text{where } a=0,3, \tag{71}$$

$$R_{11,1}\ \mathrm{V}^1+2R_{11}\ \mathrm{V}^1{}_{,1}=0, \tag{72}$$

$$R_{11}\ \mathrm{V}^1{}_{,2}+\left(f^2R_{11}\right)\mathrm{V}^2{}_{,1}=0, \tag{73}$$

$$\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ \mathrm{V}^1+2R_{11}\ \mathrm{V}^2{}_{,2}=0. \tag{74}$$

Here, we have two sub-cases

a) $R_{11}=0$ then V^1 and V^2 unconstraint functions of variables.
b) $R_{11}\neq 0$ then $\mathrm{V}^1=g\left(\chi,\phi\right)$ and $\mathrm{V}^2=h\left(\chi,\phi\right)$.

Now,differentiate equation (74) with respect to ϕ and using (73), we get

$$\mathrm{V}^2{}_{,22}=\frac{\left(f^2R_{11}\right)_{,1}}{2R_{11}}\ \mathrm{V}^2{}_{,1}\ . \tag{75}$$

Let, $\mathrm{V}^2=A\left(\chi\right)B\left(\phi\right)$ then (75) yields

$$\frac{\left(f^2R_{11}\right)_{,1}}{2R_{11}}\frac{A_{,1}}{A}=\frac{B_{,22}}{B}=\lambda. \qquad . \tag{76}$$

i)If $\lambda=0$ then $\mathrm{V}^2=c_1\phi+c_2$ and $\mathrm{V}^1=\dfrac{-2f^2R_{11}}{\left(f^2R_{11}\right)_{,1}}c_1$.

ii)If $\lambda>0$ then $\mathrm{V}^2=F\left(\chi\right)\left[c_1e^{\sqrt{\lambda}\,\phi}+c_2e^{-\sqrt{\lambda}\,\phi}\right]$and $\mathrm{V}^1=\dfrac{-2f^2R_{11}}{\left(f^2R_{11}\right)_{,1}}\sqrt{\lambda}\left[c_1e^{\sqrt{\lambda}\,\phi}-c_2e^{-\sqrt{\lambda}\,\phi}\right]$.

iii)If $\lambda<0$ then $\mathrm{V}^2=F\left(\chi\right)\left[c_1\cos\left(\sqrt{\lambda}\,\phi\right)+c_2\sin\left(\sqrt{\lambda}\,\phi\right)\right]$

and $\mathrm{V}^1=\dfrac{-2f^2R_{11}}{\left(f^2R_{11}\right)_{,1}}F\left(\chi\right)\sqrt{\lambda}\left[c_2\cos\left(\sqrt{\lambda}\,\phi\right)-c_1\sin\left(\sqrt{\lambda}\,\phi\right)\right]$.

where $F\left(\chi\right)$ arbitrary functions of χ .

Case 9: $\mathbf{V}=\left(0,\mathrm{V}^1,0,\mathrm{V}^3\right)$, where $\mathrm{V}^1\neq 0$ and $\mathrm{V}^3\neq 0$

Ricci symmetries equations (14)-(23) gives

$$R_{11}\ \mathrm{V}^1{}_{,a}=0, \quad \text{where } a=0,2, \tag{77}$$

$$R_{33}\ \mathrm{V}^3{}_{,b}=0, \quad \text{where } b=0,2,3, \tag{78}$$

$$R_{11,1}\ V^1 + 2R_{11}\ V^1{}_{,1} = 0, \tag{79}$$

$$R_{11}\ V^1{}_{,3} + R_{33}\ V^3{}_{,1} = 0, \tag{80}$$

$$\frac{\left(f^2 R_{11}\right)_{,1}}{f^2} V^1 = 0. \tag{81}$$

For these equations, we have the following sub-cases

a) $R_{11} = 0, \quad R_{33} = 0,$

b) $R_{11} = 0, \quad R_{33} \neq 0,$

c) $R_{11} \neq 0, \quad R_{33} = 0,$

d) $R_{11} \neq 0, \quad R_{33} \neq 0$.

For sub-case (a), V^1 and V^3 unconstraint functions of variables $t, \chi, \phi, z.$

For sub-case (b), $V^3 = const.$ and $V^1 = g(t, \chi, \phi, z).$

For sub-case (c), we have

$$R_{11}\ V^1{}_{,0} = 0, \tag{82}$$

$$R_{11}\ V^1{}_{,2} = 0, \tag{83}$$

$$R_{11}\ V^1{}_{,3} = 0, \tag{84}$$

$$\frac{\left(f^2 R_{11}\right)_{,1}}{f^2} V^1 = 0, \tag{85}$$

$$R_{11,1}\ V^1 + 2R_{11}\ V^1{}_{,1} = 0. \tag{86}$$

Equation (86) yields $V^1 = \dfrac{c}{\sqrt{R_{11}}}$ and $V^3 = V^3(t, \chi, \phi, z).$

For sub-case (d), equation (79) implies $V^1 = \dfrac{c_1}{\sqrt{R_{11}}}$ and using (80),we get $V^3 = const. = c_2.$

Case 10: $\mathbf{V} = \left(0, 0, V^2, V^3\right),$ where $V^2 \neq 0$ and $V^3 \neq 0$

Then we have the following RC equations, using equations (14)-(23)

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,a}=0,\quad \text{where } a=0,1, \tag{87}$$

$$R_{33}\ \mathrm{V}^3{}_{,b}=0,\quad \text{where } b=0,1,3, \tag{88}$$

$$R_{11}\,\mathrm{V}^2{}_{,2}=0, \tag{89}$$

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,3}+R_{33}\ \mathrm{V}^3{}_{,2}=0. \tag{90}$$

We have sub-cases given below

$$\text{a) } R_{11}=0,\quad R_{33}=0,$$

$$\text{b) } R_{11}=0,\quad R_{33}\neq 0,$$

$$\text{c) } R_{11}\neq 0,\quad R_{33}=0,$$

$$\text{d) } R_{11}\neq 0,\quad R_{33}\neq 0\ .$$

For sub-case (a), components of V are unconstrained functions of variables.

