Article

Integrals Involving Generalized Hypergeometric Function $_4F_3$

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Abstract

The aim of this note is to evaluate two interesting integrals involving generalized hypergeometric function $_4F_3$ by employing the extension of Gauss summation theorem available in the literature. Two known integrals have been obtained as special cases of our main findings.

Keyword: Generalized hypergeometric function, Gauss summation theorem and its extension

1. Introduction

We start with the following integrals recorded in [2, p. 71, eqs. (3.1.8) and (3.1.9)]:

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta} \ (\sin\theta)^{\alpha-1} \ (\cos\ \theta)^{\beta-1} \ {}_2F_1\left[\begin{matrix} a, \ b \\ \beta \end{matrix}; \ e^{i\theta}\cos\theta\right] d\theta =$$

$$= \quad e^{\frac{i\pi\alpha}{2}} \ \frac{\Gamma(\alpha)\,\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\,\Gamma(\alpha+\beta-b)}, \tag{1}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta} \ (\sin\theta)^{\alpha-1} \ (\cos\ \theta)^{\beta-1} \ {}_2F_1\left[\begin{matrix} a, \ b \\ \alpha \end{matrix}; \ e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta =$$

$$= \quad e^{\frac{i\pi\alpha}{2}} \ \frac{\Gamma(\alpha)\,\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\,\Gamma(\alpha+\beta-b)}, \tag{2}$$

provided $Re(\alpha) > 0$, $Re(\beta) > 0$ and $Re\,(\alpha+\beta-a-b) > 0$.

It is interesting to mention here that the above two integrals can be established with the help of the following integral due to MacRobert [1, eq. (2), p. 450]:

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta} \ (\sin\theta)^{\alpha-1} \ (\cos\ \theta)^{\beta-1} \ d\theta \quad = \quad e^{\frac{i\pi\alpha}{2}} \ \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{3}$$

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with $Re(\alpha) > 0$ and $Re(\beta) > 0$, and by using classical Gauss summation theorem [4]:

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta \\ \gamma\end{matrix};\ 1\right] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)} \quad , \tag{4} $$

provided $Re(\gamma - \alpha - \beta) > 0$.

The following extension of classical Gauss summation theorem (2) is available in the literature [2]:

$$ {}_3F_2\left[\begin{matrix}a, & b, d+1 \\ c+1, & d\end{matrix};\ 1\right] = \frac{\Gamma(c+1)\ \Gamma(c-a-b)}{\Gamma(c-a+1)\ \Gamma(c-b+1)} \left\{(c-a-b) + \frac{a\,b}{d}\right\} , \tag{5} $$

with $Re(c-a-b) > 0$ and $d \neq 0, -1, -2, \ldots$

The aim of this short note is to evaluate two interesting integrals involving generalized hypergeometric function $_4F_3$ by employing extension of Gauss summation theorem (5). The integrals (1) and (2) have been obtained as special cases of our main findings.

2. Main results

The two integrals involving generalized hypergeometric function $_4F_3$ to be evaluated in this note are given in the following:

Theorem: For $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re\,(c-a-b\,) > 0$ and $d \neq 0, -1, , \ldots,$

the next results hold true:

$$ \int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}\ (\sin\theta)^{\alpha-1}\ (\cos\ \theta)^{\beta-1}\ {}_4F_3\left[\begin{matrix}a, & b, \alpha+\beta, d+1 \\ & \beta, c+1, d\end{matrix};\ e^{i\theta}\cos\theta\right] d\theta = $$

$$ = \quad e^{\frac{i\pi\alpha}{2}}\ \frac{\Gamma(\,\alpha\,)\,\Gamma(\,\beta\,)\,\Gamma(c+1\,)\ \Gamma(c-a-b\,)}{\Gamma(\,\alpha+\beta\,)\,\Gamma(c-a+1)\,)\,\Gamma(c-b+1)\,)} \left\{(c-a-b) + \frac{a\,b}{d}\right\}, \tag{6} $$

and

$$ \int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}\ (\sin\theta)^{\alpha-1}\ (\cos\ \theta)^{\beta-1}\ {}_4F_3\left[\begin{matrix}a, & b, \alpha+\beta, d+1 \\ & \alpha, c+1, d\end{matrix};\ e^{i(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta = $$

$$ = \quad e^{\frac{i\pi\alpha}{2}}\ \frac{\Gamma(\,\alpha\,)\,\Gamma(\,\beta\,)\,\Gamma(c+1\,)\ \Gamma(c-a-b\,)}{\Gamma(\,\alpha+\beta\,)\,\Gamma(c-a+1)\,)\,\Gamma(c-b+1)\,)} \left\{(c-a-b) + \frac{a\,b}{d}\right\}. \tag{7} $$

Proof: In order to establish our first result (6), we proceed as follows. Denoting the left-hand side of (6) by I, expressing the generalized hypergeometric function $_4F_3$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series), we have:

$$I = \sum_{n=0}^{\infty} \frac{(a)_n \ (b)_n \ (\alpha+\beta)_n \ (d+1)_n}{(\beta)_n \ (c+1)_n \ (d)_n \ n!} \int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta+n)\theta} \ (\sin\theta)^{\alpha-1} \ (\cos\theta)^{\beta+n-1} \ d\theta.$$

Evaluating the integral with the help of (3) and using the result $(\alpha)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we obtain, after some simplification:

$$\mathrm{I} = e^{\frac{i\pi\alpha}{2}} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(a)_n \ (b)_n \ (d+1)_n}{(c+1)_n \ (d)_n \ n!},$$

summing up the series, we have $\mathrm{I} = e^{\frac{i\pi\alpha}{2}} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \ {}_3F_2\left[\begin{matrix} a, \ b, d+1 \\ c+1, \ d \end{matrix} ; 1\right]$. Finally, using the result (5), we arrive at the right-hand side of (6). This completes the proof of (6). In exactly the same manner, the result (7) can also be established.

3. Special cases

In (6) and (7), if we take $d = c = \alpha + \beta$, we at once recover the results (1) and (2), respectively. Thus the results (6) and (7) can be regarded as natural extensions of (1) and (2).

References

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