Article

Spin Coefficients Formalism

J. López-Bonilla[1*], R. López-Vázquez[1], J. Morales[2] & G. Ovando[2]

[1]ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, Mexico city
[2]CBI-AMA, UAM-Azcapotzalco, Av. San Pablo 180, Col. Reynosa-Tamps. 02 200, México DF

Abstract

In this article, we study some differential aspects of the spinor and Newman-Penrose techniques. Particularly, we study the evolution of the null tetrad and the spin frame at each event of the space-time which permits the introduction of the spin and rotation coefficients. We then consider Lorentz transformation which leads to special rotations of the NP's tetrad and spin frame and demonstrate the corresponding changes in the spin coefficients and in the NP components of Faraday, Ricci without trace, Lanczos and Weyl tensors.

Keywords: Newman-Penrose formalism, two-spinors, rotations, null tetrad, spin coefficients.

1. Introduction

The material in this work has relationship with the publications [1, 2], and we shall use their notation and conventions. Here we consider some differential properties of the spinorial and Newman-Penrose (NP) [3-7] formalisms. In Sec. 2, we study the evolution of the null tetrad and the spin frame at each event of the space-time which permits the introduction of the spin and rotation coefficients [8]. In Sec.3, we consider Lorentz transformation which leads to special rotations of the NP's tetrad and spin frame and demonstrate the corresponding changes in the spin coefficients and in the NP components of Faraday, Ricci without trace, Lanczos and Weyl tensors.

2. Null tetrad and its spin and rotation coefficients

The differential operator $\nabla_\mu \equiv ;\mu$ means covariant derivative and it has tensorial character, hence it can be written in terms of the null tetrad, and its corresponding projections onto the NP vectors are the intrinsic derivatives:

$$D \equiv l^\mu \nabla_\mu \,, \qquad \Delta \equiv n^\mu \nabla_\mu \,, \qquad \delta \equiv m^\mu \nabla_\mu \,, \qquad \bar{\delta} \equiv \bar{m}^\mu \nabla_\mu \,, \tag{1}$$

therefore

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, México DF
E-mail: jlopezb@ipn.mx

$$\nabla_\mu = l_\mu\, \Delta + n_\mu\, D - m_\mu\, \bar{\delta} - \bar{m}_\mu\, \delta\,, \tag{2}$$

whose spinor form is given by:

$$\nabla_{A\dot{B}} = \sigma^\mu{}_{A\dot{B}}\, \nabla_\mu = o_A o_{\dot{B}}\, \Delta + \iota_A \iota_{\dot{B}}\, D - o_A \iota_{\dot{B}} \bar{\delta} - \iota_A o_{\dot{B}}\, \delta\,, \tag{3}$$

thus:

$$D = o^A o^{\dot{B}}\, \nabla_{A\dot{B}}\,, \qquad \Delta = \iota^A \iota^{\dot{B}}\, \nabla_{A\dot{B}}\,, \qquad \delta = o^A \iota^{\dot{B}}\, \nabla_{A\dot{B}}\,, \qquad \bar{\delta} = \iota^A o^{\dot{B}}\, \nabla_{A\dot{B}}\,. \tag{4}$$

If we know the action of (1) on o^A and ι^A, then we shall learn their operation on the null tetrad. For example, Do_A is a spinor with certain components with respect to the spin frame, which introduces the NP notation [7]:

$$\kappa = o^A D o_A\,, \qquad \varepsilon = \iota^A D o_A \qquad \therefore \qquad D o_A = \varepsilon\, o_A - \kappa\, \iota_A\,, \qquad o_A \iota^A = 1\,,$$

similarly:

$$\Delta o_A = \gamma\, o_A - \tau\, \iota_A\,, \qquad \delta o_A = \beta\, o_A - \sigma\, \iota_A\,, \qquad \bar{\delta} o_A = \alpha\, o_A - \rho\, \iota_A\,,$$

$$D\iota_A = \pi\, o_A - \varepsilon\, \iota_A\,, \qquad \Delta\iota_A = \nu\, o_A - \gamma\, \iota_A\,, \qquad \delta\iota_A = \mu\, o_A - \beta\, \iota_A\,, \qquad \bar{\delta}\iota_A = \lambda\, o_A - \alpha\, \iota_A\,,$$

or in matrix form:

$$D\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} = \begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix}, \qquad \Delta\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} = \begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix},$$
$$\delta\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} = \begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix}, \qquad \bar{\delta}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} = \begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix}. \tag{5}$$

Besides:

$$\kappa = o^A D\left(o_A o_{\dot{B}} \iota^{\dot{B}}\right) = o^A\left[\iota^{\dot{B}} D(o_A o_{\dot{B}}) + o_A o_{\dot{B}} D\iota^{\dot{B}}\right] = o^A \iota^{\dot{B}} D(o_A o_{\dot{B}}) = m^\mu D l_\mu\,,$$

where we accept that:

$$\nabla_\mu\, \sigma_{\nu\, A\dot{B}} = 0\,, \qquad \nabla_\mu\, \varepsilon_{AB} = 0\,, \qquad \nabla_\mu\, \varepsilon^{AB} = 0\,, \tag{6}$$

therefore the spin coefficients admit the tensorial expressions:

$$\kappa = m^\mu D l_\mu\,, \quad \sigma = m^\mu \delta l_\mu\,, \quad \rho = m^\mu \bar{\delta} l_\mu\,, \quad \tau = m^\mu \Delta l_\mu\,, \quad \nu = n^\mu \Delta \bar{m}_\mu\,, \quad \lambda = n^\mu \bar{\delta} \bar{m}_\mu\,,$$

$$\pi = n^\mu D \bar{m}_\mu\,, \quad \mu = n^\alpha \delta \bar{m}_\alpha\,, \quad \varepsilon = \tfrac{1}{2}\left(n^\mu D l_\mu - \bar{m}^\mu D m_\mu\right), \quad \alpha = \tfrac{1}{2}\left(n^\mu \bar{\delta} l_\mu - \bar{m}^\mu \bar{\delta} m_\mu\right), \tag{7}$$

$$\gamma = \tfrac{1}{2}\left(n^\mu \Delta l_\mu - \bar{m}^\mu \Delta m_\mu\right), \qquad \beta = \tfrac{1}{2}\left(n^\mu \delta l_\mu - \bar{m}^\mu \delta m_\mu\right),$$

and

$$n^\mu D l_\mu = \varepsilon + \bar{\varepsilon}\,, \qquad n^\mu \Delta l_\mu = \gamma + \bar{\gamma}\,, \qquad n^\mu \delta l_\mu = \bar{\alpha} + \beta\,,$$
$$m^\mu D \bar{m}_\mu = \varepsilon - \bar{\varepsilon}\,, \qquad m^\mu \Delta \bar{m}_\mu = \gamma - \bar{\gamma}\,, \qquad m^\mu \delta \bar{m}_\mu = -\bar{\alpha} + \beta\,. \tag{8}$$