For sub-case (b), $\mathrm{V}^3=const.$ and $\mathrm{V}^2=h\left(t,\chi,\phi,z\right)$.

For sub-case (c), $\mathrm{V}^2=const.$ and $\mathrm{V}^3=g\left(t,\chi,\phi,z\right)$.

For sub-case (d), $\mathrm{V}^2{}_{,a}=0,\qquad a=0,1,2,$ (91)

and $\mathrm{V}^3{}_{,b}=0,\qquad \mathrm{b}=0,1,3.$ (92)

Therefore, equations (91) and (92) gives $\mathrm{V}^2=c_1,\ \mathrm{V}^3=c_2$.

Case 11: $\mathbf{V}=\left(\mathrm{V}^0,\mathrm{V}^1,\mathrm{V}^2,0\right)$, where $\mathrm{V}^0\neq 0$, $\mathrm{V}^1\neq 0,\ \mathrm{V}^2\neq 0$

Put $\mathrm{V}^3=0$ in Ricci symmetriesequations (14)-(23), lead to

$$R_{00,0}\ \mathrm{V}^0+2R_{00}\ \mathrm{V}^0{}_{,0}=0, \tag{93}$$

$$R_{00,0}\,\mathrm{V}^0{}_{,1}+R_{11}\ \mathrm{V}^1{}_{,0}=0, \tag{94}$$

$$R_{00}\ \mathrm{V}^0{}_{,2}+\left(f^2R_{11}\right)\mathrm{V}^2{}_{,0}=0, \tag{95}$$

$$R_{00}\ \mathrm{V}^0{}_{,3}=0, \tag{96}$$

$$R_{11,0}\ \mathrm{V}^0+R_{11,1}\ \mathrm{V}^1+2R_{11}\ \mathrm{V}^1{}_{,1}=0, \tag{97}$$

$$R_{11}\ V^1{}_{,2}+\left(f^2R_{11}\right)V^2{}_{,1}=0, \tag{98}$$

$$R_{11}\ V^1{}_{,3}=0, \tag{99}$$

$$\left(f^2R_{11}\right)V^2{}_{,3}=0, \tag{100}$$

$$R_{33,0}\ V^0=0, \tag{101}$$

$$R_{11,0}\ V^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}V^1+2R_{11}\ V^2{}_{,2}=0. \tag{102}$$

This case can be divided in four sub-cases

a) $R_{00}=0,\quad R_{11}=0,$

b) $R_{00}\neq 0,\quad R_{11}=0,$

c) $R_{00}=0,\quad R_{11}\neq 0,$

d) $R_{00}\neq 0,\quad R_{11}\neq 0$.

For sub-case (a) each equation satisfied identically so that non-zero components of vector field are arbitrary functions of variables.

For sub-case (b), we get solution, $V^0=\frac{c_1}{\sqrt{R_{00}}}$, $V^1=g\left(t,\chi,\phi,z\right)$ and $V^2=h\left(t,\chi,\phi,z\right)$.

Now, for sub-case (c), we have

$$R_{11}\ V^1{}_{,0}=0, \tag{103}$$

$$\left(f^2R_{11}\right)V^2{}_{,0}=0, \tag{104}$$

$$R_{11,0}\ V^0+R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}=0, \tag{105}$$

$$R_{11}\ V^1{}_{,2}+\left(f^2R_{11}\right)V^2{}_{,1}=0, \tag{106}$$

$$R_{11}\ V^1{}_{,3}=0, \tag{107}$$

$$\left(f^2R_{11}\right)V^2{}_{,3}=0, \tag{108}$$

$$R_{11,0}\ V^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}V^1+2R_{11}\ V^2{}_{,2}=0. \tag{109}$$

Equations (103),(104), (107)and (108) gives $V^1 = g(\chi,\phi)$, $V^2 = h(\chi,\phi)$ and $V^3 = k(t,\chi,\phi,z)$.

