Article

Kantowski-Sachs Cosmological Models with Variable Displacement Vector in Lyra Geometry

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Abstract

It is proposed to study the Kantwoski-Sachs metric based on Lyra geometry with time dependent displacement vector. To get a deterministic model in terms of cosmic time, it is assumed that the expansion scalar θ is proportional to the shear scalar σ. Further it is assumed that the barotropic perfect fluid satisfies the condition $p = \gamma\rho$, $0 \le \gamma \le 1$. The solution of the field equations for $\gamma = 1$ (i.e., stiff fluid distribution), $\gamma = 0$ (i.e., dust distribution), and $\gamma = \frac{1}{3}$ (i.e., disordered radiation) is presented. The properties of cosmological and physical parameters are discussed in detail.

Keywords: Barotropic, Lyra, displacement vector, Kantowski-Sachs metric, gravitation.

1. Introduction

Spatially homogeneous space-times either belong to Bianchi Types or Kantowski-Sachs models and interpreted as cosmological models in General Relativity (Raychaudhary[1]). FRW (Friedmann-Robertson-Walker) models are unstable near the singularity (Patridge and Wilkinson [2]) and fail to describe early universe. Therefore spatially homogeneous and anisotropic Bianchi models are undertaken to study the universe at its early stage of evolution. Einstein introduced his General Theory of Relativity in which gravitation is described in terms of geometry of space time and it motivated him to geometrize other physical fields. Weyl [3] made one of the best attempts in this direction. He introduced a generalization of Riemannian geometry in an attempt to unify gravitation and electromagnetism. Weyl's theory was not taken seriously because it was based on the non-integrability of length transfer. Later Lyra [4] suggested a modification of Riemannian geometry which has a close resemblance to Weyl's geometry. In Lyra's geometry, the connection is metric preserving as in Riemannian geometry and length transfer is integrable. Lyra introduced a gauge function which removed the non-integrability condition of the length of

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a vector under parallel transport. Thus Riemannian geometry was modified by Lyra and was given a new name called Lyra's geometry. Halford [5] in his study has shown that the constant displacement vector field 'ϕ_i' in Lyra geometry plays the role of cosmological constant'Λ' in General Relativity. Soleng [6] investigated cosmological models based on Lyra geometry and pointed out that the displacement field includes either a creation field and is equal to Hoyle-Narlikar creation field [7, 8] or contains a special vacuum field which with a gauge vector can be considered as a cosmological term.

It is also known that the Universe was initially dominated by radiation that was followed by a matter dominated phase. So in our view for a complete picture of the evolution with initial inflation and present day acceleration we should consider the model with the source field being a mixture of a perfect fluid and dark energy and realize it with one of the compounds being dominated at a definite period of evolution as follows: Vander Waals gas, radiation, dust, dark energy. This model seems to be more realistic and we plan to encounter it in near future. The cosmological models based on Lyra geometry have been investigated by number of authors viz. (Soleng[6], Chakravorty and Ghosh [9], Rahaman and Bera [10], Pradhan and Vishwakarma [11], Pradhanet al. [12, 13], Casana et al. [14], Singh [15], Kumar and Singh [16], Mohanty et al. [17], Bali and Chandnani [18, 19], Bhamra[20], Karade and Borikar[21], Kalyanshetti and Wagmode[22], Reddy and Innaiah[23], Beesham[24], Reddy and Venkateswarlu[25], and Rahaman et al. [26].).

In the present paper we consider kantwoski-sachs metric with zero-curvature in Lyra's manifold with perfect fluid. The solutions are obtained for two different early phases of universe viz. Inflationary phase and radiation-dominated phase by using 'gamma-law' equation of state $p=\gamma\rho$. The γ function varies continuously with cosmic time as the universe expands. The gauge function in Lyra's manifold is allowed to depend on cosmic time.