It is easy to obtain the action of the intrinsic derivatives (1) on the null tetrad:

$$D l_\mu = (n^\nu D l_\nu) l_\mu + (l^\nu D l_\nu) n_\mu - (\bar{m}^\nu D l_\nu) m_\mu - (m^\nu D l_\nu) \bar{m}_\mu \overset{(7),(8)}{=} (\varepsilon + \bar{\varepsilon})\, l_\mu - \bar{\kappa}\, m_\mu - \kappa\, \bar{m}_\mu,$$

$etc.$, that is:

$$D\vec{t} = \begin{pmatrix} \varepsilon + \bar{\varepsilon} & 0 & -\bar{\kappa} & -\kappa \\ 0 & -(\varepsilon + \bar{\varepsilon}) & \pi & \bar{\pi} \\ \bar{\pi} & -\kappa & \varepsilon - \bar{\varepsilon} & 0 \\ \pi & -\bar{\kappa} & 0 & \bar{\varepsilon} - \varepsilon \end{pmatrix} \vec{t}\,,$$

$$\Delta\,\vec{t} = \begin{pmatrix} \gamma + \bar{\gamma} & 0 & -\bar{\tau} & -\tau \\ 0 & -(\gamma + \bar{\gamma}) & \nu & \bar{\nu} \\ \bar{\nu} & -\tau & \gamma - \bar{\gamma} & 0 \\ \nu & -\bar{\tau} & 0 & \bar{\gamma} - \gamma \end{pmatrix} \vec{t}\,,$$

(9)

$$\delta\vec{t} = \begin{pmatrix} \bar{\alpha} + \beta & 0 & -\bar{\rho} & -\sigma \\ 0 & -(\bar{\alpha} + \beta) & \mu & \bar{\lambda} \\ \bar{\lambda} & -\sigma & \beta - \bar{\alpha} & 0 \\ \mu & -\bar{\rho} & 0 & \bar{\alpha} - \beta \end{pmatrix} \vec{t}\,,$$

$$\bar{\delta}\vec{t} = \begin{pmatrix} \alpha + \bar{\beta} & 0 & -\bar{\sigma} & -\rho \\ 0 & -(\alpha + \bar{\beta}) & \lambda & \bar{\mu} \\ \bar{\mu} & -\rho & \alpha - \bar{\beta} & 0 \\ \lambda & -\bar{\sigma} & 0 & \bar{\beta} - \alpha \end{pmatrix} \vec{t}\,,$$

where $\vec{t} = \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}$. From (2):

$$\nabla_\nu l_\mu = l_{\mu;\nu} = l_\nu \Delta l_\mu + n_\nu D l_\mu - m_\nu \bar{\delta} l_\mu - \bar{m}_\nu \delta l_\mu\,, \ldots \tag{10}$$

therefore:

$$l^\nu{}_{;\nu} = \varepsilon + \bar{\varepsilon} - \rho - \bar{\rho},\ n^\nu{}_{;\nu} = \mu + \bar{\mu} - \gamma - \bar{\gamma},\ m^\nu{}_{;\nu} = \beta - \bar{\alpha} - \tau + \bar{\pi},\ \bar{m}^\nu{}_{;\nu} = \pi - \bar{\tau} - \alpha + \bar{\beta}\,. \tag{11}$$

It is simple to calculate the commutators of (1) when they are applied to an arbitrary scalar function:

$$[D,\Delta]f = (Dn^{\mu} - \Delta l^{\mu})\nabla_{\mu}f, \qquad [D,\delta]f = (Dm^{\mu} - \delta l^{\mu})\nabla_{\mu}f,$$

$$[\Delta,\delta]f = (\Delta m^{\mu} - \delta n^{\mu})\nabla_{\mu}f, \qquad [\delta,\bar{\delta}]f = (\delta\bar{m}^{\mu} - \bar{\delta}m^{\mu})\nabla_{\mu}f,$$

hence

$$[D,\Delta] = -(\gamma+\bar{\gamma})\,D - (\varepsilon+\bar{\varepsilon})\,\Delta + (\pi+\bar{\tau})\,\delta + (\bar{\pi}+\tau)\,\bar{\delta},$$

$$[D,\delta] = (\bar{\pi}-\bar{\alpha}-\beta)\,D - \kappa\,\Delta + (\bar{\rho}-\bar{\varepsilon}+\varepsilon)\,\delta + \sigma\,\bar{\delta}\,,$$

$$[\Delta,\delta] = \bar{\nu}\,D + (\bar{\alpha}+\beta-\tau)\,\Delta + (\gamma-\bar{\gamma}-\mu)\,\delta - \bar{\lambda}\,\bar{\delta}\,, \tag{12}$$

$$[\delta,\bar{\delta}] = (\mu-\bar{\mu})\,D + (\rho-\bar{\rho})\,\Delta + (\bar{\beta}-\alpha)\,\delta - (\beta-\bar{\alpha})\,\bar{\delta}\,.$$

If into (10) we employ Δl_{μ}, Dl_{μ}, $\bar{\delta}l_{\mu}$ and δl_{μ} given by (9), we obtain to $l_{\mu;\nu}$ in terms of the null tetrad which allows to introduce the concept of 'rotation coefficients'. In fact, first we order the NP tetrad in the form [2]:

$$\left(Z_{(a)}{}^{\mu}\right) = (l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}), \quad a = 1,\dots,4\,, \tag{13}$$

and we construct the matrix:

$$Z = \left(Z_{(a)(b)}\right) = \left(Z_{(a)}{}^{\mu}Z_{(b)\mu}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{14}$$

with its inverse:

$$Z^{-1} = \left(Z^{(a)(b)}\right) = Z\,, \qquad Z^{(a)(c)}Z_{(c)(b)} = \delta^{(a)}_{(b)}\,, \tag{15}$$

which permits to define the dual tetrad of (13):

$$Z^{(a)\mu} = Z^{(a)(b)}Z_{(b)}{}^{\mu} \qquad \therefore \qquad \left(Z^{(a)\mu}\right) = (n^{\mu}, l^{\mu}, -\bar{m}^{\mu}, -m^{\mu}), \tag{16}$$

then:

$$Z^{(a)}{}_{\mu}Z_{(a)\nu} = g_{\mu\nu} = l_{\mu}n_{\nu} + l_{\nu}n_{\mu} - \bar{m}_{\mu}m_{\nu} - \bar{m}_{\nu}m_{\mu}\,, \qquad Z^{(b)}{}_{\mu}Z_{(c)}{}^{\mu} = \delta^{(b)}_{(c)}\,. \tag{17}$$