Now, using (105) and (109), we get

$$F(\chi)\, V^1 - V^1_{,1} + V^2_{,2} = 0 \text{ where } F(\chi) = \frac{f^2_{,1}}{2f^2}. \tag{110}$$

Using equation (106) we have

$$V^2_{,21} = -\frac{V^1_{,22}}{f^2(\chi)}, \tag{111}$$

$$F_{,1}\, V^1 + F\, V^1_{,1} - V^1_{,11} - \frac{V^1_{,22}}{f^2} = 0. \tag{112}$$

Set $V^1 = X(\chi)Y(\phi)$ then equation (112) gives

$$F_{,1} + F\frac{X_{,1}}{X} - \frac{X_{,11}}{X} = \frac{Y_{,22}}{Yf^2} = \lambda. \tag{113}$$

For $\lambda = 0$, we have the solution

$$V^1 = f.(c_1\phi + c_2),\ V^2 = -c_1 H(\chi) + c_2 \text{ and } V^3 = h(t,\chi,\phi,z), \text{ where } H(\chi) = \int \frac{1}{f} d\chi.$$

For sub-case (d), we have

$$R_{00,0}\, V^0 + 2R_{00}\, V^0_{,0} = 0, \tag{114}$$

$$R_{00,0}\, V^0_{,1} + R_{11}\, V^1_{,0} = 0, \tag{115}$$

$$R_{00}\, V^0_{,2} + \left(f^2 R_{11}\right) V^2_{,0} = 0, \tag{116}$$

$$R_{00}\, V^0_{,3} = 0, \tag{117}$$

$$R_{11,0}\, V^0 + R_{11,1}\, V^1 + 2R_{11}\, V^1_{,1} = 0, \tag{118}$$

$$R_{11}\, V^1_{,2} + \left(f^2 R_{11}\right) V^2_{,1} = 0, \tag{119}$$

$$R_{11}\, V^1_{,3} = 0, \tag{120}$$

$$\left(f^2 R_{11}\right) V^2_{,3} = 0, \tag{121}$$

$$R_{33,0}\ \mathrm{V}^0=0, \tag{122}$$

$$R_{11,0}\ \mathrm{V}^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ \mathrm{V}^1+2R_{11}\ \mathrm{V}^2{}_{,2}=0, \tag{123}$$

From equations (117), (120) and (121), we get

$$\mathrm{V}^0=g\left(t,\chi,\phi,z\right),\mathrm{V}^1=h\left(t,\chi,\phi,z\right),\mathrm{V}^2=k\left(t,\chi,\phi,z\right).$$

Now, equation (114) yields, $\mathrm{V}^0=\frac{A_1\left(\chi,\phi\right)}{\sqrt{R_{00}}}.$ (124)

Put (124) in (115), (116) gives

$$\mathrm{V}^1=-A_{1,1}\int\frac{\sqrt{R_{00}}}{R_{11}}dt+A_2\left(\chi,\phi\right), \tag{125}$$

$$\mathrm{V}^2=-\frac{A_{1,2}}{f^2}\int\frac{\sqrt{R_{00}}}{R_{11}}dt+A_3\left(\chi,\phi\right). \tag{126}$$

Now, put (124)-(126) in above constraint equations (114)-(123) then we get

$A_1=c_1\chi+c_2,\ A_2=0,\ A_3=F\left(\chi\right)=$ arbitrary function of χ.

Case 12: $\mathbf{V}=\left(\mathrm{V}^0,\mathrm{V}^1,0,\mathrm{V}^3\right)$, where $\mathrm{V}^0\neq0,\mathrm{V}^1\neq0$ and $\mathrm{V}^3\neq0$

Put $\mathrm{V}^2=0$ in set of equations (14)-(23)

$$R_{00,0}\ \mathrm{V}^0+2R_{00}\ \mathrm{V}^0{}_{,0}=0, \tag{127}$$

$$R_{00}\ \mathrm{V}^0{}_{,1}+R_{11}\ \mathrm{V}^1{}_{,0}=0, \tag{128}$$

$$R_{00}\ \mathrm{V}^0{}_{,3}+R_{33}\ \mathrm{V}^3{}_{,0}=0, \tag{129}$$

$$R_{11,0}\ \mathrm{V}^0+R_{11,1}\ \mathrm{V}^1+2R_{11}\ \mathrm{V}^1{}_{,1}=0, \tag{130}$$

$$R_{11}\ \mathrm{V}^1{}_{,3}+R_{33}\ \mathrm{V}^3{}_{,1}=0, \tag{131}$$

$$R_{11,0}\ \mathrm{V}^0+\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ \mathrm{V}^1=0, \tag{132}$$

$$R_{33,0}\ V^0+2R_{33}\ V^3{}_{,3}=0, \tag{133}$$

$$R_{aa}\ V^a{}_{,2}=0, \text{where } a=0,1,3. \tag{134}$$

From equations (127)-(134), we have the following sub-cases,

a) $R_{00}=0,\quad R_{11}=0,\ R_{33}=0,$

b) $R_{00}=0,\quad R_{11}=0,\ R_{33}\neq 0,$

c) $R_{00}=0,\quad R_{11}\neq 0,\ R_{33}=0,$

d) $R_{00}\neq 0,\quad R_{11}=0,\ R_{33}=0,$

e) $R_{00}=0,\quad R_{11}\neq 0,\ R_{33}\neq 0,$

f) $R_{00}\neq 0,\quad R_{11}\neq 0,\ R_{33}=0,$

g) $R_{00}\neq 0,\quad R_{11}=0,\ R_{33}\neq 0,$

h) $R_{00}\neq 0,\quad R_{11}\neq 0,\ R_{33}\neq 0.$

For sub-case (a) all constraint equation are satisfied identically.

For sub-case (b), we have, $V^0=V^0(t,\chi,\phi,z)$, $V^1=V^1(t,\chi,\phi,z)$ and $V^3=const.$

For sub-case (c), we get, $V^0=V^0(t,\chi,\phi,z)$, $V^1=c_1 f$ and $V^3=V^3(t,\chi,\phi,z)$.

