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Some Local Expressions of Special Curves in Three Dimentional Finsler Manifold F^3

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Abstract

In the Euclidean space E^3, there exist three classes of curves, so-called rectifying, normal and osculating curves satisfying the planes spanned by $\{T, B\}$, $\{N, B\}$ and $\{T, N\}$ as well as to each unit speed curve $\alpha : I \subset IR \to E^3$ whose orthogonal unit vector fields T, N, B, called respectively the tangent, the principal normal and the binormal vector fields.

In this paper, we give the definition of rectifying, normal and osculating curves in 3-dimensional Finsler manifold. Furthermore, we obtain some characterizations related to these curves.

Keywords: Rectifying curve, normal curve, osculating curve, Finsler manifold.

1. Introduction

In practical world, curves can be defined in many different ways such as profile or contours of the objects. One can see it as a trace of a pen in the simplest term. Because of having many applications to different application areas from physics to mechanics, robotics and many branch of life sciences, curve theory is one of the major study field for differential geometers . Mathematically, a curve means a continous map from an interval to $\mathbb{R}^n$. There are numerous type of curves and each of them has special definitions. One of the new type of that of curves are rectifying, normal and osculating curves which is defined by means of the position vectors which lies their rectifying, normal and osculating planes, respectively. Rectifying curves are introduced in 2003 by B.Y.Chen [1],rectifying ,osculating and normal curves are sepecially defined curves that satisfy Cesa'ro's fixed point condition [2]. Rectifying curves in different spaces studied in [3-8]. Normal curves in the Minkowski space-time are introduced in [9] as the space curves whose position vector always lies in its normal space, which indicates orthogonal complement of the tangent vector field of the curve. Also osculating curves in the Minkowski space-time are defined in [10] by same authors as the space curves whose position vector (with respect to some chosen origin) always lies in its osculating space, which represents the orthogonal complement of the first binormal or second binormal vector field of the curve.

Finsler geometry is the most natural generalization of Riemannian geometry. It started in 1918 when P. Finsler wrote his thesis on curves and surfaces in what he called generalized metric spaces. Due to its importance it has a huge research field from geometry to biology, physics and also engineering and computer sciences, [11-13].From the frame work of differential geometry, the position vectors of the planes has meaningful properities . By this purpose we study the position vectors of the planes in 3-dimensional Finsler manifold.

2. Basic Notions & Properties

The following part of the study is on the basic concepts of the Finsler manifolds

Definition 2.1. Let M be a real m-dimensional smooth manifold and TM be the tangent bundle of M.Denote by Π the canonical projection of TM on M.

Let M' be an non-empty open submanifold of TM such that $\Pi\left(M'\right) = M$ and $\theta(M) \cap M' = \varnothing$, where θ is the zero section of TM.

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We now consider a smooth function $F : M^{'} \to (0, \infty)$ and take $F^* = F^2$. Then suppose that for any coordinate system $\left\{ \left(u^{'}, \Phi^{'} \right); x^i, y^i \right\}$ in $M^{'}$, the following conditions are fulfilled:

(F_1) F is positively homogeneus of degree one with respect to $\left(y^1, ..., y^m \right)$ i.e. we have

$$F\left(x^1, ...x^m, ky^1, ..., ky^m\right) = kF\left(x^1, ..., x^m, y^1, ..., y^m\right)$$

for any $(x, y) \in \Phi^{'}\left(U^{'}\right)$ and any $k > 0$.

(F_2) At any point $(x, y) \in \Phi^{'}\left(U^{'}\right)$

$$g_{ij}(X, Y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y^i \partial y^j}(X, Y), \ i, j \in \{1, ..., m\}$$

are the components of a positive definite quadratic form on $\mathbb{R}^m$,[16].

We say that the triple $\mathbb{F}^m = \left(M, \ M^{'}, F\right)$ with satisfying (F_1) and (F_2) is a Finsler manifold and F is the fundamental function of $\mathbb{F}^m$.

Definition 2.2. Let $\mathbb{F}^{m+1} = \left(M, M^{'}, F\right)$ be a Finsler manifold and $\mathbb{F}^{'} = \left(C, C^{'}, F\right)$ be a 1-dimensional Finsler submanifold of $\mathbb{F}^{m+1}$, where C is a smooth curve in M given locally by the equations

$$x^i = x^i(s) \qquad , i \in \{1, ..., m+n\}, \quad s \in (a, b)$$

s being the arclength parameter on C. Denote by (s, v) the coordinates on $C^{'}$. Then we have

$$y^i(s, v) = v\frac{dx^i}{ds} \qquad\qquad i \in \{0, ..., m\}$$

Moreover $\left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial v}\right\}$ is a natural field of frames on C where $\frac{\partial}{\partial v}$ is a unit Finsler vector field, [16].