Einstein field equation based on Lyra's geometry in normal gauge may be written as

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{3}{2}\phi_\mu\phi_\nu-\frac{3}{4}g_{\mu\nu}\phi_k\phi^k=-T_{\mu\nu} \tag{1}$$

where ϕ_i is the displacement vector defined as

$$\phi_i=\left(0,0,0,\beta(t)\right) \tag{2}$$

and $T_{\mu\nu}$ represents the energy tensor for the fluid filling the space time which is represented by the perfect fluid

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu-p\,g_{\mu\nu} \tag{3}$$

where ρ and P are respectively the energy density and pressure of cosmic fluid. u^μ is the velocity vector satisfying $u^\mu u_\mu=1$.

2. Kantowski-Sachs Metric and Field equations

We consider the Kantowski-Sachs metric in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{4}$$

where A and B are functions of time only . In view of Equation (3) for the Kantwoski-Sachs space time Equation (4) the field equation (1) leads to

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} + \frac{3}{4}\beta^2 = -p \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -p \tag{6}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} - \frac{3}{4}\beta^2 = \rho \tag{7}$$

A over head dot indicates differentiation with respect to cosmic time t.

The energy conservation equation $T^{\mu}_{\mu;\nu} = 0$ leads to

$$\dot{\rho} + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)(\rho + p) = 0\,. \tag{8}$$

and

$$\left(R^{\nu}_{\mu} - \frac{1}{2}g^{\nu}_{\mu}R\right)_{;\nu} + \frac{3}{2}\left(\phi_{\mu}\phi^{\nu}\right)_{;\nu} - \frac{3}{4}\left(g^{\nu}_{\mu}\phi_k\phi^k\right)_{;\nu} = 0\,. \tag{9}$$

Equation (9) leads to

$$\frac{3}{2}\phi_{\mu}\left[\frac{\partial\phi^{\mu}}{\partial x^{\mu}} + \phi^{l}\Gamma^{\nu}_{l\nu}\right] + \frac{3}{2}\phi^{\nu}\left[\frac{\partial\phi_{\mu}}{\partial x^{\nu}} - \phi_l\Gamma^{l}_{\mu\nu}\right] - \frac{3}{4}g^{\nu}_{\mu}\phi_k\left[\frac{\partial\phi^{k}}{\partial x^{\nu}} + \phi^{l}\,\Gamma^{k}_{l\nu}\right]$$
$$-\frac{3}{4}g^{\nu}_{\mu}\phi_k\left[\frac{\partial\phi_k}{\partial x^{\nu}} + \phi_l\Gamma^{l}_{l\nu}\right] = 0\,. \tag{10}$$

Equation (10) is identically satisfied for $i = 1,2,3$ but for $i = 4$, it is reduced to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = 0\,. \tag{11}$$

3. Exact Solutions of the field equations

The field equations (5) – (7) are a system of three equations with five unknown parameters A, B, β , ρ and p . We need two additional conditions to get a deterministic solution of the above system of equations.

Firstly, we consider the case when the source field is given by a perfect fluid. Here the energy density ρ and the pressure p of the perfect fluid are connected by a linear equation of state of the form

$$p = \gamma\rho \tag{12}$$

where γ $(0 \le \gamma \le 1)$ is constant.

Now subtracting equation (7) from equation (5) and using equation (12), we get

$$\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B} - \frac{3}{4}\beta^2 = \rho\left(\frac{1+\gamma}{2}\right) \tag{13}$$

subtracting equation (13) from equation (6),we obtain

$$\frac{\ddot{A}}{A} + \frac{2\dot{A}\dot{B}}{AB} = \rho\left(\frac{1-\gamma}{2}\right) \tag{14}$$

On account of equations (8) and (12), equation (14) can be solved to get

$$\rho = \frac{c_1}{\left(AB^2\right)^{1+\gamma}} \tag{15}$$

where c_1 is an arbitrary constant.

