Article

On a New Class of Integral Involving Generalized Hypergeometric Function

Arjun K. Rathie[1], J. López-Bonilla[2*] & R. López-Vázquez[2]

[1]Dept. of Math., Central Univ. of Kerala, School of Math. & Physical Sciences, Periye, P.O., Kasaragod- 671 328, Kerala State, India
[2]ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, Mexico city

Abstract

The aim of this note is to evaluate an interesting integral involving generalized hypergeometric function by employing the extension of Gauss's summation theorem available in the literature. A few known integrals have been obtained as special cases of our main findings.

Keyword: Generalized hypergeometric function, Gauss summation theorem.

1. Introduction

We start with the following integral recorded in [1, p. 399, eq. (5)]:

$$\int_0^1 x^{\rho-1}\,(1-x)^{\sigma-1}\;{}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};\,x\right]dx = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\;{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\rho\\ \gamma,\ \rho+\sigma\end{matrix};1\right], \qquad (1)$$

provided $Re(\rho) > 0,\ Re(\sigma) > 0$ *and* $Re(\gamma+\sigma-\alpha-\beta) > 0.$

It is interesting to mention here that if in (1) we take $\sigma = \beta - \rho$ or $\rho = \gamma$, $\sigma = \rho$ we see that, in each case, the ${}_3F_2$ appearing on the right-hand side of (1) reduces to ${}_2F_1$ which can be evaluated by classical Gauss's summation theorem [3]:

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};\,1\right] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)}\ , \qquad Re(\gamma-\alpha-\beta) > 0, \qquad (2)$$

---

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, México DF
E-mail: jlopezb@ipn.mx

and we arrive at the following integrals involving hypergeometric function which are also recorded in [1, p. 399, Eqs. (3) and (4)]:

$$\int_0^1 x^{\rho-1}\,(1-x)^{\sigma-1}\quad {}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};\,x\right]\ dx\ =\ \frac{\Gamma(\gamma)\,\Gamma(\rho)\ \ \Gamma(\beta-\rho)\,\Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta)\ \ \Gamma(\gamma-\alpha)\,\Gamma(\gamma-\rho)}\ , \tag{3}$$

such that $Re(\rho)\ >0,\ Re(\beta-\ \rho)\ >0\ \ and\ \ Re(\gamma\ -\alpha\ -\ \rho)\ >0$, and

$$\int_0^1 x^{\gamma-1}\,(1-x)^{\rho-1}\quad {}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};x\right]\ dx\ =\ \frac{\Gamma(\gamma)\,\Gamma(\rho)\ \ \Gamma(\beta-\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}\ , \tag{4}$$

with $Re(\gamma)\ >0,\ Re(\rho)\ >0\ \ and\ \ Re(\gamma\ +\ \ \rho\ -\alpha\ -\ \beta)\ >0.$

The following extension of classical Gauss's summation theorem (2) is available in the literature [2]:

$${}_3F_2\left[\begin{matrix}a,\ \ b\ ,d+1\\ c+1\ ,\ \ \mathrm{d}\end{matrix}\ ;\ 1\right]\ =\ \frac{\Gamma(\mathrm{c}+1)\ \ \Gamma(\mathrm{c}-\mathrm{a}-\mathrm{b})}{\Gamma(\mathrm{c}-\mathrm{a}+1)\ \Gamma(\mathrm{c}-\mathrm{b}+1)}\ \left\{(c-a-b)\ \ +\ \ \frac{a\,b}{d}\right\}\ , \tag{5}$$

provided $Re(c-a-b)\ >0\ and\ \ d\ \ \neq\ \ 0,-1,-2,\ldots$

The aim of this short note is to evaluate an interesting integral involving generalized hypergeometric function by employing extension of Gauss's summation theorem (5). A few interesting integrals [including the results (1), (3), and (4)] have been obtained as special cases of our main findings.

2. Main result

The interesting integral involving generalized hypergeometric function to be evaluated in this work is given in the following:

Theorem: For $Re(\rho)\ >0,\ \ Re(\sigma)\ >0,\ Re\big(\gamma\ +\ \ \sigma-\alpha-\ \beta\big)\ >0\ \ and\ \ d\ \neq 0,-1,,\ldots,$

the following result holds true:

$$\int_0^1 x^{\rho-1}\,(1-x)^{\sigma-1}\quad {}_3F_2\left[\begin{matrix}\alpha,\ \beta\ ,\ \ d+1\\ \gamma+1\ ,\ \ d\end{matrix}\ ;\ x\right]dx\ \ =\ \ \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\quad {}_4F_3\left[\begin{matrix}\alpha,\ \beta\ ,\ \ \rho,d+1\\ \gamma+1\ ,\ \ \rho+\sigma,\ \ \mathrm{d}\end{matrix}\ ;1\right]\ . \tag{6}$$