Definition 2.3. Let $\mathbb{F}^3 = \left(M, \ M^{'}, F\right)$ be a 3-dimensional Finsler manifold and C be a smooth curve in M given locally by the parametric equations

$$x^i = x^i(s) \ \ ; \quad \left(x^{'1}(s), \ x^{'2}(s), \ x^{'3}(s)\right) \neq (0, 0, 0)$$

where s is the arclength parameter on C.

Then we have

$$\begin{aligned} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} &= kn, \\ \nabla^*_{\frac{\partial}{\partial s}} n &= -k\frac{\partial}{\partial v} + \tau b \\ \nabla^*_{\frac{\partial}{\partial s}} b &= \tau n. \end{aligned} \tag{2.1}$$

where n and b called principal normal Finsler vector field and binormal Finsler vector field on C respestively. We are entiteled to call $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame for the curve C in $\mathbb{F}^3$. As in the Riemannian case we call k the curvature and τ the torsion of C respesctively, [16].

3. Rectifying, Normal & Osculating Curves in Three Dimentional Finsler ManifoldF^3

Definition 3.1. Let C be a smooth curve of $\mathbb{F}^3 = \left(M, \ M^{'}, F\right)$. If the position vector of C always lies in its rectifying plane, then it is called a rectifying curve in F^3.

By this definition, for a curve in F^3, the position vector of C satisfies

$$C(s) = \lambda(s) \frac{\partial}{\partial v}(s) + \mu(s) b(s) \tag{3.1}$$

where $\lambda(s), \mu(s)$ are some differentiable functions.

Theorem 3.1. Let C be a rectifying curve in 3-dimensional Finsler manifold, with curvature $k \rangle 0$, and let s be its arc length function. Then;

i) The distance function $\rho(s) = |C(s)|$ satisfies

$$\rho^2(s) = |\langle C(s), C(s)\rangle| = \left|s^2 + 2as + a^2 + b^2\right|$$

for some $a \epsilon R$, $b \epsilon R - \{0\}$.

ii) The tangential component of the position vector of the curve given by $\left\langle C(s), \frac{\partial}{\partial v}(s)\right\rangle = s + a$ satisfies for some constant a.

iii) The normal component $C^N(s)$ of the position vector of the curve has constant length and the distance function $\rho(s)$ is non-constant.

iv) The torsion $\tau(s) \neq 0$ and the binormal component of the position vector is constant, i.e. $\langle C(s), b(s)\rangle$ is constant.

Conversely, if $C(s)$ is a curve with $k(s) \rangle 0$ and if one of (i), (ii), (iii) or (iv) holds, then C is a rectifying curve of $\mathbb{F}^3 = \left(M,\ M', F\right)$.

Proof. Let us assume that $C(s)$ is a rectifying curve of $\mathbb{F}^3 = \left(M,\ M', F\right)$. Then, from Definition 3.1, one can write the position vector of C by

$$C(s) = \lambda(s) \frac{\partial}{\partial v}(s) + \mu(s) b(s) \tag{3.2}$$

where $\lambda(s)$, $\mu(s)$ are some differentiable functions.

i) Differentiating the equation (3.2) with respect to s and using the equations (2.1), one can have

$$\begin{aligned} \nabla^*_{\frac{\partial}{\partial s}} \lambda &= 1 \\ \lambda k + \mu\tau &= 0 \\ \nabla^*_{\frac{\partial}{\partial s}} \mu &= 0. \end{aligned} \tag{3.3}$$

Thus, one can get

$$\begin{aligned} \lambda &= s + a, \quad a\epsilon R, \\ \mu &= b, \quad b\epsilon R, \\ \mu\tau &= -\lambda k \neq 0 \end{aligned} \tag{3.4}$$

and hence $\mu = b \neq 0$. From the equation (3.2), one can easily find that

$$\rho^2(s) = |\langle C(s), C(s)\rangle| = \left|s^2 + 2as + a^2 + b^2\right|. \tag{3.5}$$

ii) Considering equation (3.2), one can get

$$\left\langle C, \frac{\partial}{\partial v}\right\rangle = \lambda \tag{3.6}$$

which means that the tangential component of the position vector C is given by

$$\left\langle C, \frac{\partial}{\partial v} \right\rangle = s + a, \quad a\epsilon R. \tag{3.7}$$

iii) From the equation (3.2), it follows that the normal component of C^N of the position vector C is given by

$$C^N = \mu b. \tag{3.8}$$

Therefore,

$$\left\| C^N \right\| = |\mu| = |b| \neq 0. \tag{3.9}$$

Thus, statement (iii) is proved.

iv) Considering equation (3.2), one can easily get

$$\langle C, b \rangle = \mu = cons.,$$

and since $k \rangle 0$, the statement (iv) is proved.