Now substituting equation (15) in equation (14)

$$\left(AB^2\right)^{\gamma}\left(\dot{A}B^2\right)^{\bullet} = c_1\left(\frac{1-\gamma}{2}\right) \tag{16}$$

Secondly, we assume for a spatially homogeneous KS metric, the normal congruence to homogeneous expansion implies that $\frac{\sigma}{\theta}$ is constant, i.e., “the expansion scalar θ is proportional to shear scalar σ ”. This leads to

$$A = B^m, m \ne 1 \, . \tag{17}$$

Now substituting equation (17) into equation (16) yields

$$\frac{\ddot{B}}{B} + (1+m)\frac{\dot{B}^2}{B^2} = \frac{c_1(1-\gamma)}{2m\, B^{(m+2)(\gamma+1)}} \, . \tag{18}$$

which on solving , admits the solution

$$B(t) = \left(\sqrt{\frac{c_1(m+2)}{4m}}\,(\gamma+1)\,t + c_2\right)^{\frac{2}{(m+2)(\gamma+1)}} \tag{19}$$

and consequently we get

$$A(t) = \left(\sqrt{\frac{c_1(m+2)}{4m}}\,(\gamma+1)\,t + c_2 \right)^{\frac{2m}{(m+2)(\gamma+1)}}. \tag{20}$$

Thus the space-time given by equation can be expressed as

$$\begin{aligned} ds^2 = dt^2 - &\left(\sqrt{\frac{c_1(m+2)}{4m}}\,(\gamma+1)\,t + c_2 \right)^{\frac{4m}{(m+2)(\gamma+1)}} dr^2 - \\ &- \left(\sqrt{\frac{c_1(m+2)}{4m}}\,(\gamma+1)\,t + c_2 \right)^{\frac{4}{(m+2)(\gamma+1)}} \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \end{aligned} \tag{21}$$

The expressions for the proper energy density (ρ), the isotropic pressure (p), and the displacement vector *(β)* for the above model are obtained as

$$\rho = \frac{4mc_1}{\left(c_1(m+2)(\gamma+1)t + c_2\right)} \tag{22}$$

$$p = \frac{4m\gamma c_1}{\left(c_1(m+2)(\gamma+1)t + c_2\right)}$$

$$\beta = \frac{c_3}{\left(\sqrt{c_1(m+2)}\,(\gamma+1)\,t + 2\sqrt{m}\,c_2\right)}. \tag{23}$$

From equations (22) and (23), it is observed that the isotropic pressure, density and the displacement vector are decreasing function of time and they approaches to small positive value at late time.

It is observed that the displacement vectors $\beta(t)$ in both cases coincide with the nature of the cosmological constant Λ. In recent time Λ-term has attracted theoreticians and observers for many reasons. The nontrivial role of the vacuum in the early universe generates a Λ-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter; the density parameter and the age of the universe (for example, see Refs. [27] and [28]). In recent past there has been an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [29–32]. Therefore the study of cosmological models based on Lyra geometry may be relevant for inflationary models. There seems to be a good possibility of Lyra's geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology.

The behaviour of the displacement vector 'β' is clearly shown in Figure-1 for dust filled universe, stiff universe and radiation universe with respect to cosmic time 't' for a specific value of *m*.The displacement vector $\beta(t)$ is large at beginning of Universe and reduces fast during its evolution analogous to cosmological constant Λ. Figure-2 depicts the variation in density parameter versus time during the evolution of dust filled universe, stiff universe and radiation universe. The density parameter approaches the same values at late time.

