Article

On an Integral Involving the Hypergeometric Function $_3F_2$

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Abstract

The aim of this note is to evaluate an interesting integral involving generalized hypergeometric function $_3F_2$ by employing the extension of Saalschutz summation theorem obtained recently by Rakha and Rathie. A known integral available in the literature has been deduced as special case of our main findings.

Keywords: Generalized hypergeometric function, Saalschutz summation theorem and its extension

1. Introduction

We start with the following integral recorded in [1, p. 398, eq. (1)]:

$$\int_0^1 x^{\rho-1}\,(1-x)^{\beta-\gamma-n}\ \ {}_2F_1\left[\begin{matrix}-n,\ \beta\\ \gamma\end{matrix};\,x\right]dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\rho)\,\Gamma(\beta-\gamma+\rho+1)}\ , \tag{1}$$

provided $Re(\rho) > 0$, $Re(\beta-\gamma) > n-1$, $n = 0, 1, 2, \ldots$

This result can be established with the help of the following classical Saalschutz's summation theorem [2]:

$$ {}_3F_2\left[\begin{matrix}-n,\ a,\ b\\ c,\ 1+a+b-c-n\end{matrix};\,1\right] = \frac{(c-a)_n\ \ (c-b)_n}{(c)_n\ \ (c-a-b)_n}\ . \tag{2}$$

It is interesting to mention here that (2) reduces to the following classical Gauss's summation theorem:

$$ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\ 1\right] = \frac{\Gamma(c)\quad\Gamma(c-a-b)}{\Gamma(c-a)\quad\Gamma(c-b)} \tag{3}$$

provided $Re\left(c-a-b\right) > 0$, by letting $c \to \infty$.

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Recently, an interesting extension of the classical Saalschutz summation theorem (1) was given by Rakha and Rathie [3] in the following form:

$$ {}_4F_3\left[\begin{matrix}-n, a, & b, & d+1\\ c+1, & 1+a+b-c-n\end{matrix}\ ;\ 1\right] = \frac{(c-a)_n\ (c-b)_n}{(c)_n\ (c-a-b)_n}\ \frac{(1+g)_n}{(g)_n}\quad , \tag{4}$$

where $g = \frac{f\,(b-c)}{b-f}$ and $f = \frac{d\,(a-c)}{a-d}$. Clearly (4) reduces to (2) by taking $d = c$.

The aim of this note is to evaluate an interesting integral involving generalized hypergeometric function $_3F_2$ by employing the extension of Saalschutz summation theorem (4). The known integral (1) available in the literature has been obtained as special case of our main findings.

2. Main result

The interesting integral involving generalized hypergeometric function to be evaluated in this note is given in the following theorem:

Theorem: For $Re(\rho) > 0,\ Re(\beta-\gamma) > n-1;\ n = 0, 1, 2, \ldots and\ d \neq 0, -1, -2, \ldots$, the following result holds true:

$$\int_0^1 x^{\rho-1}\ (1-x)^{\beta-\gamma-n}\ {}_3F_2\left[\begin{matrix}-n, & \beta, & d+1\\ \gamma+1, & d\end{matrix}\ ;\ x\right] dx$$

$$= \frac{\Gamma(\rho)\,\Gamma(\beta-\gamma+1)}{\Gamma(\rho+\beta-\gamma+1)}\ \frac{(\gamma-\rho)_n}{(\gamma+1)_n}\ \frac{(1+g)_n}{(g)_n}\ , \tag{5}$$

where $g = \frac{f\,(\beta-\gamma)}{\beta-f}$ and $f = \frac{d\,(\rho-\gamma)}{\rho-d}$.

Proof: In order to establish our main result (5), we proceed as follows. Denoting the left-hand side of (5) by I, expressing the generalized hypergeometric function $_3F_2$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series), we have:

$$\mathrm{I} = \sum_{r=0}^{\infty} \frac{(-n)_r\ (\beta)_r\ (d+1)_r}{(\gamma+1)_r\ (d)_r\ r!} \int_0^1 x^{\rho+r-1}\ (1-x)^{\beta-\gamma-n}\ dx,$$

evaluating the beta integral and using the result, we have, after some simplification:

$$\mathrm{I} = \frac{\Gamma(\rho)\,\Gamma(\beta-\gamma-n+1)}{\Gamma(\rho+\beta-\gamma+1-n))}\ \sum_{r=0}^{\infty} \frac{(-n)_r\ (\rho)_r\ (\beta)_r\ (d+1)_r}{(\gamma+1)_n\ (1+\rho+\beta-\gamma-n)_n\ (d)_r\ r!}\ ,$$

finally, summing up the series, we get:

$$I = \frac{\Gamma(\rho)\,\Gamma(\beta-\gamma-n+1)}{\Gamma(\rho+\beta-\gamma+1-n))} \quad {}_4F_3\left[\begin{matrix} -n, \ \rho, \ \beta, \ d+1 \\ \gamma+1, \ 1+\rho+\beta-\gamma-n \end{matrix} ; \ 1\right].$$

We now observe that the $_4F_3$ can be evaluated with the help of the result (4), and then using the elementary identity $\Gamma(\alpha-n) = (-1)^n \frac{\Gamma(\alpha)}{(1-\alpha)_n}$ and after some simplification, we arrive at the right-hand side of (5). This completes the proof of (5).

A special case:

In (5), if we take $d = \gamma \implies f = \gamma \ \ and\, so \ \ g = \gamma$, we immediately recover the known integral (1). Thus the integral (5) can be regarded as a natural extension of the known integral (1).

References

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