For sub-case (d), $V^0=\dfrac{c_1}{\sqrt{R_{00}}}$, $V^1=V^1(t,\chi,\phi,z)$ and $V^3=V^3(t,\chi,\phi,z)$.

For sub-case (e), we get, $V^0=V^0(t,\chi,\phi,z)$, $V^1=V^1(\chi,z)$ and $V^3=V^3(\chi,z)$ with following constraint equations

$$R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}-\frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ V^1=0, \tag{135}$$

$$R_{11}\,V^1{}_{,3}+R_{33}\ V^3{}_{,1}=0, \tag{136}$$

$$R_{33,0}\ V^0+2R_{33}\ V^3{}_{,3}=0. \tag{137}$$

From equations (135) and (136),

$$V^1 = f.F(z),\ V^3 = -F(z)_{,3}\int\frac{R_{11}f}{R_{33}}d\chi \text{ and } V^0 = -\frac{2R_{33}}{R_{33,0}}F(z)_{,33}\int\frac{R_{11}f}{R_{33}}d\chi.$$

where $F(z) = c_1\cos\left(\sqrt{\lambda}\,z\right) + c_2\sin\left(\sqrt{\lambda}\,z\right)$.

For sub-case (f), we have constraint equations

$$R_{00,0}\ V^0 + 2R_{00}\ V^0{}_{,0} = 0, \tag{138}$$

$$R_{00}\ V^0{}_{,1} + R_{11}\ V^1{}_{,0} = 0, \tag{139}$$

$$R_{00}\ V^0{}_{,2} = 0, \tag{140}$$

$$R_{00}\ V^0{}_{,3} = 0, \tag{141}$$

$$R_{11,0}\ V^0 + R_{11,1}\ V^1 + 2R_{11}\ V^1{}_{,1} = 0, \tag{142}$$

$$R_{11}\ V^1{}_{,2} = 0, \tag{143}$$

$$R_{11}\ V^1{}_{,3} = 0, \tag{144}$$

$$R_{11,0}\ V^0 + \frac{\left(f^2R_{11}\right)_{,1}}{f^2}\ V^1 = 0. \tag{145}$$

Using (140), (141), (143) and (144), we have

$$V^0 = h(t,\chi),\ V^1 = g(t,\chi) \text{ and } V^3 = V^3(t,\chi,\phi,z).$$

Now, equation (138) gives, $V^0 = \dfrac{A_1(\chi)}{\sqrt{R_{00}}}$. (146)

Using (146) and (139), we have

$$V^1 = -A_{1,1}\int\frac{\sqrt{R_{00}}}{R_{11}}dt + A_2(\chi). \tag{147}$$

On substituting (146) and (147) in (142) and (145), we get following constraint equations

$$R_{11,0}\ A_1 = 0, \tag{148}$$

$$R_{11,1}\ A_2\sqrt{R_{00}} = 0, \tag{149}$$

$$\frac{\left(f^2{}_{,1}R_{11}\right)}{f^2}\sqrt{R_{00}}\,A_2=0, \tag{150}$$

$$R_{11,1}A_{1,1}=0, \tag{151}$$

$$R_{11}A_{1,11}=0, \tag{152}$$

$$R_{11,0}\,A_1=0, \tag{153}$$

$$R_{11}A_{2,1}=0, \tag{154}$$

$$R_{11,1}A_2=0. \tag{155}$$

Using (155), we have following sub-cases,

i) $R_{11,1}\neq 0,\ A_2=0$, $R_{11,0}=0,$

ii) $R_{11,1}=0,\ A_2\neq 0,\ R_{11,0}=0.$

In first sub-case, we have $A_1=c_1\neq 0,\ A_2=0$ which is contradiction to $V^1\neq 0$ so this sub-case is not possible and in second sub-case, $A_1=c_1\chi+c_2,\ A_2=0.$

For sub-case (g), we have

$$R_{00,0}\ V^0+2R_{00}\ V^0{}_{,0}=0, \tag{156}$$

$$R_{00}\,V^0{}_{,a}=0, \tag{157}$$

$$R_{00}\,V^0{}_{,3}+R_{33}\ V^3{}_{,0}=0, \tag{158}$$

$$R_{33}\ V^3{}_{,a}=0,\ \text{where } a=1,2, \tag{159}$$

$$R_{33,0}\ V^0+2R_{33}\ V^3{}_{,3}=0. \tag{160}$$

Now, differentiate equation (160), we get

$$V^3{}_{,0}=-\frac{2R_{33}}{R_{33,0}}\ V^3{}_{,33}, \tag{161}$$

and using (158) and (161) , we get

$$V^3{}_{,33}=\frac{R_{33,0}}{2R_{00}}\ V^3{}_{,0}\,. \tag{162}$$

Assume $V^3 = A(t)B(z)$ then equation (162) gives

$$\frac{R_{33,0}}{2R_{00}} \quad \frac{A_{,0}}{A} = \frac{B_{,33}}{B} = \lambda. \tag{163}$$

For $\lambda = 0,\ V^3 = c_1 z + c_2$ and $V^0 = -\frac{2R_{33}}{R_{33,0}} c_1.$

For $\lambda < 0,\ V^3 = \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\left[c_1 \cos\left(\sqrt{\lambda}\, z\right) + c_2 \sin\left(\sqrt{\lambda}\, z\right)\right]$ and

$$V^0 = -\frac{2R_{33}}{R_{33,0}} \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\sqrt{\lambda}\left[c_2 \cos\left(\sqrt{\lambda}\, z\right) - c_1 \sin\left(\sqrt{\lambda}\, z\right)\right].$$