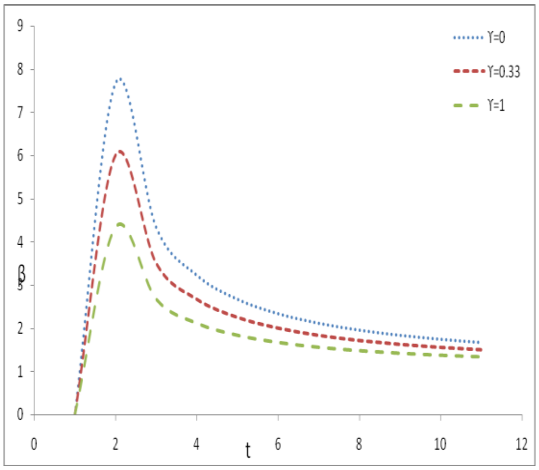

Fig. 1 The plot of displacement vector(β) vs. cosmic time(t)with $m=0.25$

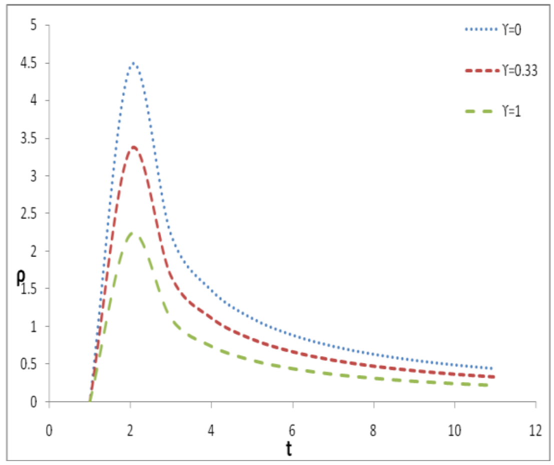

Fig. 2 The plot of proper energy density (ρ) vs. cosmic time (t) with $m=0.25$

Figure-3 and Figure-4 show the variation of isotropic pressure with respect to cosmic time for some specific values of the constant 'm'. It is clear that the behaviour of stiff model and radiation model is almost similar in Lyra geomwtry when the displacement vector is time dependent.

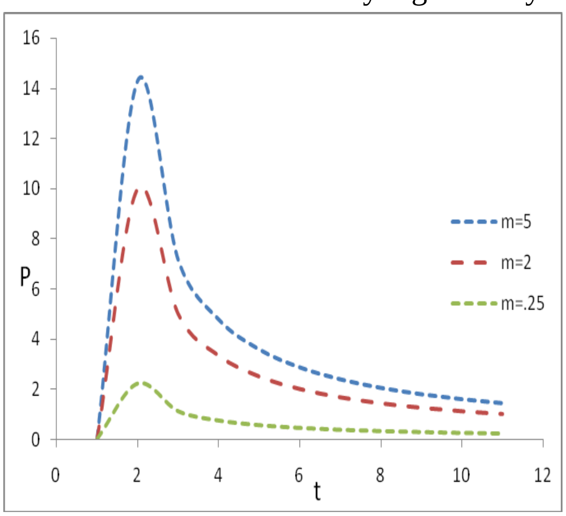

Fig. 3 The plot of isotropic pressure (P) of stiff universe vs.cosmic time (t) with $m=0.25,2,5$

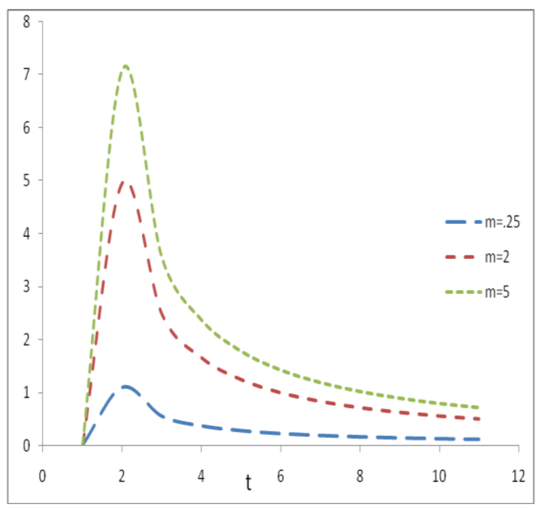

Fig. 4 The plot of isotropic pressure (P) of radiation Universe vs.cosmic time (t) with $m=0.25,2,5$

The scalar of expansion, the shear scalar, spatial volume, Hubble's parameter, the relative anisotropy parameter and the deceleration parameter are given by