The covariant derivative of (13) can be written in terms of the null tetrad:

$$Z_{(a)\mu;\nu} = \gamma_{bac}\,Z^{(b)}{}_{\mu}\,Z^{(c)}{}_{\nu}\,, \tag{18}$$

with the rotation coefficients:

$$\gamma_{abc} = -\gamma_{bac} = Z_{(b)\mu;\nu}\,Z_{(a)}{}^{\mu}\,Z_{(c)}{}^{\nu}\,, \tag{19}$$

hence (7) acquire the expressions:

$$\kappa = \gamma_{311}, \quad \sigma = \gamma_{313}, \quad \nu = \gamma_{242}, \quad \rho = \gamma_{314}, \quad \lambda = \gamma_{244}\,, \quad \tau = \gamma_{312}\,, \quad \pi = \gamma_{241}\,, \quad \mu = \gamma_{243}\,,$$

$$\varepsilon = \frac{1}{2}(\gamma_{211} - \gamma_{431}), \gamma = \frac{1}{2}(\gamma_{212} - \gamma_{432}), \alpha = \frac{1}{2}(\gamma_{214} - \gamma_{434}), \beta = \frac{1}{2}(\gamma_{213} - \gamma_{433}), \varepsilon + \bar{\varepsilon} = \gamma_{211} \tag{20}$$

$$\varepsilon - \bar{\varepsilon} = \gamma_{341}\,, \quad \gamma + \bar{\gamma} = \gamma_{212}\,, \quad \gamma - \bar{\gamma} = \gamma_{342}\,, \quad \beta + \bar{\alpha} = \gamma_{213}\,, \quad \beta - \bar{\alpha} = \gamma_{343}\,,$$

that is:

$$\gamma_{abc}: \begin{array}{ccccc} ab \setminus c: & 1 & 2 & 3 & 4 \\ .. & & & & \\ 31 & \kappa & \tau & \sigma & \rho \\ 24 & \pi & \nu & \mu & \lambda \\ 21 & \varepsilon + \bar{\varepsilon} & \gamma + \bar{\gamma} & \beta + \bar{\alpha} & \alpha + \bar{\beta} \\ 34 & \varepsilon - \bar{\varepsilon} & \gamma - \bar{\gamma} & \beta - \bar{\alpha} & \alpha - \bar{\beta} \end{array} \tag{21}$$

and the four matrix relations (9) are compacted to:

$$O_r\, \vec{t} = \begin{pmatrix} \gamma_{21r} & 0 & \gamma_{14r} & \gamma_{13r} \\ 0 & \gamma_{12r} & \gamma_{24r} & \gamma_{23r} \\ \gamma_{23r} & \gamma_{13r} & \gamma_{34r} & 0 \\ \gamma_{24r} & \gamma_{14r} & 0 & \gamma_{43r} \end{pmatrix} \vec{t}\,, \; O_1 = D, \quad O_2 = \Delta\,, \quad O_3 = \delta\,, \quad O_4 = \bar{\delta}\,. \tag{22}$$

In (19) we have covariant derivatives, however, we shall show how to determine γ_{abc} without to calculate Christoffel symbols. In fact, we introduce the quantities:

$$C_{abr} = -C_{arb} = \gamma_{abr} - \gamma_{arb} = \left(Z_{(a)\mu;\nu} - Z_{(a)\nu;\mu}\right) Z_{(b)}{}^{\nu} Z_{(r)}{}^{\mu}, \tag{23}$$

therefore:

$$C_{abr} = \left(Z_{(a)\mu,\nu} - Z_{(a)\nu,\mu}\right) Z_{(b)}{}^{\nu} Z_{(r)}{}^{\mu}, \qquad \gamma_{abr} = \frac{1}{2}(C_{abr} + C_{bra} - C_{rab}), \tag{24}$$

and

$$\kappa = C_{113}, \sigma = C_{313}, \nu = C_{242}, \rho = \frac{1}{2}(C_{314} + C_{143} + C_{413}), \beta = \frac{1}{4}(C_{213} + C_{132} + C_{312} + 2C_{343})$$
$$\mu = \frac{1}{2}(C_{243} + C_{432} + C_{342}),\; \pi = \frac{1}{2}(C_{241} + C_{412} + C_{142}),\; \varepsilon = \frac{1}{4}(C_{143} + C_{341} + C_{413} + 2C_{112}),$$
$$\lambda = C_{442}, \quad \tau = \frac{1}{2}(C_{312} + C_{123} + C_{213}), \quad \gamma = \frac{1}{4}(C_{243} + C_{342} + C_{423} + 2C_{212}), \tag{25}$$
$$\alpha = \frac{1}{4}(C_{214} + C_{142} + C_{412} + 2C_{443}).$$

Penrose [9] comments that the spin coefficients formalism was born in [10]. The next Section studies Lorentz transformations (which imply special rotations of the null tetrad and spin frame) and the corresponding changes into the spin coefficients and the NP projections of Faraday, Ricci, Lanczos and Weyl tensors.

3. Rotations of the null tetrad via Lorentz transformations

At an event of the space-time we can rotate the real tetrad [1, 11]:

$$\tilde{e}^{(a)}{}_{\mu} = L^{a}{}_{b}\, e^{(b)}{}_{\mu}\,, \tag{26}$$

with the dual tetrad $e^{(0)}{}_{\nu} = e_{(0)\nu}$ and $e^{(j)}{}_{\nu} = -e_{(j)\nu}$, $j = 1, 2, 3$. The Lorentz matrix can be generated via the quaternionic relation [1, 13-16]:

$$\boldsymbol{R}' = \boldsymbol{A}\,\boldsymbol{R}\,\overline{\boldsymbol{A}^{*}}\,, \tag{27}$$

or using the Cartan matrix [17]:

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \qquad \alpha\,\delta - \beta\,\gamma = 1\,, \tag{28}$$

into relation [1, 18-20]:

$$L^{\mu}{}_{\nu} = \sigma^{\mu}{}_{D\dot{E}}\,\sigma_{\nu}{}^{A\dot{C}}\,B^{-1D}{}_{A}\,B^{\dagger-1}{}_{\dot{C}}{}^{\dot{E}}\,, \tag{29}$$

which implies the expressions [21-24]:

$$\begin{aligned}
&L^{0}{}_{0} = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad L^{1}{}_{0} = \tfrac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, \quad L^{2}{}_{0} = \tfrac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc\,,\\
&L^{0}{}_{1} = \tfrac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \quad L^{1}{}_{1} = \tfrac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \quad L^{2}{}_{1} = \tfrac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,\\
&L^{0}{}_{2} = \tfrac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \quad L^{1}{}_{2} = \tfrac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \quad L^{2}{}_{2} = \tfrac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc\,,\\
&L^{0}{}_{3} = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^{1}{}_{3} = \tfrac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, \quad L^{2}{}_{3} = \tfrac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc\,,\\
&L^{3}{}_{0} = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^{3}{}_{1} = \tfrac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \quad L^{3}{}_{2} = \tfrac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc\,,\\
&L^{3}{}_{3} = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad \alpha\delta - \beta\gamma = 1,
\end{aligned} \tag{30}$$

where *cc* means the complex conjugate of all the previous terms. The components (30) verify the property:

$$L^{\tau}{}_{\mu}\, g_{\tau\lambda}\, L^{\lambda}{}_{\nu} = g_{\mu\nu}\,, \qquad \tilde{g} = Diag(1,-1,-1,-1). \tag{31}$$

If in (30) we make the changes [$\tau = \frac{1}{\sqrt{1-\bar{\Gamma}\bar{\Omega}}}$]:

$$\alpha = -\tau \exp\left(-\frac{A+iB}{2}\right), \quad \beta = \tau \exp\left(-\frac{A+iB}{2}\right)\bar{\Gamma}, \quad \gamma = \tau \exp\left(\frac{A+iB}{2}\right)\bar{\Omega}, \quad \delta = -\tau \exp\left(\frac{A+iB}{2}\right), \tag{32}$$

such that A, B are real and Γ, Ω are complex with $\Gamma\Omega \neq 1$ (it is simple to see that (32) respects the requirement $\alpha\delta - \beta\gamma = 1$), we obtain [25-27] [$\mathrm{Q} = \frac{1}{2|1-\Gamma\Omega|}$]:

$$L^{0}{}_{0} = \mathrm{Q}\,[e^{\mathrm{A}}(1+\Omega\bar{\Omega}) + e^{-\mathrm{A}}(1+\Gamma\bar{\Gamma})], \qquad L^{1}{}_{0} = -\mathrm{Q}\left[e^{i\mathrm{B}}(\Gamma+\bar{\Omega}) + cc\right],$$

$$L^{2}{}_{0} = -i\mathrm{Q}\left[e^{-i\mathrm{B}}(\Omega+\bar{\Gamma}) - cc\right], \qquad L^{0}{}_{1} = -\mathrm{Q}\,[e^{\mathrm{A}}(\Omega+\bar{\Omega}) + e^{-\mathrm{A}}(\Gamma+\bar{\Gamma})],$$

$$L^{1}{}_{1} = \mathrm{Q}\left[e^{i\mathrm{B}}(1+\bar{\Omega}\Gamma) + cc\right], \qquad L^{2}{}_{1} = i\mathrm{Q}\left[e^{-i\mathrm{B}}(1+\Omega\bar{\Gamma}) - cc\right],$$

$$L^{0}{}_{2} = i\mathrm{Q}\,[e^{\mathrm{A}}(\bar{\Omega}-\Omega) + e^{-\mathrm{A}}(\Gamma-\bar{\Gamma})], \qquad L^{1}{}_{2} = i\mathrm{Q}\left[e^{i\mathrm{B}}(1-\bar{\Omega}\Gamma) - cc\right],$$

$$L^{2}{}_{2} = -\mathrm{Q}\left[e^{-i\mathrm{B}}(\Omega\bar{\Gamma}-1) + cc\right], \qquad L^{0}{}_{3} = \mathrm{Q}\,[e^{\mathrm{A}}(\Omega\bar{\Omega}-1) + e^{-\mathrm{A}}(1-\Gamma\bar{\Gamma})], \tag{33}$$

$$L^{1}{}_{3} = \mathrm{Q}\left[e^{i\mathrm{B}}(\Gamma-\bar{\Omega}) + cc\right], \qquad L^{2}{}_{3} = i\mathrm{Q}\left[e^{-i\mathrm{B}}(\bar{\Gamma}-\Omega) - cc\right],$$

$$L^{3}{}_{0} = \mathrm{Q}\,[-e^{\mathrm{A}}(1+\Omega\bar{\Omega}) + e^{-\mathrm{A}}(1+\Gamma\bar{\Gamma})], \qquad L^{3}{}_{1} = \mathrm{Q}\,[e^{\mathrm{A}}(\Omega+\bar{\Omega}) - e^{-\mathrm{A}}(\Gamma+\bar{\Gamma})],$$

$$L^{3}{}_{2} = i\mathrm{Q}\,[e^{\mathrm{A}}(\Omega-\bar{\Omega}) + e^{-\mathrm{A}}(\Gamma-\bar{\Gamma})], \qquad L^{3}{}_{3} = \mathrm{Q}\,[e^{\mathrm{A}}(1-\Omega\bar{\Omega}) + e^{-\mathrm{A}}(1-\Gamma\bar{\Gamma})],$$

whose application in (26) gives the most general rotation of the null tetrad and spin frame [3, 4, 8, 28, 29]:

$$\tilde{l}^{\mu} = 2\mathrm{Q}\, e^{\mathrm{A}}(l^{\mu} + \Omega\bar{\Omega}\, n^{\mu} + \bar{\Omega}\, m^{\mu} + \Omega\, \bar{m}^{\mu}), \quad \tilde{n}^{\mu} = 2\mathrm{Q}\, e^{-\mathrm{A}}(n^{\mu} + \Gamma\bar{\Gamma}\, l^{\mu} + \Gamma\, m^{\mu} + \bar{\Gamma}\, \bar{m}^{\mu}),$$

$$\tilde{m}^{\mu} = 2\mathrm{Q}\, e^{-i\mathrm{B}}(\bar{\Gamma}\, l^{\mu} + \Omega\, n^{\mu} + m^{\mu} + \bar{\Gamma}\Omega\, \bar{m}^{\mu}), \quad \tilde{\bar{m}}^{\mu} = 2\mathrm{Q}\, e^{i\mathrm{B}}\,(\Gamma\, l^{\mu} + \bar{\Omega}\, n^{\mu} + \bar{m}^{\mu} + \Gamma\bar{\Omega}\, m^{\mu}) \tag{34}$$

$$\tilde{o}^{C} = \frac{1}{\sqrt{1-\Gamma\Omega}} e^{\frac{\mathrm{A}-i\mathrm{B}}{2}}(o^{C} + \Omega\, \iota^{C}), \qquad \tilde{\iota}^{C} = \frac{1}{\sqrt{1-\Gamma\Omega}} e^{\frac{-\mathrm{A}+i\mathrm{B}}{2}}(\iota^{C} + \Gamma\, o^{C}),$$

hence in the literature it is natural to employ three types of rotations:

Class I: $\Omega = 0.$ l^μ preserves its direction.

$$\tilde{l}^\mu = e^{A}\, l^\mu, \qquad \tilde{n}^\mu = e^{-A}(n^\mu + \Gamma\bar{\Gamma}\, l^\mu + \Gamma\, m^\mu + \bar{\Gamma}\,\bar{m}^\mu),$$

$$\tilde{m}^\mu = e^{-iB}(\bar{\Gamma}\, l^\mu + m^\mu), \qquad \tilde{\bar{m}}^\mu = e^{iB}(\Gamma\, l^\mu + \bar{m}^\mu),$$

Class II: $\Gamma = 0.$ n^ν maintains its direction. (35)

$$\tilde{l}^\nu = e^{A}(l^\nu + \Omega\bar{\Omega}\, n^\nu + \bar{\Omega}\, m^\nu + \Omega\,\bar{m}^\nu), \qquad \tilde{n}^\nu = e^{-A}\, n^\nu,$$

$$\tilde{m}^\nu = e^{-iB}(\Omega\, n^\nu + m^\nu), \qquad \tilde{\bar{m}}^\nu = e^{iB}(\bar{\Omega}\, n^\nu + \bar{m}^\nu),$$

Class III: $\Omega = \Gamma = 0.$

$$\tilde{l}^\mu = e^{A}\, l^\mu\,, \qquad \tilde{n}^\mu = e^{-A}\, n^\mu\,, \qquad \tilde{m}^\mu = e^{-iB}\, m^\mu\,, \qquad \tilde{\bar{m}}^\mu = e^{iB}\,\bar{m}^\mu\,,$$

which are important because it is usual to realize rotations to align l^μ or/and n^μ with the principal directions for the conformal tensor [8, 30-32] or for the Faraday electromagnetic tensor [33], this generates great simplification in many relativistic calculations.

It is useful to indicate the changes of diverse quantities when the null tetrad experiments these special rotations:

<u>NP components of the Faraday tensor:</u>

Class I: $\tilde{\phi}_0 = e^{A-iB}\phi_0\,, \quad \tilde{\phi}_1 = \phi_1 + \Gamma\phi_0\,, \quad \tilde{\phi}_2 = e^{-A+iB}(\phi_2 + 2\Gamma\phi_1 + \Gamma^2\phi_0).$

Class II: $\tilde{\phi}_0 = e^{A-iB}(\phi_0 + 2\Omega\phi_1 + \Omega^2\phi_2), \quad \tilde{\phi}_1 = \phi_1 + \Omega\phi_2\,, \quad \tilde{\phi}_2 = e^{-A+iB}\phi_2.$ (36)

Class III: $\tilde{\phi}_0 = e^{A-iB}\phi_0\,, \quad \tilde{\phi}_1 = \phi_1\,, \quad \tilde{\phi}_2 = e^{-A+iB}\phi_2.$

<u>NP projections of the Ricci tensor without trace:</u>

Class I: $\tilde{\phi}_{00} = e^{2A}\phi_{00}\,, \ \tilde{\phi}_{01} = e^{A-iB}(\phi_{01} + \bar{\Gamma}\phi_{00}), \quad \tilde{\phi}_{02} = e^{-2iB}(\phi_{02} + 2\bar{\Gamma}\phi_{01} + \bar{\Gamma}^2\phi_{00}),$

$$\tilde{\phi}_{11} = \phi_{11} + \Gamma\bar{\Gamma}\phi_{00} + \Gamma\phi_{01} + \bar{\Gamma}\phi_{10},$$

$$\tilde{\phi}_{12} = e^{-A-iB}(\phi_{12} + \Gamma\phi_{02} + 2\bar{\Gamma}\phi_{11} + 2\Gamma\bar{\Gamma}\phi_{01} + \bar{\Gamma}^2\phi_{10} + \Gamma\bar{\Gamma}^2\phi_{00}),$$

$$\tilde{\phi}_{22} = e^{-2A}(\phi_{22} + 2\Gamma\phi_{12} + 2\bar{\Gamma}\phi_{21} + 4\Gamma\bar{\Gamma}\phi_{11} + \Gamma^2\phi_{02} + \bar{\Gamma}^2\phi_{20} + 2\Gamma^2\bar{\Gamma}\phi_{01} + 2\Gamma\bar{\Gamma}^2\phi_{10} + \Gamma^2\bar{\Gamma}^2\phi_{00}).$$

Class II: $\tilde{\phi}_{22} = e^{-2A}\phi_{22}, \tilde{\phi}_{12} = e^{-A-iB}(\phi_{12} + \Omega\phi_{22}), \tilde{\phi}_{02} = e^{-2iB}(\phi_{02} + 2\Omega\phi_{12} + \Omega^2\phi_{22})$

$$\tilde{\phi}_{11} = \phi_{11} + \bar{\Omega}\phi_{12} + \Omega\phi_{21} + \Omega\bar{\Omega}\phi_{22} \tag{37}$$

$$\tilde{\phi}_{01} = e^{A-iB}(\phi_{01} + 2\Omega\phi_{11} + \bar{\Omega}\phi_{02} + \Omega^2\phi_{21} + 2\Omega\bar{\Omega}\phi_{12} + \Omega^2\bar{\Omega}\phi_{22})$$

$$\tilde{\phi}_{00} = e^{2A}(\phi_{00} + 2\bar{\Omega}\phi_{01} + 2\Omega\phi_{10} + \Omega^2\phi_{20} + \bar{\Omega}^2\phi_{02} + 4\Omega\bar{\Omega}\phi_{11} + \Omega^2\bar{\Omega}^2\phi_{22} + 2\Omega^2\bar{\Omega}\phi_{21} + 2\Omega\bar{\Omega}^2\phi_{12}).$$