Conversely, suppose that statement (i) or statement (ii) holds. Then one can have

$$\left\langle C, \frac{\partial}{\partial v} \right\rangle = s + a, \quad a\epsilon R. \tag{3.10}$$

Differentiating equation (3.10) with respect to s, one can obtain

$$k \langle C, n \rangle = 0. \tag{3.11}$$

Since $k \rangle 0$, it follows that

$$\langle C, n \rangle = 0 \tag{3.12}$$

which means that C is a rectifying curve.

If statement (iii) holds, one can write

$$\langle C, C \rangle = \left\langle C, \frac{\partial}{\partial v} \right\rangle^2 + A, \tag{3.13}$$

where A is constant. Differentiating (3.13) with respect to s, then one can have

$$\left\langle C, \frac{\partial}{\partial v} \right\rangle = \left\langle C, \frac{\partial}{\partial v} \right\rangle (1 + k \langle C, n \rangle). \tag{3.14}$$

Since the distance function $\rho \neq cons\tan t$, one can obtain

$$\left\langle C, \frac{\partial}{\partial v} \right\rangle \neq 0. \tag{3.15}$$

Moreover, since $k \rangle 0$ and from (3.14), one can get

$$\langle C, n \rangle = 0, \tag{3.16}$$

which means that C is a rectifying curve.

Theorem 3.2. Let C be a curve of $\mathbb{F}^3 = \left(M,\ M', F \right)$. Then the curve C is a rectifying curve if and only if there holds

$$\frac{\tau}{\kappa} = c_1 s + c_2 \tag{3.17}$$

where $c_1 \epsilon\ R-\{0\}\,,\ c_2\ \epsilon R.$

Proof. Let us first suppose that the curve C is rectifying. From the equations (3.3) and (3.4), one can easily find that (3.17).

Conversely, let us suppose that $\frac{\tau}{\kappa}=c_1s+c_2$, $c_1\ \epsilon\ R-\{0\}\,,\ c_2\ \epsilon R$. Then one may choose

$$\begin{aligned} c_1 &= \frac{1}{b}3.18 \\ c_2 &= \frac{a}{b} \end{aligned} \tag{0.1}$$

where $b\ \epsilon\ R-\{0\}\,,\ a\epsilon R.$

Hence one can get

$$\frac{\tau}{\kappa}=\frac{s+a}{b}. \tag{3.19}$$

Considering the Frenet equations (3.1), one can conclude that

$$\frac{d}{ds}\left[C\left(s\right)-\left(s+a\right)\frac{\partial}{\partial v}\left(s\right)-bb\left(s\right)\right]=0 \tag{3.20}$$

which means that C is a rectifying curve.

Definition 3.2. Let C be smooth curve of $\mathbb{F}^3=\left(M,\ M',F\right)$. If the position vector of $C\left(s\right)$ always lies in its normal plane, then it is called a normal curve in $\mathbb{F}^3$.

By this definition, for a curve in $\mathbb{F}^3$, the position vector of C satisfies

$$C\left(s\right)=\xi\left(s\right)n\left(s\right)+\eta\left(s\right)b\left(s\right) \tag{3.21}$$

where $\xi\left(s\right)$ and $\eta\left(s\right)$ are differentiable functions.

Theorem 3.3. Let C be a smooth curve of $\mathbb{F}^3=\left(M,\ M',F\right)$, with curvature $k,\tau\epsilon R$. Then C is a normal curve if and only if the principal normal and binormal components of the position vector are respectively given by

$$\triangledown^*_{\frac{\partial}{\partial s}}\triangledown^*_{\frac{\partial}{\partial s}}\xi+2\triangledown^*_{\frac{\partial}{\partial s}}\eta\tau+\eta\triangledown^*_{\frac{\partial}{\partial s}}\tau+\xi\tau^2=k \tag{3.22}$$

$$\triangledown^*_{\frac{\partial}{\partial s}}\triangledown^*_{\frac{\partial}{\partial s}}\eta+2\triangledown^*_{\frac{\partial}{\partial s}}\xi\tau+\xi\triangledown^*_{\frac{\partial}{\partial s}}\tau+\eta\tau^2=0. \tag{3.23}$$

