Article

Bianchi Type-III Massive String Cosmological Models with Vacuum Energy Density in General Relativity

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Abstract

A massive string cosmological model with vacuum energy density in spatially homogeneous and anisotropic Bianchi Type III space-time is investigated. To get the deterministic model of the universe, we assume that the shear (σ) is proportional to the expansion (θ) which leads to $B = C^n$ and $\Lambda \sim 1/R^3$ where B and C are metric potentials, n is a constant and R is scale factor. The models start with a big-bang and the expansion decreases with time. The energy density (ρ) and string tension density are initially large but decreases with time. The model in general represents anisotropic space and for large values of T, it isotropizes. The first model represents decelerating and accelerating phases of the universe when $2n+1 < 0$ and $2n+1 > 0$ respectively while the special model represents decelerating phase of universe.

Keywords: Bianchi Type III, massive string, cosmological, vacuum energy density.

1. Introduction

Cosmic strings have received a considerable interest in cosmology because it is believed that these strings give rise to density perturbation leading to the formation of galaxies (Zel'dovich[1]). In the early universe (string-dominated era), the strings produce fluctuations in the density of particles. We may speculate that as the strings vanish and particles become important then the fluctuations grow in such a way that finally we shall end up with galaxies. The presence of strings in early universe can be explained using grand unified field (GUT) theories as discussed by Kibble[2], Everet[3] and Vilenkin[4]. These strings have stress energy and classified as geometrical and massive strings. The pioneering work in the formulation of the energy-momentum tensor for massive string is due to Letelier[5] who explained that the massive strings are formed by geometric strings (Nambu string[6]) with particles attached along its extension. Letelier[7] first used this idea in obtaining some cosmological solutions for massive strings using Bianchi Type I and Kantowski-Sachs space-time. A number of string cosmological models in different Bianchi space-times are studied by Banerjee et al.[8], Chakraborty[9], Tikekar and Patel[10,11], Bali et al.[12,13] in different contexts.

The relevance of cosmological constant (vacuum energy density) related with the observations are given by Zel'dovich[14], Krauss and Turner[15]. Two independent groups led by Riess et al.[16]

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and Perlmutter et al.[17] investigated that the universe is not only expanding but it is also accelerating.

This discovery provided the first direct evidence that Λ is non zero $\Lambda \sim 1.7 \times 10^{-121}$ Planck unit.

Thus the present time accelerating behaviour of universe is due to dominance of vacuum energy density. Many authors viz. Bertolami[18], Hu[19], Arbab[20], Bali and Jain[21], Ram and Verma[22], Barrow and Shaw[23] have investigated cosmological models with decaying vacuum energy density in different contexts. Wang[24] investigated Bianchi Type III massive string cosmological model with magnetic field.

In this paper, we have investigated Bianchi Type III massive string cosmological model with decaying vacuum energy density in General Relativity. To get the deterministic model of the universe, we have used the condition that shear (σ) is proportional to expansion (θ) which leads to $B = C^n$ and $\Lambda \sim 1/R^3$ where B and C are metric potentials, n is a constant, Λ the vacuum energy density and R the scale factor. The physical aspects of the models are also discussed.

2. The Metric and Field Equations

We consider Bianchi Type III space-time in the form as

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2ax} dy^2 + C^2 dz^2 \tag{1}$$

where a is constant. A, B and C are the metric potentials and are functions of 't' only.

The energy momentum tensor for a cloud of string dust is given by Letelier[5]

$$T_i^j = \rho v_i v^j - \lambda x_i x^j \tag{2}$$

where v_i and x_i satisfy condition

$$v^i v_i = -x^i x_i = -1, x^i v_i = 0 \tag{3}$$

where ρ is the proper energy density for a cloud string with particles attached to them, λ the string tension density, v^i the four-velocity of the particles and x^i is a unit space like vector representing the direction of string.