For $\lambda > 0,\ V^3 = \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\left[c_1 e^{\sqrt{\lambda}\, z} + c_2 e^{-\sqrt{\lambda}\, z}\right]$ and

$$V^0 = -\frac{2R_{33}}{R_{33,0}} \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\sqrt{\lambda}\left[c_1 e^{\sqrt{\lambda}\, z} - c_2 e^{-\sqrt{\lambda}\, z}\right].$$

For sub-case (h), we have following constraint equations

$$R_{00,0}\ V^0 + 2R_{00}\ V^0{}_{,0} = 0, \tag{164}$$

$$R_{00}\ V^0{}_{,1} + R_{11}\ V^1{}_{,0} = 0, \tag{165}$$

$$R_{00}\ V^0{}_{,2} = 0, \tag{166}$$

$$R_{00}\ V^0{}_{,3} + R_{33}\ V^3{}_{,0} = 0, \tag{167}$$

$$R_{11,0}\ V^0 + R_{11,1}\ V^1 + 2R_{11}\ V^1{}_{,1} = 0, \tag{168}$$

$$R_{11}\ V^1{}_{,2} = 0, \tag{169}$$

$$R_{11}\ V^1{}_{,3} + R_{33}\ V^3{}_{,1} = 0, \tag{170}$$

$$R_{33}\ V^3{}_{,2} = 0, \tag{171}$$

$$R_{11,0}\ V^0 + \frac{\left(f^2 R_{11}\right)_{,1}}{f^2}\ V^1 = 0, \tag{172}$$

$$R_{33,0}\ \mathrm{V}^0+2R_{33}\ \mathrm{V}^3{}_{,3}=0. \tag{173}$$

Then we arrive at solution, $\mathrm{V}^3=c_1$.

Using equations (166),(167),(169) and (170), we have, $\mathrm{V}^0=g(t,\chi)$ and $\mathrm{V}^1=h(t,\chi)$.

Now, equation (164) gives $\mathrm{V}^0=\dfrac{A_1(\chi)}{\sqrt{R_{00}}}$. (174)

Using equations (168) and (172), $\mathrm{V}^1=A_2(t)\,f$. (175)

Substitute (174) and (175) in (165), we get

$$\frac{A_{1,1}}{R_{11}f}=-\frac{A_{2,0}}{\sqrt{R_{00}}}=\lambda. \tag{176}$$

If $\lambda=0$, then $A_1=c_2$ and $A_2=c_3$.

If $\lambda\neq0$, then $A_1=\lambda\int R_{11}f\,d\chi+c_2$ and $A_2=-\lambda\int\sqrt{R_{00}}\,dt+c_3$.

Case 13: $\mathbf{V}=\left(\mathrm{V}^0,0,\mathrm{V}^2,\mathrm{V}^3\right)$, where $\mathrm{V}^0\neq0$, $\mathrm{V}^2\neq0$, $\mathrm{V}^3\neq0$

Put $v^2=0$ in set of equations (14)-(23), leads to

$$R_{00,0}\ \mathrm{V}^0+2R_{00}\ \mathrm{V}^0{}_{,0}=0, \tag{177}$$

$$R_{00}\ \mathrm{V}^0{}_{,1}=0, \tag{178}$$

$$R_{00}\ \mathrm{V}^0{}_{,2}+\left(f^2R_{11}\right)\mathrm{V}^2{}_{,0}=0, \tag{179}$$

$$R_{00}\ \mathrm{V}^0{}_{,3}+R_{33}\ \mathrm{V}^3{}_{,0}=0, \tag{180}$$

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,1}=0, \tag{181}$$

$$R_{33}\ \mathrm{V}^3{}_{,1}=0, \tag{182}$$

$$\left(f^2R_{11}\right)\mathrm{V}^2{}_{,3}+R_{33}\ \mathrm{V}^3{}_{,2}=0, \tag{183}$$

$$2R_{11}\ \mathrm{V}^2{}_{,2}=0, \tag{184}$$

$$R_{33,0}\ \mathrm{V}^0+2R_{33}\ \mathrm{V}^3{}_{,3}=0. \tag{185}$$

We have the following sub-cases

$$\text{a) } R_{00}=0, \quad R_{11}=0,\ R_{33}=0,$$

$$\text{b) } R_{00}=0, \quad R_{11}=0,\ R_{33}\neq 0,$$

$$\text{c) } R_{00}=0, \quad R_{11}\neq 0,\ R_{33}=0,$$

$$\text{d) } R_{00}\neq 0, \quad R_{11}=0,\ R_{33}=0,$$

$$\text{e) } R_{00}=0, \quad R_{11}\neq 0,\ R_{33}\neq 0,$$

$$\text{f) } R_{00}\neq 0, \quad R_{11}\neq 0,\ R_{33}=0,$$

$$\text{g) } R_{00}\neq 0, \quad R_{11}=0,\ R_{33}\neq 0,$$

$$\text{h) } R_{00}\neq 0, \quad R_{11}\neq 0,\ R_{33}\neq 0.$$

Sub-case (a) which is similar to sub-case (a) of case 12.