Class III: $\tilde{\phi}_{00} = e^{2A}\phi_{00}, \quad \tilde{\phi}_{01} = e^{A-iB}\phi_{01}, \quad \tilde{\phi}_{02} = e^{-2iB}\phi_{02},$

$$\tilde{\phi}_{11} = \phi_{11}, \quad \tilde{\phi}_{12} = e^{-A-iB}\phi_{12}, \quad \tilde{\phi}_{22} = e^{-2A}\phi_{22}.$$

<u>NP components of the conformal tensor:</u>

Class I: $\tilde{\psi}_0 = e^{2(A-iB)}\psi_0, \ \tilde{\psi}_1 = e^{A-iB}(\psi_1 + \Gamma\psi_0), \tilde{\psi}_3 = e^{-A+iB}(\psi_3 + 3\Gamma\psi_2 + 3\Gamma^2\psi_1 + \Gamma^3\psi_0)$

$$\tilde{\psi}_2 = \psi_2 + 2\Gamma\psi_1 + \Gamma^2\psi_0, \quad \tilde{\psi}_4 = e^{2(-A+iB)}(\psi_4 + 4\Gamma\psi_3 + 6\Gamma^2\psi_2 + 4\Gamma^3\psi_1 + \Gamma^4\psi_0).$$

Class II: $\tilde{\psi}_1 = e^{A-iB}(\psi_1 + 3\Omega\psi_2 + 3\Omega^2\psi_3 + \Omega^3\psi_4), \quad \tilde{\psi}_3 = e^{-A+iB}(\psi_3 + \Omega\psi_4),$ (38)

$$\tilde{\psi}_2 = \psi_2 + 2\Omega\psi_3 + \Omega^2\psi_4, \quad \tilde{\psi}_4 = e^{2(-A+iB)}\psi_4,$$

$$\tilde{\psi}_0 = e^{2(A-iB)}(\psi_0 + 4\Omega\psi_1 + 6\Omega^2\psi_2 + 4\Omega^3\psi_3 + \Omega^4\psi_4).$$

Class III: $\tilde{\psi}_0 = e^{2(A-iB)}\psi_0, \quad \tilde{\psi}_1 = e^{A-iB}\psi_1, \quad \tilde{\psi}_2 = \psi_2,$

$$\tilde{\psi}_3 = e^{-A+iB}\psi_3, \quad \tilde{\psi}_4 = e^{2(-A+iB)}\psi_4,$$

and in their deduction are useful the relations:

$$C_{(1)(4)(1)(2)} + C_{(1)(4)(4)(3)} = C_{(2)(4)(2)(3)} = C_{(1)(4)(1)(3)} = C_{(4)(2)(4)(1)} = 0, \quad \psi_1 = C_{(4)(3)(1)(3)},$$
(39)
$$\psi_3 = C_{(3)(4)(2)(4)}, \quad \psi_2 + \bar{\psi}_2 = C_{(1)(2)(1)(2)} = C_{(3)(4)(3)(4)}, \quad \bar{\psi}_2 - \psi_2 = C_{(1)(2)(3)(4)}.$$

NP projections of the Lanczos tensor:

Class I: $\widetilde{\Omega}_0 = e^{2A-iB}\Omega_0$, $\widetilde{\Omega}_1 = e^{A}(\Gamma\Omega_0 + \Omega_1)$, $\widetilde{\Omega}_2 = e^{iB}(\Omega_2 + 2\Gamma\Omega_1 + \Gamma^2\Omega_0)$,

$\widetilde{\Omega}_4 = e^{A-2iB}(\Omega_4 + \bar{\Gamma}\Omega_0)$, $\widetilde{\Omega}_3 = e^{-A+2iB}(\Omega_3 + 3\Gamma\Omega_2 + 3\Gamma^2\Omega_1 + \Gamma^3\Omega_0)$,

$$\widetilde{\Omega}_5 = e^{-iB}(\Omega_5 + \Gamma\Omega_4 + \bar{\Gamma}\Omega_1 + \Gamma\bar{\Gamma}\Omega_0),$$

$$\widetilde{\Omega}_6 = e^{-A}(\Omega_6 + 2\Gamma\Omega_5 + \Gamma^2\Omega_4 + \bar{\Gamma}\Omega_2 + 2\Gamma\bar{\Gamma}\Omega_1 + \Gamma^2\bar{\Gamma}\Omega_0)$$

$$\widetilde{\Omega}_7 = e^{-2A+iB}(\Omega_7 + 3\Gamma\Omega_6 + 3\Gamma^2\Omega_5 + \Gamma^3\Omega_4 + \bar{\Gamma}\Omega_3 + 3\Gamma\bar{\Gamma}\Omega_2 + 3\Gamma^2\bar{\Gamma}\Omega_1 + \Gamma^3\bar{\Gamma}\Omega_0).$$

Class II: $\widetilde{\Omega}_3 = e^{-A+2iB}(\Omega_3 + \bar{\Omega}\Omega_7)$, $\widetilde{\Omega}_5 = e^{-iB}(\Omega_5 + 2\Omega\Omega_6 + \Omega^2\Omega_7)$, $\widetilde{\Omega}_6 = e^{-A}(\Omega_6 + \Omega\Omega_7)$ (40)

$\widetilde{\Omega}_2 = e^{iB}(\Omega_2 + \Omega\Omega_3 + \bar{\Omega}\Omega_6 + \Omega\bar{\Omega}\Omega_7)$, $\widetilde{\Omega}_4 = e^{A-2iB}(\Omega_4 + 3\Omega\Omega_5 + 3\Omega^2\Omega_6 + \Omega^3\Omega_7)$,

$\widetilde{\Omega}_1 = e^{A}(\Omega_1 + 2\Omega\Omega_2 + \Omega^2\Omega_3 + \bar{\Omega}\Omega_5 + 2\Omega\bar{\Omega}\Omega_6 + \bar{\Omega}\Omega^2\Omega_7)$, $\widetilde{\Omega}_7 = e^{-2A+iB}\Omega_7$,

$$\widetilde{\Omega}_0 = e^{2A-iB}(\Omega_0 + 3\Omega\Omega_1 + 3\Omega^2\Omega_2 + \Omega^3\Omega_3 + \bar{\Omega}\Omega_4 + 3\Omega\bar{\Omega}\Omega_5 + 3\Omega^2\bar{\Omega}\Omega_6 + \Omega^3\bar{\Omega}\Omega_7).$$

Class III: $\widetilde{\Omega}_0 = e^{2A-iB}\Omega_0$, $\widetilde{\Omega}_1 = e^{A}\Omega_1$, $\widetilde{\Omega}_2 = e^{iB}\Omega_2$, $\widetilde{\Omega}_3 = e^{-A+2iB}\Omega_3$,