If the particle density of the configuration is denoted by ρ_p, then we have

$$\rho = \rho_p + \lambda \tag{4}$$

Einstein's field equations for a system of strings with vacuum energy density are given by

$$R_i^j - \frac{1}{2} R g_i^j - \Lambda g_i^j = -T_i^j \tag{5}$$

where R_i^j is the Ricci tensor; R = $g^{ij} R_{ij}$ is the Ricci scalar.

In a comoving co-ordinate system, we have

$$v^i = (0,0,0,1), x^i = (0,0,1/C,0) \tag{6}$$

By equation (2), we have

$$T_1^1 = 0 = T_2^2; T_3^3 = -\lambda; T_4^4 = -\rho \tag{7}$$

The field equation (5) with equation (7) subsequently leads to the following system of equations:

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = \Lambda \tag{8}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} = \Lambda \tag{9}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{a^2}{A^2} = \Lambda + \lambda \tag{10}$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - \frac{a^2}{A^2} = \Lambda + \rho \tag{11}$$

$$\frac{A_4}{A} - \frac{B_4}{B} = 0 \tag{12}$$

where the suffix '4' denotes ordinary differentiation with respect to 't'.

The particle density ρ_p is given by

$$\rho_p = \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - \frac{B_{44}}{B} - \frac{A_{44}}{A} \tag{13}$$

in accordance with equation (4).

The velocity v^i as specified by equation (6) is irrotational. The expansion (θ) and components of shear tensor (σ_i^j) after using equation (12) are given by

$$\theta = \frac{2B_4}{B} + \frac{C_4}{C} \tag{14}$$

$$\sigma_1^1 = \frac{1}{3}\left[\frac{B_4}{B} - \frac{C_4}{C}\right] \tag{15}$$

$$\sigma_2^2 = \frac{1}{3}\left[\frac{B_4}{B} - \frac{C_4}{C}\right] \tag{16}$$

$$\sigma_3^3 = \frac{2}{3}\left[\frac{C_4}{C} - \frac{B_4}{B}\right] \tag{17}$$

$$\sigma_4^4 = 0 \tag{18}$$

Therefore, shear is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}[(\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2 + (\sigma_4^4)^2]$$

which leads to

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) \text{ using equation (12)} \tag{19}$$

The field equations (8) – (12) are a system of five equations with six unknown parameters A, B, C, ρ, λ and Λ. One additional constraint relating these parameters is required to obtain explicit solutions of the system.

Following Bali and Jain[21], we assume that the expansion (θ) in the model is proportional to the shear (σ), which is physically plausible condition. This condition leads to

$$B = C^n \text{ (taking } \ell = 1) \tag{20}$$

where ℓ is proportionality constant and n is a constant. Equation (12) leads to

$$A = mB \tag{21}$$

where m is constant of integration.

3. Solutions of the Field Equations

From equation (8) with equations (20) and (21), we have

$$(n+1)\frac{C_{44}}{C} + n^2\frac{C_4^2}{C^2} = \Lambda \tag{22}$$

Now spatial volume is given by

$$R^3 = ABC = mC^{2n+1} \tag{23}$$

using (20) and (21).
We also suppose that vacuum energy density

$$\Lambda = \frac{\alpha}{R^3} \tag{24}$$

which leads to

$$\Lambda = \frac{\alpha}{m}C^{-(2n+1)} \tag{25}$$

Thus, equation (22) leads to

$$2C_{44} + \frac{2n^2}{n+1}\frac{C_4^2}{C} = \frac{2\alpha}{m(n+1)}C^{-2n} \tag{26}$$

To get the deterministic solution of equation (24), we assume

$$C_4 = f(C)$$

which leads to

$$C_{44} = f f' \text{ and } f' = \frac{df}{dC}$$

Thus equation (26) leads to

$$\frac{d}{dC}(f^2)+\frac{2n^2}{n+1}\frac{f^2}{C}=\beta C^{-2n} \tag{27}$$

where

$$\beta=\frac{2\alpha}{m(n+1)} \tag{28}$$

From equation (27), we get

$$\left(\frac{dC}{dt}\right)^2=\left(\frac{1+n}{1-n}\right)\beta C^{-2n+1}+bC^{-\frac{2n^2}{n+1}} \tag{29}$$

where b is constant of integration.

