Report

Empty Type D Metrics & Their Lanczos Potential

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Abstract

We employ the Newman-Penrose formalism to construct the Lanczos spintensor for any type D vacuum spacetime.

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1. Introduction

We shall use the notation and conventions of [1-3]. Lanczos [4, 5] showed that the conformal tensor, in an arbitrary spacetime, is generated by the potential $K_{\mu\nu\alpha}$ via the following expression for the complex Weyl tensor [2]:

$$S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i\ {}^*C_{\mu\nu\alpha\beta} = S_{\mu\nu\alpha;\beta} - S_{\mu\nu\beta;\alpha} + S_{\alpha\beta\mu;\nu} - S_{\alpha\beta\nu;\mu} + \frac{1}{2}[(H_{\mu\beta} + H_{\beta\mu})g_{\nu\alpha} - \tag{1}$$
$$- (H_{\mu\alpha} + H_{\alpha\mu})g_{\nu\beta} + (H_{\nu\alpha} + H_{\alpha\nu})g_{\mu\beta} - (H_{\nu\beta} + H_{\beta\nu})g_{\mu\alpha}],$$

with the complex spintensor $S_{\mu\nu\alpha} \equiv K_{\mu\nu\alpha} + i\ {}^*K_{\mu\nu\alpha}$ and $H_{\mu\nu} \equiv S_{\mu\ \ \nu;\alpha}^{\ \alpha}$.

The Lanczos potential satisfies the algebraic conditions [1]:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}\,, \qquad K_{\mu\ \ \nu}^{\ \nu} = 0\,, \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0\,, \tag{2}$$

hence it has 16 independent real components coded into the complex quantities $\Omega_r,\ r = 0,\ldots,7$, that is, the projections of $S_{\mu\nu\alpha}$ [1] onto the null tetrad $(l_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ of Newman-Penrose (NP) [1-3, 5-7]:

$$S_{\mu\nu\alpha} = 2\,[\Omega_0 U_{\mu\nu} n_\alpha + \Omega_1(M_{\mu\nu} n_\alpha - U_{\mu\nu} m_\alpha) + \Omega_2(V_{\mu\nu} n_\alpha - M_{\mu\nu} m_\alpha) - \Omega_3 V_{\mu\nu} m_\alpha - \tag{3}$$

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$$-\Omega_4 U_{\mu\nu}\bar{m}_\alpha + \Omega_5(U_{\mu\nu}l_\alpha - M_{\mu\nu}\bar{m}_\alpha) + \Omega_6\big(M_{\mu\nu}l_\alpha - V_{\mu\nu}\bar{m}_\alpha\big) + \Omega_7 V_{\mu\nu}l_\alpha],$$

where $V_{\mu\nu} = l_\mu \text{ x } m_\nu$, $U_{\mu\nu} = \bar{m}_\mu \text{ x } n_\nu$, $M_{\mu\nu} = n_\mu \text{ x } l_\nu + m_\mu \text{ x } \bar{m}_\nu$ and:

$$\begin{array}{llll} 2\,\Omega_0 = S_{(1)(3)(1)}, & 2\,\Omega_1 = S_{(1)(3)(4)}, & 2\,\Omega_2 = S_{(4)(2)(1)}, & 2\,\Omega_3 = S_{(4)(2)(4)}, \\ 2\,\Omega_4 = S_{(1)(3)(3)}, & 2\,\Omega_5 = S_{(1)(3)(2)}, & 2\,\Omega_6 = S_{(4)(2)(3)}, & 2\,\Omega_7 = S_{(4)(2)(2)}, \end{array} \tag{4}$$

then $K_{\mu\nu\alpha} = \frac{1}{2}\big(S_{\mu\nu\alpha} + \overline{S_{\mu\nu\alpha}}\big)$ is determined if we know the Ω_r for a given NP tetrad.

In [8] were obtained the NP components (4) for arbitrary spacetimes of Petrov types 0, N and III, with respect to any canonical tetrad [7, 9]. It is interesting to note that in these cases the Ω_r are proportional to the corresponding spin coefficients [3, 6, 7].

Here we deduce the Lanczos potential for the 11 metrics of Kinnersley [10] associated to an arbitrary type D vacuum spacetimes: We work with an adequate canonical null tetrad to achieve certain relationships between its spin coefficients, which allows find a general expression for the Ω_r .