For sub-case (b), $\mathrm{V}^3 = const.$, $\mathrm{V}^0 = \mathrm{V}^0(t,\chi,\phi,z)$ and $\mathrm{V}^2 = \mathrm{V}^2(t,\chi,\phi,z)$.

For sub-case (c), $\mathrm{V}^2 = const.$, $\mathrm{V}^0 = \mathrm{V}^0(t,\chi,\phi,z)$ and $\mathrm{V}^3 = \mathrm{V}^3(t,\chi,\phi,z)$.

For sub-case (d),

$$R_{00,0}\ \mathrm{V}^0 + 2R_{00}\ \mathrm{V}^0{}_{,0} = 0, \tag{186}$$

$$R_{00}\ \mathrm{V}^0{}_{,a} = 0\ , \quad a = 1,2,3. \tag{187}$$

From equation (186), $\mathrm{V}^0 = \dfrac{c}{\sqrt{R_{00}}}$, $\mathrm{V}^2 = \mathrm{V}^2(t,\chi,\phi,z)$ and $\mathrm{V}^3 = \mathrm{V}^3(t,\chi,\phi,z)$.

For sub-case (e),we have $\mathrm{V}^2 = c_1$, $\mathrm{V}^3 = c_2$ and
$\mathrm{V}^0 = \mathrm{V}^0(t,\chi,\phi,z)$ with constriant condition $R_{33,0} = 0$.

For sub-case (f),we get

$$R_{00,0}\ \mathrm{V}^0 + 2R_{00}\ \mathrm{V}^0{}_{,0} = 0, \tag{188}$$

$$R_{00}\ \mathrm{V}^0{}_{,2} + \left(f^2 R_{11}\right)\mathrm{V}^2{}_{,0} = 0, \tag{189}$$

$$R_{00}\ \mathrm{V}^0{}_{,a} = 0, \tag{190}$$

$$\left(f^2 R_{11}\right) V^2{}_{,a} = 0, \quad \text{where } a = 1,3, \tag{191}$$

$$R_{11}\ V^2{}_{,2} = 0. \tag{192}$$

Using (190) and (191), we get, $V^0 = g(t,\phi)$, $V^2 = h(t)$ and $V^3 = V^3(t,\chi,\phi,z)$.

From (188) and (189), $V^0 = \dfrac{c_1}{\sqrt{R_{00}}}$ and $V^2 = c_2$.

For sub-case (g), we obtain

$$R_{33,0}\ V^0 + 2R_{33}\ V^3{}_{,3} = 0, \tag{193}$$

$$R_{00}\ V^0{}_{,3} + R_{33}\ V^3{}_{,0} = 0, \tag{194}$$

$$R_{00,0}\ V^0 + 2R_{00}\ V^0{}_{,0} = 0, \tag{195}$$

$$R_{00}\ V^0{}_{,a} = 0, \tag{196}$$

$$R_{33}\ V^3{}_{,a} = 0, \quad \text{where } a = 1,2,. \tag{197}$$

Equation (196) and (197) gives $V^0 = h(t,z)$, $V^3 = g(t,z)$ and $V^2 = V^2(t,\chi,\phi,z)$.

Now, using (193) and (194), we have, $\dfrac{R_{33,0}}{2R_{00}}\ V^3{}_{,0} = V^3{}_{,33}$. (198)

We assume, $V^3 = A(t)B(z)$.

Then equation (198) gives

$$\frac{R_{33,0}}{2R_{00}}\ \frac{A_{,0}}{A} = \frac{B_{,33}}{B} = \lambda. \tag{199}$$

For $\lambda = 0$, $V^3 = c_1 z + c_2$ and $V^0 = -\dfrac{2R_{33}}{R_{33,0}} c_1$.

For $\lambda < 0$, $V^3 = \exp\left\{\int \dfrac{2\lambda R_{00}}{R_{33,0}} dt\right\}\left[c_1 \cos\left(\sqrt{\lambda}\, z\right) + c_2 \sin\left(\sqrt{\lambda}\, z\right)\right]$ and

$$V^0 = -\frac{2R_{33}}{R_{33,0}} \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\sqrt{\lambda}\left[c_2 \cos\left(\sqrt{\lambda}\, z\right) - c_1 \sin\left(\sqrt{\lambda}\, z\right)\right].$$

For $\lambda > 0$, $V^3 = \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\left[c_1 e^{-\sqrt{\lambda}\, z} + c_2 e^{\sqrt{\lambda}\, z}\right]$ and

$$V^0 = -\frac{2R_{33}}{R_{33,0}} \exp\left\{\int \frac{2\lambda R_{00}}{R_{33,0}} dt\right\}\left[c_2 e^{\sqrt{\lambda}\, z} - c_1 e^{-\sqrt{\lambda}\, z}\right].$$