Hence metric (1) reduces to the form

$$ds^2=-\frac{dT^2}{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1}+b\,T^{-\frac{2n^2}{n+1}}}+T^{2n}dX^2+T^{2n}e^{-\frac{2aX}{m}}dY^2+T^2dZ^2 \tag{30}$$

where we have used the transformation C = T and cosmic time t is defined as

$$t=\int\frac{dT}{\sqrt{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1}+bT^{-\frac{2n^2}{n+1}}}} \tag{31}$$

and mx = X, y = Y, z =Z.

4. Special Model

To get the deterministic solution in terms of cosmic time t, we put b = 0 in equation (29), we get

$$\left(\frac{dC}{dt}\right)^2=\left(\frac{1+n}{1-n}\right)\beta C^{-2n+1}$$

and solution is

$$C^{2n+1}=\left(\frac{2n+1}{2}\right)^2T^2 \tag{32}$$

where $\delta t+L=T$; L is constant of integration and

$$\delta=\sqrt{\left(\frac{1+n}{1-n}\right)\beta} \tag{33}$$

Therefore, the metric (1) leads to

$$ds^2 = -\frac{dT^2}{\delta^2} + T^{\frac{4n}{2n+1}}\left\{dX^2 + e^{-\frac{2aX}{m}\left(\frac{2n+1}{2}\right)^{-\frac{2n}{2n+1}}}dY^2\right\} + T^{\frac{4}{2n+1}}dZ^2 \tag{34}$$

where

$$\left(\frac{2n+1}{2}\right)^{\frac{2n}{2n+1}}mx = X, \left(\frac{2n+1}{2}\right)^{\frac{2n}{2n+1}}y = Y, \left(\frac{2n+1}{2}\right)^{\frac{2}{2n+1}}z = Z.$$

5. Physical and Geometrical Aspects

The expansion (θ), vacuum energy density (Λ), shear (σ), the energy density (ρ), string tension density (λ), the particle density (ρ_p) and decelerating parameter (q) for the model (30) are given by

$$\theta = \left(\frac{2B_4}{B} + \frac{C_4}{C}\right)$$

$$= (2n+1)\frac{C_4}{C}$$

$$= \frac{(2n+1)}{T}\sqrt{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1} + bT^{-\frac{2n^2}{n+1}}} \tag{35}$$

$$\Lambda = \frac{\alpha}{m}C^{-(2n+1)}$$

$$= \frac{\alpha}{m}T^{-(2n+1)} \tag{36}$$

$$\sigma = \frac{(n-1)}{\sqrt{3}}\frac{C_4}{C}$$

$$= \frac{(n-1)}{\sqrt{3}T}\sqrt{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1} + b\,T^{-\frac{2n^2}{n+1}}} \tag{37}$$

$$\rho = (n^2+2n)\frac{C_4^2}{C^2} - \frac{a^2}{m^2C^{2n}} - \frac{\alpha}{m}C^{-(2n+1)}$$

$$= \frac{(n^2+2n)}{T^2}\left\{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1} + bT^{-\frac{2n^2}{n+1}}\right\} - \frac{a^2}{m^2T^{2n}} - \frac{\alpha}{m}T^{-(2n+1)} \tag{38}$$

$$\lambda = 2n\frac{C_{44}}{C} + (3n^2 - 2n)\frac{C_4^2}{C^2} - \frac{a^2}{m^2C^{2n}} - \frac{\alpha}{m}C^{-(2n+1)}$$

$$= \frac{n(n^2-1)}{1-n}\beta T^{-2n-1} + \left(\frac{n^3+n^2-2n}{n+1}\right)b\,T^{-\frac{2n^2+2n+2}{n+1}} - \frac{a^2}{m^2T^{2n}} - \frac{\alpha}{m}T^{-(2n+1)} \quad (39)$$