$\widetilde{\Omega}_4 = e^{A-2iB}\Omega_4$, $\widetilde{\Omega}_5 = e^{-iB}\Omega_5$, $\widetilde{\Omega}_6 = e^{-A}\Omega_6$, $\widetilde{\Omega}_7 = e^{-2A+iB}\Omega_7$,

where we used the expressions:

$$K_{(1)(2)(1)} = \Omega_1 + \bar{\Omega}_1\,, \qquad K_{(1)(4)(4)} = \Omega_6 + \bar{\Omega}_6\,, \qquad K_{(1)(2)(4)} = \Omega_2 + \bar{\Omega}_5\,,$$

$$K_{(4)(3)(1)} = \Omega_1 - \bar{\Omega}_1\,, \qquad K_{(4)(3)(2)} = \Omega_6 - \bar{\Omega}_6\,, \qquad K_{(4)(3)(4)} = \Omega_2 - \bar{\Omega}_5\,. \qquad (41)$$

Spin coefficients [27, 34]:

Class I: $\tilde{\kappa} = e^{2A-iB}\kappa$, $\tilde{\rho} = e^{A}(\rho + \Gamma\kappa)$, $\tilde{\tau} = e^{-iB}(\tau + \Gamma\sigma + \bar{\Gamma}\varrho + \Gamma\bar{\Gamma}\kappa)$, $\tilde{\sigma} = e^{A-2iB}(\sigma + \bar{\Gamma}\kappa)$

$\tilde{\alpha} = e^{iB}\left[\frac{1}{2}\left(\Gamma D + \bar{\delta}\right)(A - iB) + \alpha + \Gamma(\varepsilon + \rho + \Gamma\kappa)\right]$, $\tilde{\pi} = e^{iB}(D\Gamma + \pi + 2\Gamma\varepsilon + \Gamma^2\kappa)$,

$\tilde{\beta} = e^{-iB}\left[\frac{1}{2}(\bar{\Gamma}D + \delta)(A - iB) + \beta + \Gamma(\sigma + \bar{\Gamma}\kappa) + \bar{\Gamma}\varepsilon\right]$, $\tilde{\varepsilon} = e^{A}\left[\frac{1}{2}D(A - iB) + \varepsilon + \Gamma\kappa\right]$

$$\tilde{\lambda} = e^{-A+2iB}\left[\bar{\delta}\Gamma + \Gamma(D\Gamma + \pi + 2\alpha) + \lambda + \Gamma^2(\rho + 2\varepsilon + \Gamma\kappa)\right],$$

$$\tilde{\mu} = e^{-A}[\delta\Gamma + \bar{\Gamma}(D\Gamma + \pi + 2\Gamma\varepsilon + \Gamma^2\kappa) + \mu + \Gamma(2\beta + \Gamma\sigma)],$$

$$\tilde{\gamma} = e^{-A}\left[\frac{1}{2}\left(\Delta + \Gamma\delta + \bar{\Gamma}\bar{\delta} + \Gamma\bar{\Gamma}D\right)(A - iB) + \gamma + \Gamma(\tau + \beta + \Gamma\sigma) + \bar{\Gamma}(\alpha + \Gamma^2\kappa) + \Gamma\bar{\Gamma}(\varepsilon + \rho)\right],$$

$$\tilde{\nu} = e^{-2A+iB}[\Delta\Gamma + \Gamma(\delta\Gamma + \bar{\Gamma}D\Gamma + 2\gamma + \mu) + \bar{\Gamma}\left(\bar{\delta}\Gamma + \lambda\right) + \nu + \Gamma^2(2\beta + \tau + \Gamma\sigma) +$$
$$+\Gamma\bar{\Gamma}(2\alpha + \pi + \Gamma(2\varepsilon + \rho) + \Gamma^2\kappa)]\,. \tag{42}$$

Class II: $\tilde{\nu} = e^{-2A+iB}\nu, \quad \tilde{\mu} = e^{-A}(\mu + \Omega\nu), \quad \tilde{\lambda} = e^{-A+2iB}(\lambda + \bar{\Omega}\nu),$

$$\tilde{\pi} = e^{iB}(\pi + \Omega\lambda + \bar{\Omega}\mu + \Omega\bar{\Omega}\nu),$$

$$\tilde{\alpha} = e^{iB}\left[\frac{1}{2}\left(\bar{\Omega}\Delta + \bar{\delta}\right)(A - iB) + \alpha + \Omega\lambda + \bar{\Omega}(\gamma + \Omega\nu)\right], \qquad \tilde{\tau} = e^{iB}(-\Delta\Omega + \tau + 2\Omega\gamma + \Omega^2\nu),$$

$$\tilde{\beta} = e^{-iB}\left[\frac{1}{2}(\Omega\Delta + \delta)(A - iB) + \beta + \Omega(\gamma + \mu) + \Omega^2\nu\right], \quad \tilde{\gamma} = e^{-A}\left[\frac{1}{2}\Delta(A - iB) + \gamma + \Omega\nu\right],$$

$$\tilde{\rho} = e^{A}\left[-\bar{\delta}\Omega + \bar{\Omega}(-\Delta\Omega + \tau + 2\Omega\gamma) + \rho + \Omega^2(\lambda + \bar{\Omega}\nu)\right],$$

$$\tilde{\sigma} = e^{A-2iB}[-\delta\Omega + \Omega(-\Delta\Omega + \tau + 2\beta) + \sigma + \Omega^2(\mu + 2\gamma + \Omega\nu)],$$

$$\tilde{\varepsilon} = e^{A}\left[\frac{1}{2}\left(D + \Omega\bar{\delta} + \bar{\Omega}\delta + \Omega\bar{\Omega}\Delta\right)(A - iB) + \varepsilon + \Omega(\pi + \alpha - \Omega\lambda) + \Omega\bar{\Omega}(\gamma + \mu + \Omega\nu)\right],$$

$$\tilde{\kappa} = e^{2A-iB}[-D\Omega + \Omega\left(-\bar{\delta}\Omega - \bar{\Omega}\Delta\Omega + 2\varepsilon + \rho\right) + \bar{\Omega}(-\delta\Omega + \sigma) + \kappa + \Omega^2(2\alpha + \pi + \Omega\lambda) +$$
$$+\Omega\bar{\Omega}(2\beta + \tau + \Omega(2\gamma + \mu) + \Omega^2\nu)].$$