For sub-case (h), we have

$$R_{00,0}\ V^0 + 2R_{00}\ V^0{}_{,0} = 0, \tag{200}$$

$$R_{00}\ V^0{}_{,2} + \left(f^2 R_{11}\right) V^2{}_{,0} = 0, \tag{201}$$

$$R_{00}\ V^0{}_{,3} + R_{33}\ V^3{}_{,0} = 0, \tag{202}$$

$$\left(f^2 R_{11}\right) V^2{}_{,3} + R_{33}\ V^3{}_{,2} = 0, \tag{203}$$

$$R_{33,0}\ V^0 + 2R_{33}\ V^3{}_{,3} = 0, \tag{204}$$

$$R_{00}\ V^0{}_{,1} = 0, \tag{205}$$

$$R_{11}\ V^2{}_{,2} = 0, \tag{206}$$

$$\left(f^2 R_{11}\right) V^2{}_{,1} = 0, \tag{207}$$

$$R_{33}\ V^3{}_{,1} = 0. \tag{208}$$

We have solutions as

$$V^2 = c_1, \tag{209}$$

$$V^0 = \frac{G(z)}{\sqrt{R_{00}}}, \tag{210}$$

$$V^3 = -G(z)_{,3} \int \frac{\sqrt{R_{00}}}{R_{33}} dt + c_2. \tag{211}$$

Now, using equations (204),(210) and (211), we have

$$\frac{R_{33,0}}{2R_{33}\sqrt{R_{00}}} \frac{1}{F(t)} = \frac{G(z)_{,33}}{G(z)} = \lambda, \tag{212}$$

where $F(t)=\int \frac{\sqrt{R_{00}}}{R_{33}}dt.$

If $\lambda=0,\ G(z)=c_3z+c_4.$

If $\lambda>0,\ G(z)=c_3e^{-\sqrt{\lambda}z}+c_4e^{\sqrt{\lambda}z}.$

If $\lambda<0,\ G(z)=c_3\cos\left(\sqrt{\lambda}z\right)+c_4\sin\left(\sqrt{\lambda}z\right).$

Case -14: $\mathbf{V}=\left(0,V^1,V^2,V^3\right)$, where $V^1\neq 0$, $V^2\neq 0$, $V^3\neq 0$

Put $V^0=0$ in equations (14)-(23), we get

$$R_{11}\ V^0{}_{,1}=0, \tag{213}$$

$$\left(f^2R_{11}\right)V^2{}_{,0}=0, \tag{214}$$

$$R_{33}\ V^3{}_{,0}=0, \tag{215}$$

$$R_{33}\ V^3{}_{,3}=0, \tag{216}$$

$$R_{11,1}\ V^1+2R_{11}\ V^1{}_{,1}=0, \tag{217}$$

$$R_{11}\ V^1{}_{,2}+\left(f^2R_{11}\right)V^2{}_{,1}=0, \tag{218}$$

$$R_{11}\ V^1{}_{,3}+R_{33}\ V^3{}_{,1}=0, \tag{219}$$

$$\left(f^2R_{11}\right)V^2{}_{,3}+R_{33}\ V^3{}_{,2}=0, \tag{220}$$

$$\frac{\left(f^2R_{11}\right)_{,1}}{f^2}V^1+2R_{11}\ V^2{}_{,2}=0. \tag{221}$$

Further using set of equations (213)-(221), we have following sub-cases

a) $R_{11}=0,\quad R_{33}=0,$

b) $R_{11}=0,\quad R_{33}\neq 0,$

c) $R_{11}\neq 0,\quad R_{33}=0,$

d) $R_{11}\neq 0,\quad R_{33}\neq 0.$

For sub-case (a), all constraint equations are satisfied identically.

For sub-case (b), $V^1 = V^1(t,\chi,\phi,z)$ $V^2 = V^2(t,\chi,\phi,z)$ and $V^3 = const.$

For sub-case (c), constraint equations are

$$R_{11}\ V^1{}_{,0} = 0, \tag{222}$$

$$\left(f^2 R_{11}\right) V^2{}_{,0} = 0, \tag{223}$$

$$R_{11,1}\ V^1 + 2R_{11}\ V^1{}_{,1} = 0, \tag{224}$$

$$R_{11}\ V^1{}_{,2} + \left(f^2 R_{11}\right) V^2{}_{,1} = 0, \tag{225}$$

$$R_{11}\ V^1{}_{,3} = 0, \tag{226}$$

$$\left(f^2 R_{11}\right) V^2{}_{,3} = 0, \tag{227}$$

$$\frac{\left(f^2 R_{11}\right)_{,1}}{f^2} V^1 + 2R_{11}\ V^2{}_{,2} = 0. \tag{228}$$

Equations (222), (223), (226) and (227) gives

$V^1 = g(\chi,\phi),\ V^2 = h(\chi,\phi)$ and $V^3 = V^3(t,\chi,\phi,z).$

Now, differentiate equation (228) with respect to ϕ and using (225), we get

$$\frac{\left(f^2 R_{11}\right)_{,1}}{2R_{11}} V^2{}_{,1} = V^2{}_{,22}\ . \tag{229}$$

Let, $V^2 = A(\chi)B(\phi)$ then equation (229) yields,

$$\frac{\left(f^2 R_{11}\right)_{,1}}{2R_{11}}\ \frac{A_{,1}}{A} = \frac{B_{,22}}{B} = \lambda. \tag{230}$$

For $\lambda = 0,\ V^2 = c_1\phi + c_2$ and $V^1 = -\dfrac{2\left(f^2 R_{11}\right)}{\left(f^2 R_{11}\right)_{,1}} c_1.$