Proof: In order to establish our main result (6), we proceed as follows. Denoting the left-hand side of (6) by I, expressing the generalized hypergeometric function ${}_3F_2$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series), we have:

$$I = \sum_{n=0}^{\infty} \frac{(\alpha)_n \ (\beta)_n \ (d+1)_n}{(\gamma+1)_n \ (d)_n \ n!} \int_0^1 x^{\rho+n-1} \ (1-x)^{\sigma-1} \ dx \, .$$

Evaluating the beta integral and using the result $(\alpha)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we obtain, after some simplification:

$$I = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)} \sum_{n=0}^{\infty} \frac{(\alpha)_n \ (\beta)_n \ (\rho)_n \ (d+1)_n}{(\gamma+1)_n \ (\rho+\sigma)_n \ (d)_n \ n!} \, .$$

Finally, summing up the series, we easily arrive at the right–hand side of (6). This completes the proof of our main result (6).

3. Special cases

In this section, we shall consider some of the very interesting special cases of our main result.

Corollary 1. In (6), if we take $d = \gamma$, we at once get the known expression (1). Thus the result (6) may be regarded as an extension of (1).

Corollary 2. In (6), if we select $\sigma = \beta - \rho$, we get after some simplification:

$$\int_0^1 x^{\rho-1} \ (1-x)^{\beta-\rho-1} \ {}_3F_2\left[\begin{matrix}\alpha, \ \beta, \ d+1 \\ \gamma+1, \ d\end{matrix} ; x\right] dx = \frac{\Gamma(\rho)\,\Gamma(\beta-\rho)}{\Gamma(\beta)} \ {}_3F_2\left[\begin{matrix}\alpha, \ \rho, \ d+1 \\ \gamma+1, \ d\end{matrix} ; 1\right]. \quad (7)$$

We now observe that the ${}_3F_2$ appearing on the right-hand side of (7) can be evaluated with the help of the result (5) by taking $a = \beta$, $b = \rho$ and $c = \gamma$ and after some simplification, we deduce the following new and interesting result:

$$\int_0^1 x^{\rho-1} (1-x)^{\beta-\rho-1} \ {}_3F_2\left[\begin{matrix}\alpha, \ \beta, \ d+1 \\ \gamma+1, \ d\end{matrix} ; x\right] dx =$$

$$= \frac{\Gamma(\rho)\,\Gamma(\beta-\rho)}{\Gamma(\beta)}\,\frac{\Gamma(\gamma+1)\,\Gamma(\gamma-\alpha-\rho)}{\Gamma(\gamma-\alpha+1)\,\Gamma(\gamma-\rho+1)}\left\{(\gamma-\alpha-\rho)+\frac{\alpha\,\rho}{d}\right\}. \tag{8}$$

Further, in (8), if we take $d = \gamma$, we get at once the known result (3). Thus the formula (8) may be regarded as the extension of the result (3).

Corollary 3. In (6), if we take $\rho = \gamma$ and $\sigma = \rho$, we obtain:

$$\int_0^1 x^{\gamma-1}\,(1-x)^{\rho-1}\;{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ d+1\\ \gamma+1,\ d\end{matrix};x\right]dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)}{\Gamma(\gamma+\rho)}\;{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \gamma,d+1\\ \gamma+1,\ \gamma+\rho,\ d\end{matrix};1\right], \tag{9}$$

In this, if we take $d = \gamma$, we get:

$$\int_0^1 x^{\gamma-1}\,(1-x)^{\rho-1}\;{}_3F_2\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};x\right]dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)}{\Gamma(\gamma+\rho)}\;{}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma+\rho\end{matrix};1\right]. \tag{10}$$

We now observe that the ${}_2F_1$ appearing on the right-hand side of (10) can be evaluated with the help of Gauss's summation theorem (2), and after some simplification, we get the known result (4). Similarly, other results can also be obtained.

References

[1] A. Erdelyi et al, *Tables of Integral Transforms*, Tata McGraw-Hill, New York (1960)

[2] A. P. Prudnikov, Yu A. Brychkov, O. I. Mariachev, *Integrals and Series*, vol. 3: *More Special Functions*, Gordon and Breach Science Pub, New York (1990)

[3] E. D. Rainville, *Special Functions*, Chelsea Pub. Co, New York (1960)