Article

On the Legendre Polynomials

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Abstract

We give a simple deduction of the Laplace's integral formula for the Legendre polynomials, and we show that they permit to obtain the Lanczos derivative for higher orders.

Keywords: Legendre polynomials, Lanczos derivative, Laplace's integral formula, Lanczos quadrature.

1. Introduction

The Legendre's polynomials [1] $P_n(t), -1 \leq t \leq 1$, can be defined via the following recurrence relation [2-4]:

$$(n+1)P_{n+1} = (2n+1)t\, P_n - n\, P_{n-1}, \quad P_0 = 1, \quad P_1 = t, \quad n = 1, 2, \ldots, \quad (1)$$

hence:

$$P_2 = \frac{1}{2}(3t^2 - 1), \qquad P_3 = \frac{1}{2}(5t^3 - 3t), \qquad P_4 = \frac{1}{8}(35t^4 - 30t^2 + 3), \ldots \quad (2)$$

These polynomials also are determined univocally through the conditions [5, 6]:

$$\int_{-1}^{1} P_m(t) P_n(t) dt = 0, \quad m \neq n, \qquad P_n(1) = 1, \quad \forall\, n, \quad (3)$$

and the Laplace's integral formula [3-5] gives an alternative way to generate the expressions (2):

$$P_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (t + \sqrt{t^2 - 1} \cos \beta)^n \, d\beta, \qquad n = 0, 1, 2 \ldots \quad (4)$$

In Sec. 2 we exhibit a simple manner to obtain (4).

On the other hand, the Cioranescu [7]-(Haslam-Jones) [8]-Lanczos [9] derivative is defined by [10, 11]:

$$f'_L(x, \varepsilon) = \frac{3}{2\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t\, f(x+t)\, dt, \qquad \varepsilon \ll 1, \quad (5)$$

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and if $\varepsilon \to 0$ then, in general, (5) tends to the ordinary derivative:

$$f_L'(x) \equiv lim_{\varepsilon\to 0}\ f_L'(x,\varepsilon) = f'(x), \tag{6}$$

that is, derivation by integration. In Sec. 3 we employ the Lanczos quadrature technique [9] to generalize (5) for higher orders, in which we will obtain the relation:

$$f_L^{(n)}(x) = lim_{\varepsilon\to 0}\ \frac{(2n+1)!!}{2\varepsilon^{n+1}} \int_{-\varepsilon}^{\varepsilon} P_n\left(\frac{t}{\varepsilon}\right) f(x+t)\, dt, \qquad n = 1, 2, \ldots \tag{7}$$

in harmony with the result of Rangarajan and Purushothaman [12]. The $P_n(x)$ are orthogonal polynomials [3], therefore Diekema and Koornwinder [13] consider that the name 'orthogonal derivative' is adequate for (7). We note that Lanczos [9] uses the Legendre polynomials to deduce his quadrature method, then in (7) is natural the participation of these polynomials.

2. The Laplace's integral formula

It is easy to verify that:

$$V(x,y,z) = \int_{-\pi}^{\pi} f(z + i\, x \cos\beta + i\, y \sin\beta)\, d\beta, \quad i = \sqrt{-1}, \tag{8}$$

where $f(\eta)$ is an arbitrary function, satisfies the Laplace equation in three dimensions:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{9}$$

For the case $f(\eta) = \eta$, from (8) we obtain:

$$V = \int_{-\pi}^{\pi} (z + i\, x \cos\beta + i\, y \sin\beta)\, d\beta = 2\pi\, z = 2\pi\, r \cos\theta = 2\pi\, r\, P_1(\cos\theta),$$

in spherical coordinates r and θ. If $f(\eta) = \eta^2$, then (8) gives:

$$V = \pi(2z^2 - x^2 - y^2) = \pi\, r^2 (3\cos^2\theta - 1) = 2\pi\, r^2 P_2(\cos\theta),$$

in fact, we know [5] that $r^m P_m(\cos\theta)$ are harmonic functions in the near zone. Thus, (8) leads to $V = 2\pi\, r^n P_n(\cos\theta)$ when $f(\eta) = \eta^n$, therefore:

$$P_n(\cos\theta) = \frac{1}{2\pi\, r^n} \int_{-\pi}^{\pi} (z + i\, x \cos\beta + iy \sin\beta)^n\, d\beta = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos\theta + i \sin\theta \cos(\beta - \varphi)]^n\, d\beta \tag{10}$$

because $x = r\sin\theta\cdot\cos\varphi$, $y = r\sin\theta\cdot\sin\varphi$, $z = r\cos\theta$. In (10) φ is arbitrary, then we may employ $\varphi = \pi$ and to introduce the variable $t = \cos\theta$, that is, $\sin\theta = i\sqrt{t^2-1}$, in this manner (10) implies the Laplace's integral formula (4) published in [14].