The scalar of expansion: $$\theta = \frac{\sqrt{c_1(m+2)}(m+2)(\gamma+1)}{\left(\sqrt{c_1(m+2)}\,(\gamma+1)t + 2\sqrt{m}\,c_2\right)},$$

the shear scalar: $$\sigma = \frac{\sqrt{c_1(m+2)}(m-1)(\gamma+1)}{\left(\sqrt{c_1(m+2)}\,(\gamma+1)t + 2\sqrt{m}\,c_2\right)},$$

spatial volume: $$V = \left(\sqrt{\frac{c_1(m+2)}{4m}}\,(\gamma+1)t + c_2\right)^{\frac{2}{(\gamma+1)}}$$

Hubble parameter: $$H = \frac{\sqrt{c_1(m+2)}(m+2)(\gamma+1)}{3\left(\sqrt{c_1(m+2)}\,(\gamma+1)t + 2\sqrt{m}\,c_2\right)}$$

The relative anisotropy: $$\frac{\sigma^2}{\rho} = \frac{\sqrt{c_1(m+2)}(m-1)^2(\gamma+1)}{(m+2)\left(\sqrt{c_1(m+2)}\,(\gamma+1)t + 2\sqrt{m}\,c_2\right)}$$

$$\frac{\sigma^2}{\theta} = \frac{(m+2)(m-1)^2(\gamma+1)^2}{(m+2)\left(\sqrt{c_1(m+2)}\,(\gamma+1)t + 2\sqrt{m}\,c_2\right)}$$

and

the deceleration parameter $$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1$$

At the initial moment i.e., $t=0$, the parameters ρ, p, β, σ^2 and H tend to infinity. So the universe starts from initial singularity with infinite energy density, infinite internal pressure, infinitely large gauge function, infinite rate of shear and expansion. Moreover ρ, p, β, σ^2 and H tend to finite limit at some finite epoch. Therefore ρ, p, β, σ^2 and H are monotonically decreasing toward a nonzero finite quantity. The spatial volume V tends to zero at the initial singularity. As time proceeds the universe approaches toward a finite volume. The expansion scalar is always positive. Therefore the model describes an expanding cosmological model. The deceleration parameter q is constant in the interval. The ratio $\frac{\sigma^2}{\theta}$ the relative anisotropy of the model tends to infinity as $t \to 0$, and tend to a finite limit at subsequent time. This shows that the model is highly anisotropic at the time of the evolution of the universe.

Since recent observations of SNe Ia depicts that the present Universe is accelerating and the value of deceleration parameter lies in the range $-1 < q < 0$. It follows that in our derived models, one can choose the value of deceleration parameter consistent with the observation. Thus the derived models are realistic.

It should be noted that the universe exhibits initial singularity of the Point-type at $t=0$.

4. Conclusions

In this paper, we explore the cosmological solutions of Kantowski-Sachs universe in the framework of Lyra's Geometry. The essential difference between the cosmological theories based on Lyra geometry and Riemannian geometry lies in the fact that the constant vector field β arises naturally from the concept of gauge in Lyra geometry but the cosmological constant was introduce in temporary manner. It is therefore very important that we have a space-time metric with time-dependent displacement vectors based on Lyra's geometry in normal gauge which is capable of describing almost all these attributes for suitable values for certain parameters. The change in the parameter values may be accomplished by phase transitions. Moreover, we find that the energy density and pressure of the fluid decrease with the increase in the age of the Universe. Physical properties of $\beta(t)$ and $\rho(t)$ have been examined for the three cases and it is found that the models are defined for the special cases for $\gamma = 1$ (stiff fluid distribution), $\gamma = 0$ (dust distribution), $\gamma = \frac{1}{3}$(disordered radiation). Here, it is found that the displacement field β plays the role of a variable cosmological term $\Lambda(t)$. The study of cosmological models based on Lyra's geometry may be relevant for inflationary models. Also the space dependence of the displacement field β is important for inhomogeneous models for the early stages of the evolution of universe. Thus, we believe that the solutions presented in this paper put ample light on the understanding of the evolution of the early universe in Lyra geometry.

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