Class III: $\tilde{\kappa} = e^{2A-iB}\kappa, \quad \tilde{\rho} = e^{A}\rho, \quad \tilde{\sigma} = e^{A-2iB}\sigma, \quad \tilde{\tau} = e^{-iB}\tau, \quad \tilde{\pi} = e^{iB}\pi,$

$$\tilde{\nu} = e^{-2A+iB}\nu, \quad \tilde{\mu} = e^{-A}\mu,$$

$$\tilde{\lambda} = e^{-A+2iB}\lambda, \;\; \tilde{\varepsilon} = e^{A}\left[\frac{1}{2}D(A - iB) + \varepsilon\right], \; \tilde{\gamma} = e^{-A}\left[\frac{1}{2}\Delta(A - iB) + \gamma\right],$$

$$\tilde{\alpha} = e^{iB}\left[\frac{1}{2}\bar{\delta}(A - iB) + \alpha\right], \quad \tilde{\beta} = e^{-iB}\left[\frac{1}{2}\delta(A - iB) + \beta\right].$$

In other work we will apply the material of this paper and from [1, 2] to obtain the NP equations, and the spinor and NP versions of the Bianchi identities and Weyl-Lanczos equations.

References

1. A. Hernández G., J. López-Bonilla, R. López-Vázquez, G. Pérez-Teruel, *Faraday tensor and Maxwell spinor*, Prespacetime Journal **6**, No. 2 (2015) 88-107
2. J. López-Bonilla, R. López-Vázquez, J. Morales, G. Ovando, *Maxwell, Lanczos and Weyl spinors,* Prespacetime Journal **6**, No. 6 (2015) 509-520
3. E. Newman, R. Penrose, *An approach to gravitational radiation by a method of spin coefficients,* J. Math. Phys. **3**, No. 3 (1962) 566-578
4. S. J. Campbell, J. Wainwright, *Algebraic computing and the Newman-Penrose formalism in general relativity,* Gen. Rel. Grav. **8**, No. 12 (1977) 987-1001
5. M. Carmeli, *Classical fields: general relativity and gauge theory,* John Wiley & Sons, New York (1982)
6. S. Chandrasekhar, *The mathematical theory of black holes,* Oxford University Press (1983)
7. P. O'Donnell, *Introduction to 2-spinors in general relativity,* World Scientific, Singapore (2003)
8. D. Kramer, H. Stephani, M. MacCallum, E. Herlt, *Exact solutions of Einstein's field equations,* Cambridge University Press (1980)
9. R. Penrose, *Lecture in honour of Leopold Infeld. Spinors in general relativity,* Acta Physica Polonica B**30**, No. 10 (1999) 2979-2987
10. P. Jordan, J. Ehlers, R. Sachs, *Beiträge zur theorie der reinen gravitationstrahlung,* Akad. Wiss. Lit. Mainz, Math-Naturwiss. Kl, No. 1 (1961)
11. R. Linares, J. López-Bonilla, J. Quino, *Real tetrads and Lorentz transformations,* Bol. Soc. Cub. Mat. Comp. **8**, No. 2 (2010) 141-153
12. F. Klein, A. Sommerfeld, *Über die theorie des Kreisles,* Teubner, Leipzig (1910) [Johnson Reprint, New York (1965)]
13. L. Silberstein, *Quaternionic form of relativity,* Phil. Mag. **23**, No. 137 (1912) 790-809
14. C. Lanczos, *The variational principles of mechanics,* University of Toronto Press (1970)
15. J. L. Synge, *Quaternions, Lorentz transformations and the Conway-Dirac-Eddington matrices,* Comm. Dublin Inst. Adv. Stud. Series A, No. 21 (1972)
16. B. Carvajal-Gámez, I. Guerrero M., J. López-Bonilla, *Quaternions, 2x2 complex matrices and Lorentz transformations*, Bibechana (Nepal) **12** (2015) 30-34
17. E. Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane,* Bull. Soc. Math. de France **41** (1913) 53-96
18. O. Veblen, *Geometry of two-component spinors,* Proc. Nat. Acad. Sci. USA **19** (1933) 462-474
19. B. Carvajal G., M. Galaz, J. López-Bonilla, *On the Lorentz matrix in terms of Infeld-van der Waerden symbols,* Scientia Magna **3**, No. 3 (2007) 82-84
20. Z. Ahsan, J. López-Bonilla, B. Man Tuladhar, *Lorentz transformations via Pauli matrices,* J. of Advances in Natural Sciences **2**, No. 1 (2014) 49-51
21. J. Aharoni, *The special relativity*, Clarendon Press, Oxford (1959)
22. J. L. Synge, *Relativity: the special theory,* North-Holland Pub, Amsterdam (1965)
23. J. López-Bonilla, J. Morales, G. Ovando, *On the homogeneous Lorentz transformations,* Bull. Allahabad Math. Soc. **17** (2002) 53-58
24. I. Guerrero-Moreno, G. Leija H., J. López-Bonilla, *Rotations in Minkowski spacetime*, Bull. of Soc. for Mathematical Services & Standards (India) **3**, No. 3 (2014) 17-27
25. R. K. Sachs, *Gravitational waves in general relativity VI: The outgoing radiation condition,* Proc. Roy. Soc. London A**264** (1961) 309-337

26. P. J. Greenberg, J. P. Knauer, Stud. Appl. Math. **53** (1974) 165
27. M. Carmeli, *Group theory and general relativity*, McGraw-Hill, New York (1977)
28. P. J. Greenberg, *The algebra of the Riemann curvature tensor in general relativity: Preliminaries,* Stud. Appl. Math. **51** (1972) 277
29. S. Chandrasekhar, *An introduction to the theory of the Kerr metric and its perturbations,* in 'General Relativity', Eds. S. W. Hawking, W. Israel, Cambridge University Press (1979) Chap. 7
30. E. Cartan, *Sur les espaces conformes généralisés et l'Univers optique,* C. R. Acad. Sci. Paris **174** (1922) 857-859
31. R. Penrose, *A spinor approach to general relativity,* Ann. of Phys. **10**, No. 2 (1960) 171-201
32. R. Debever, *Le rayonnement gravitationnel: Le tenseur de Riemann en relativité générale,* Cah. de Phys. **168-169** (1967) 303-349
33. J. López-Bonilla, R. Meneses, M. Turgut, *Faraday tensor: its algebraic structure*, J. Vect. Rel. **4**, No. 3 (2009) 23-32
34. M. Carmeli, M. Kaye, *Transformation laws of the Newman-Penrose field variables,* Ann. of Phys. **99**, No. 1 (1976) 188-195