For $\lambda < 0,\ V^2 = \exp\left\{\int \dfrac{2\lambda R_{11}}{\left(f^2 R_{11}\right)_{,1}} d\chi\right\}\left[c_1 \cos\left(\sqrt{\lambda}\,\phi\right) + c_2 \sin\left(\sqrt{\lambda}\,\phi\right)\right]$

$$V^1 = -\frac{2\left(f^2R_{11}\right)}{\left(f^2R_{11}\right)_{,1}} \exp\left\{\int \frac{2\lambda R_{11}}{\left(f^2R_{11}\right)_{,1}} d\chi\right\} \sqrt{\lambda}\left[c_2\cos(\sqrt{\lambda}\phi) - c_1\sin(\sqrt{\lambda}\phi)\right].$$

For $\lambda > 0$, $V^2 = \exp\left\{\int \frac{2\lambda R_{11}}{\left(f^2R_{11}\right)_{,1}} d\chi\right\}\left[c_1 e^{\sqrt{\lambda}\phi} + c_2 e^{-\sqrt{\lambda}\phi}\right]$

$$V^1 = -\frac{2\left(f^2R_{11}\right)}{\left(f^2R_{11}\right)_{,1}} \exp\left\{\int \frac{2\lambda R_{11}}{\left(f^2R_{11}\right)_{,1}} d\chi\right\} \sqrt{\lambda}\left[c_1 e^{\sqrt{\lambda}\phi} - c_2 e^{-\sqrt{\lambda}\phi}\right].$$

For sub-case (d), we arrive at the solutions

$$V^3 = c_1,\ V^1 = \frac{G(\phi)}{\sqrt{R_{11}}} \text{ and } V^2 = -G(\phi)_{,3}\int \frac{1}{f^2\sqrt{R_{11}}} d\chi + c_2.$$

Using (228), we get

$$\frac{\left(f^2R_{11}\right)_{,1}}{2\left(f^2R_{11}\right)} \frac{1}{F(\chi)\sqrt{R_{11}}} = \frac{G(\phi)_{,22}}{G(\phi)} = \lambda. \tag{231}$$

when $\lambda = 0$ then $G(\phi) = c_3\phi + c_4$.

when $\lambda > 0$ then $G(\phi) = c_3 e^{\sqrt{\lambda}\phi} + c_4 e^{-\sqrt{\lambda}\phi}$.

when $\lambda < 0$ then $G(\phi) = c_3\cos\left(\sqrt{\lambda}\,\phi\right) + c_4\sin\left(\sqrt{\lambda}\,\phi\right)$.

3. Discussion

In this classification of axially symmetric space-time according Ricci symmetries, we find ten Ricci symmetries (collineation) equations.We have solved these equations for degenerate case and non-degenerate cases.

In case 1, on solving six Ricci collineation equations (using equations (14)-(23)) we have obtained one RC for non-degenerate Ricci tensor and in case of degenerate every direction is the Ricci collineation.

In cases 2, 3 and 4 we obtained the similar result as in case 1.

In case 5, we have nine Ricci collineation equations and four sub-cases. The sub-cases (a), (b) and (c) produce degenerate case where we got infinite number of RCs and sub-case (d), gives non-degenerate case and in this case we have found two RCs .

In cases 6,7,8,9 and 10 we have nine, eight, seven, eight and seven Ricci collineation equations respectively. Applying different constraint conditions, we have four sub-cases for cases 6, 7, 9 and 10 and two sub-cases for case 8 which gives the same result as in case 5.

In case 11, we have ten RC equations and four sub-cases (a), (b), (c) and (d) using constraint conditions on Ricci tensor R_{00} and R_{11}. Out of these four sub-cases, sub-cases (a), (b) and (c) gives degenerate case where we have infinite number of RCs and sub-case (d) obtain non-degenerate case in which we have two RCs.

In case 12, we have ten RC equations and eight sub-cases (a) to (h) depending constraint conditions on R_{00}, R_{11} and R_{33}. Out of these eight sub-cases, sub-cases (a) to (g) arise degenerate case givinginfinite number of RCs except sub-case (e). In this sub-case (e), we have finite number of RCs *i.e.* twoand sub-case (h) gives non-degenerate case in which we have three RCs.

In case 13, we have nine RC equations and eight sub-cases depending on constraint conditions. Thus sub-cases (a) to (g) lead to degenerate case having infinite number of RCs and sub-case (h) gives four RCs for non-degenerate case.

In case 14, we have nine RC equations and after applying constraint conditions on R_{11} and R_{33} we have four sub-cases (a) to (d). Out of these four sub-cases, sub-cases (a) to (c) gives degenerate case where we have infinite number of RCs and sub-case (d) gives non-degenerate case in which we have four RCs.

4. Conclusion

In this paper, axially symmetric space-time is classified according to their Ricci symmetries or collineations (RC). The Ricci collineations equations have been solved by considering one non-zero component in cases 1 to 4, two non-zero components in cases 5 to 10 and three components are non-zero in cases 11 to 14 respectively.

When there is one non-zero component of **V** we obtained one RC for non-degenerate Ricci tensor. In case of degenerate Ricci tensor every direction is RC.In case of two non-zero components of RC vector **V** we have two RCs for non-degenerate Ricci tensor while for degenerate case we have infinite RCs.In case three non-zero components of **V** we have two, three and four RCs when Ricci tensor is non-degenerate and in degenerate case, we have finite RCs in some sub-cases and infinite RCs in other sub-cases. It is found that if the Ricci tensor R_{ab} is regarded as a four dimensional space-time defined on M then finite dimensional Ricci symmetries associated with a Lie algebra of smooth vector field **V** is less than or equal to ten but not nine.

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