3. The orthogonal derivative

Here we shall deduce (7) from a remarkable quadrature technique obtained by Lanczos [9] where only values of the function and its derivatives at the end points *a* and *b* are needed:

$$\int_a^b F(t)dt = \sum_{k=1}^{n}\frac{(2n-k)!!}{(2n)!}\binom{n}{k}\left[F^{(k-1)}(a)-(-1)^k F^{(k-1)}(b)\right]h^k, \quad h = b-a, \tag{11}$$

The expression (11) is quite efficient since with only a few terms it gives a result very near to the exact one, and this property is useful in solving differential equations with boundary values. For *n* = 3 the relation (11) turns out to be:

$$\int_a^b F(t)dt = \frac{h}{2}[F(a)+F(b)] + \frac{h^2}{10}[F'(a)-F'(b)] + \frac{h^3}{120}[F''(a)+F''(b)], \tag{12}$$

which leads, naturally, to orthogonal derivative without the explicit use of the Least Squares Method of Legendre [15]-Gauss [16]-Laplace [17] as occurs in the deduction of Lanczos [9, 18, 19].

In fact, application of (12) for $F(t) = t\,f(x+t)$ with $a=-\varepsilon$, $b=\varepsilon \ll 1$, so $h = 2\varepsilon$, gives:

$$\frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t\,f(x+t)dt = \frac{9}{5}\cdot\frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon} - \frac{2}{5}[f'(x+\varepsilon)+f'(x-\varepsilon)] + \\ + \frac{\varepsilon}{10}[f''(x+\varepsilon)-f''(x-\varepsilon)],$$

consequently:

$$\lim_{\varepsilon\to 0}\frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t\,f(x+t)dt = \frac{9}{5}f'(x) - \frac{4}{5}f'(x) = f'(x), \tag{13}$$

in harmony with (5) and (6). If now we apply (12) for $F(t) = t^2 f(x+t)$, then:

$$\int_{-\varepsilon}^{\varepsilon} t^2 f(x+t)dt = \frac{\varepsilon^5}{15}[f''(x+\varepsilon)+f''(x-\varepsilon)] + \frac{\varepsilon^3}{3}[f(x+\varepsilon)+f(x-\varepsilon)] - \\ -\frac{2\varepsilon^4}{15}[f'(x+\epsilon)+f'(x-\varepsilon)], \tag{14}$$

but (12) with $F(t) = f(x+t)$ gives:

$$\frac{\varepsilon^3}{3}[f(x+\varepsilon)+f(x-\varepsilon)]-\frac{2\varepsilon^4}{15}[f'(x+\varepsilon)+f'(x-\varepsilon)]=\frac{\varepsilon^2}{3}\int_{-\varepsilon}^{\varepsilon}f(x+t)dt-\frac{\varepsilon^5}{45}[f''(x+\varepsilon)+ + f''(x-\varepsilon)],$$

thus (14) implies:

$$\frac{1}{\varepsilon^3}\int_{-\varepsilon}^{\varepsilon}(\frac{t^2}{\varepsilon^2}-\frac{1}{3})\,f(x+t)\,dt=\frac{2}{45}[f''(x+\varepsilon)+f''(x-\varepsilon)],$$

which alternatively gives:

$$f''(x)=lim_{\,\varepsilon\to 0}\ \frac{15}{4\varepsilon^3}\int_{-\varepsilon}^{\varepsilon}(\frac{3t^2}{\varepsilon^2}-1)\,f(x+t)\,dt=lim_{\,\varepsilon\to 0}\ \frac{5!!}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon}P_2\left(\frac{t}{\varepsilon}\right)f(x+t)\,dt. \qquad (15)$$

Similar process with $F(t)=t^3 f(x+t)$ leads to:

$$f'''(x)=\ lim_{\,\varepsilon\to 0}\ \frac{7!!}{2\varepsilon^4}\int_{-\varepsilon}^{\varepsilon}P_3\left(\frac{t}{\varepsilon}\right)f(x+t)\,dt, \qquad (16)$$

then (13), (15) and (16) permit to write (7), that is, the orthogonal derivative for higher orders, in according with the result of Rangarajan and Prushothaman [12]; also it is possible to derive (7) using mathematical induction. In this manner, the method of differentiation by integration due to Cioranescu [7]-(Haslam-Jones) [8]-Lanczos [9] is generalized to cover derivatives of arbitrary order.

4. Conclusions

Our elementary deduction of (4) shows that the Laplace's integral formula for the Legendre polynomials $P_n(x)$ is coming from the Laplace equation. Besides, we exhibit that the Lanczos quadrature method permits to generalize the orthogonal derivative for higher orders, where the $P_n(x)$ participate in the corresponding kernel obtained by Rangarajan and Purushothaman [12].

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