Proof. Let us assume that C is a normal curve of $\mathbb{F}^3=\left(M,\ M',F\right)$, then from (3.21), one can have

$$C\left(s\right)=\xi\left(s\right)n\left(s\right)+\eta\left(s\right)b\left(s\right).$$

Differentiating this with respect to s, one can get

$$\frac{\partial}{\partial v}=-k\xi\frac{\partial}{\partial v}+\left(\triangledown^*_{\frac{\partial}{\partial s}}\xi+\eta\tau\right)n+\left(\xi\tau+\triangledown^*_{\frac{\partial}{\partial s}}\eta\right)b. \tag{3.24}$$

Again differentiating this with respect to s and by using (3.1), one can have

$$\begin{aligned} kn &= \left(-\triangledown^*_{\frac{\partial}{\partial s}}k\xi-2k\triangledown^*_{\frac{\partial}{\partial s}}\xi-k\eta\tau\right)\frac{\partial}{\partial v} \\ &\quad+\left(\triangledown^*_{\frac{\partial}{\partial s}}\triangledown^*_{\frac{\partial}{\partial s}}\xi+2\triangledown^*_{\frac{\partial}{\partial s}}\eta\tau+\eta\triangledown^*_{\frac{\partial}{\partial s}}\tau+\xi\tau^2\right)n \\ &\quad+\left(2\triangledown^*_{\frac{\partial}{\partial s}}\xi\tau+\xi\triangledown^*_{\frac{\partial}{\partial s}}\tau+\triangledown^*_{\frac{\partial}{\partial s}}\triangledown^*_{\frac{\partial}{\partial s}}\eta+\eta\tau^2\right)b.3.25 \end{aligned} \tag{0.2}$$

From equation (3.25), one can obtain the differential equation system given by (3.22) and (3.23), which completes the proof.

Definition 3.3. Let C be smooth curve of $\mathbb{F}^3 = \left(M,\ M^{'}, F\right)$. If the position vector of $C\left(s\right)$ always lies in its osculating plane, then it is called a osculating curve in $\mathbb{F}^3$.

By this definition, for a curve in $\mathbb{F}^3$, the position vector of C satisfies

$$C\left(s\right) = x\left(s\right)\frac{\partial}{\partial v}\left(s\right) + y\left(s\right)n\left(s\right) \tag{3.26}$$

where $x\left(s\right)$ and $y\left(s\right)$ are differentiable functions.

Theorem 3.4. Let C be a smooth curve of $\mathbb{F}^3 = \left(M,\ M^{'}, F\right)$, with curvature $k, \tau\epsilon R$. Then C is a osculating curve if and only if the tangential normal and principal normal components of the position vector are respectively given by

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} x - \nabla^*_{\frac{\partial}{\partial s}} ky - 2k \nabla^*_{\frac{\partial}{\partial s}} y - xk^2 = 0 \tag{3.27}$$

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} y + 2k \nabla^*_{\frac{\partial}{\partial s}} x + x \nabla^*_{\frac{\partial}{\partial s}} k - k^2 y + y\tau^2 = k. \tag{3.28}$$

Proof. Let us assume that C is a osculating curve of $\mathbb{F}^3 = \left(M,\ M^{'}, F\right)$, then from (3.26), one can get

$$C\left(s\right) = x\left(s\right)\frac{\partial}{\partial v}\left(s\right) + y\left(s\right)n\left(s\right).$$

Differentiating this with respect to s and using equation (3.1), one can have

$$\begin{aligned} \frac{\partial}{\partial v} &= \left(\nabla^*_{\frac{\partial}{\partial s}} x - ky\right)\frac{\partial}{\partial v} \\ &+ \left(xk + \nabla^*_{\frac{\partial}{\partial s}} y\right) n \\ &+ \left(y\tau\right) b \end{aligned} \tag{3.29}$$

Again differentiating this with respect to s and by using equation (3.1), one can obtain

$$\begin{aligned} kn &= \left(\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} x - \nabla^*_{\frac{\partial}{\partial s}} ky - 2k \nabla^*_{\frac{\partial}{\partial s}} y - xk^2\right)\frac{\partial}{\partial v} \\ &+ \left(\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} y + 2k \nabla^*_{\frac{\partial}{\partial s}} x + x \nabla^*_{\frac{\partial}{\partial s}} k - k^2 y + y\tau^2\right) n \\ &+ \left(2 \nabla^*_{\frac{\partial}{\partial s}} y\tau + y \nabla^*_{\frac{\partial}{\partial s}} \tau + xk\tau\right) b \end{aligned} \tag{3.30}$$

From equation (3.30), one can get the differential equation system given by (3.27) and (3.28), which completes the proof.

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