2. Kinnersley's metrics

We select to l^μ and n^μ as the two Debever-Penrose principal directions [7, 9] associated with any empty type D solution, then the Goldberg-Sachs theorem [7, 11] implies:

$$\kappa = \sigma = \lambda = \nu = 0\,; \tag{5}$$

besides it is possible to show that, for each metric of Kinnersley [10], exists a scale-rotation (type III [3, 7]) onto this null tetrad such that:

$$\tau = \pi, \qquad \alpha = \beta, \qquad \gamma = q\varepsilon\,, \qquad \mu = q\varrho\,, \qquad \psi_2 = 4(\gamma\rho - \pi\beta), \qquad q = \pm 1\,. \tag{6}$$

If we employ (5), (6), $\psi_r = 0,\ r \neq 2$ and the Newman-Penrose equations in the Weyl-Lanczos relations [2]:

$$\psi_0 = 2[\delta\Omega_0 - D\Omega_4 + (-\bar{\alpha} - 3\beta + \bar{\pi})\Omega_0 + 3\sigma\Omega_1 + (\bar{\rho} + 3\varepsilon - \bar{\varepsilon})\Omega_4 - 3\kappa\Omega_5],$$

$$2\psi_1 = \Delta\Omega_0 + 3\delta\Omega_1 - \bar{\delta}\Omega_4 - 3D\Omega_5 - (3\gamma + \bar{\gamma} + 3\mu - \bar{\mu})\Omega_0 + 3(-\bar{\alpha} - \beta + \bar{\pi} + \tau)\Omega_1 +$$

$$+ 6\sigma\Omega_2 \ +\left(3\alpha - \bar{\beta} + 3\pi + \bar{\tau}\right)\Omega_4 + 3(\varepsilon - \bar{\varepsilon} - \rho + \bar{\rho})\Omega_5 - 6\kappa\Omega_6,$$

$$\psi_2 = \Delta\Omega_1 + \delta\Omega_2 - \bar{\delta}\Omega_5 - D\Omega_6 - \nu\Omega_0 - (2\mu - \bar{\mu} + \gamma + \bar{\gamma})\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} + 2\tau)\Omega_2 +$$

$$+ \sigma\Omega_3 \ +\lambda\Omega_4 + \left(\alpha - \bar{\beta} + 2\pi + \bar{\tau}\right)\Omega_5 - (\varepsilon + \bar{\varepsilon} - \bar{\rho} + 2\rho)\Omega_6 - \kappa\Omega_7, \qquad (7)$$

$$2\psi_3 = 3\Delta\Omega_2 + \delta\Omega_3 - 3\bar{\delta}\Omega_6 - D\Omega_7 - 6\nu\Omega_1 + 3(\bar{\mu} - \mu - \bar{\gamma} + \gamma)\Omega_2 + (-\bar{\alpha} + 3\beta + 3\tau +$$

$$+\bar{\pi})\Omega_3 + 6\lambda\Omega_5 + 3\left(-\alpha - \bar{\beta} + \pi + \bar{\tau}\right)\Omega_6 - (3\varepsilon + \bar{\varepsilon} - \bar{\rho} + 3\rho)\Omega_7,$$

$$\psi_4 = 2\left[\Delta\Omega_3 - \bar{\delta}\Omega_7 - 3\nu\Omega_2 + (\bar{\mu} + 3\gamma - \bar{\gamma})\Omega_3 + 3\lambda\Omega_6 + \left(-3\alpha - \bar{\beta} + \bar{\tau}\right)\Omega_7\right],$$

we find the general solution:

$$\Omega_0 = \Omega_7 = -q\frac{\pi}{4}, \quad \Omega_4 = q\Omega_3 = -\frac{\rho}{4}, \quad \Omega_1 = q\Omega_6 = -\left(\frac{\varepsilon}{3} + \frac{\rho}{12}\right), \quad \Omega_2 = \Omega_5 = -\left(\frac{\beta}{3} + \frac{\pi}{12}\right). \qquad (8)$$

The Lanczos potential in Kerr geometry obtained in [12] is a special case of (8).

In (8) we observe that the NP components of $K_{\mu\nu\alpha}$ are linear combinations of the spin coefficients, as in the cases O, N and III studied in [8]. We conjecture that the Lanczos scalars Ω_r always can be chosen, for an adequate null tetrad, as a linear combination of the corresponding twelve spin coefficients.

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