$$\rho_p = (4n - 2n^2)\frac{C_4^2}{C^2} - 2n\frac{C_{44}}{C}$$

$$= \frac{3n(n+1)}{1-n}\beta T^{-2n-1} + \frac{2n(n+2)}{n+1}b\,T^{-\frac{2n^2+2n+2}{n+1}} \quad (40)$$

$$q = -\frac{1}{(2n+1)}\left[\frac{\frac{2n^2+3n+1}{2(n-1)}\beta T^{-2n+1} - \frac{4n^2+3n+5}{n+1}b\,T^{-\frac{2n^2}{n+1}}}{\left(\frac{1+n}{1-n}\right)\beta T^{-2n+1} + bT^{-\frac{2n^2}{n+1}}}\right] \quad (41)$$

Now the expansion (θ), shear (σ), vacuum energy density (Λ), the energy density (ρ), string tension density (λ), the particle density (ρ_p), deceleration parameter (q) and Hubble parameter (H) for the model (34) are given by

$$\theta = \frac{2\delta}{T} \quad (42)$$

$$\sigma = \frac{(n-1)}{\sqrt{3}(2n+1)}\frac{2\delta}{T} \quad (43)$$

$$\Lambda = \frac{4\alpha}{m(2n+1)^2}T^{-2} \quad (44)$$

$$\rho = \frac{4m(n^2+2n)\delta^2 - 4\alpha}{m(2n+1)^2}T^{-2} - \frac{a^2}{m^2\left\{\left(\frac{2n+1}{2}\right)^{\frac{4n}{2n+1}}T^{\frac{4n}{2n+1}}\right\}} \quad (45)$$

$$\lambda = \frac{4n\delta^2(n-1)}{(2n+1)^2T^2} - \frac{a^2}{m^2\left\{\left(\frac{2n+1}{2}\right)^{\frac{4n}{2n+1}}T^{\frac{4n}{2n+1}}\right\}} - \frac{4\alpha}{m(2n+1)^2T^2} \quad (46)$$

$$\rho_p = \frac{12n\delta^2}{(2n+1)^2T^2} \quad (47)$$

$$q = -\frac{R_{44}/R}{R_4^2/R^2}$$

$$= \frac{1}{2} \tag{48}$$

$$H = \frac{R_4}{R}$$

$$= \frac{(2n+1)}{3}\sqrt{\left(\frac{1+n}{1-n}\right)}\beta\, T^{-\left(\frac{2n+1}{2}\right)} \tag{49}$$

where R is scale factor and the spatial volume (R^3) is given by

$$R^3 = ABC = m\,T^{2n+1} \tag{50}$$

The model (30) starts expanding with a big-bang at T = 0 and the expansion decreases with time. Since $\frac{\sigma}{\theta} \neq 0$, hence the model represents anisotropic space-time. However, for large values of T and n = 1, the model isotropizes. The vacuum energy density (Λ) decreases with time and for n = 1/2, it matches with the result as obtained by Chen and Wu[25]. The energy density (ρ) and string tension density (λ) are initially large and decrease with time. The model represents accelerating and decelerating phases of universe when 2n+1 > 0 and 2n+1 < 0 respectively. The model has Point Type singularity at T = 0 when n > 0. (MacCallum[26]).

The model (34) starts with a big-bang at T = 0 and the expansion in the model decreases with time. In general, it represents anisotropic space-time but for large values of T and n = 1, it isotropizes. The vacuum energy density is initially large but decreases with time. The result so obtained matches with the result by Chen and Wu[25]. The energy density (ρ) and string tension density (λ) are initially large but decrease with time. The deceleration parameter q > 0. Thus the model represents decelerating universe. The spatial volume (R^3) increases with time, hence represents inflationary scenario. The Hubble parameter (H) is initially large but decreases with time. The model has point type singularity at T = 0 when n > 0 (MacCallum[26]).

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