Article

On the Linear Differential Equation of Second Order

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Abstract

We consider the equation $p(x)y'' + q(x)y' + r(x)y = \phi(x)$ for $p'' - q' + r = 0$, where it is possible to obtain the solutions y_1 and y_2 of the corresponding homogeneous equation and a particular solution y_p for the original equation, and also for $p'' - q' + r \neq 0$, where we must know y_1 to construct y_2 and y_p via two integrations of certain differential relation.

Keywords: 2th order, linear differential equation, variation of parameters.

1. Introduction

Here we study the general solution of second order linear differential equation:

$$p(x)y'' + q(x)y' + r(x)y = \phi(x), \tag{1}$$

via an alternative (but equivalent) method to the variation of parameters technique of Newton (Principia)-Bernoulli-Euler-Lagrange [1]. It is convenient to consider two cases:

a). $p'' - q' + r = 0.$

In Sec. 2 we exhibit that the differential expression [2]:

$$\frac{d}{dx}\left[p^2\, W\, \frac{d}{dx}\left(\frac{y}{pW}\right)\right] = \phi\,, \qquad W(x) = \exp\left(-\int^x \frac{q(\xi)}{p(\xi)}\, d\xi\right), \tag{2}$$

gives the complete solution of (1).

b). $p'' - q' + r \neq 0.$

The Sec. 3 shows that two integrations of:

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$$\frac{d}{dx}\left[\frac{{y_1}^2}{W}\frac{d}{dx}\left(\frac{y}{y_1}\right)\right]=\frac{y_1\phi}{pW}, \qquad p(x){y_1}''+q(x){y_1}'+r(x)y_1=0, \tag{3}$$

allows to construct the general solution of (1).

2. Case $p''-q'+r=0$

In this situation first we calculate the wronskian W, and after two successive integrations of (2) we obtain the complete solution of (1):

$$y(x)=c_1\,y_1(x)+c_2\,y_2(x)+y_p(x), \tag{4}$$

where

$$y_1=Wp, \quad y_2=y_1\int^x\frac{d\xi}{Wp^2}=y_1\int^x\frac{W}{y_1^2}d\xi, \quad p(x){y_2}''+q(x){y_2}'+r(x)y_2=0,$$

$$y_p=y_1\int^x\frac{d\xi}{Wp^2}\int^\xi\phi(\eta)\,d\eta=y_2\int^x\phi(\xi)\,d\xi-y_1\int^x\frac{y_2\phi}{y_1}d\xi, \tag{5}$$

in harmony with the variation of parameters method [1, 3].

3. Case $p''-q'+r\neq 0$

Here we need one solution of the homogeneous equation associated to (1), then two integrations of (3) give the general solution (4) such that:

$$y_2(x)=y_1(x)\int^x\frac{W(\xi)}{[y_1(\xi)]^2}d\xi, \quad y_p(x)=y_2(x)\int^x\frac{y_1(\xi)\phi(\xi)}{p(\xi)W(\xi)}d\xi-y_1(x)\int^x\frac{y_2(\xi)\phi(\xi)}{p(\xi)W(\xi)}d\xi, \tag{6}$$

and (6) implies (5) when $y_1=Wp$. The integration of (3) justifies the traditional ansatz [1, 3] employed in the variation of parameters technique. It is easy to apply our approach to differential equations of third and fourth order [4].

The fundamental differential relation (3) can be deduced via the self-adjoint and exact operators concepts [3, 5, 6] applied to (1) (thus it is not necessary the Lagrange's ansatz), with the important participation of the expression (2) of Abel-Liouville-Ostrogradski [7] for the wronskian $W=y_1\,y_2'-y_2\,y_1'$.

References

1. Z. Ahsan, *Differential equations and their applications*, Prentice-Hall, New Delhi (2004)
2. J. Clegg, *A new factorization of a general second order differential equation,* Int. J. Math. Educ. Sci. Techn. **37** (1) (2006) 51-64
3. A. Hernández-Galeana, J. López-Bonilla, R. López-Vázquez, *On the second order linear differential equation*, Pure and Applied Maths. Lett. **2** (2014) 31-34
4. Z. Ahsan, R. Cruz-Santiago, J. López-Bonilla, *Linear differential equations of third and fourth order*, Aligarh Bull. Maths. **13** (1) (2012) 5-7
5. C. Lanczos, *Linear differential operators*, Dover, New York (1997)
6. A. Hernández-Galeana, J. López-Bonilla, J. Rivera-Rebolledo, *Second order linear differential equation in its exact form*, The SciTech, J. of Science & Technology **2** (1) (2013) 34-36
7. L. Elsgotz, *Differential equations and variational calculus*, MIR, Moscow (1983)