Article

Geometry and Physic in Gravitation Theory

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Abstract

This article is devoted to analysis of the relation of geometrical and physical quantities in the Newtonian theory of gravitation, general relativity and Lorentz-invariant gravitation theory (LIGT), and also to clarification of the physical meaning of the metric tensor and the space-time interval in the Euclidean, pseudo-Euclidean and pseudo-Riemannian spaces. The succession of the use of geometric concepts in these three theories is shown. It is shown that the math expression of interval is mutually uniquely associated with physical equations of elementary particles and LIGT. It is also shown that in LIGT the metric tensor has the physical meaning of the scale factor, defined by means of the Lorentz-invariant transformations. Evidence are given of that the metric tensor in general relativity should have the same meaning.

Key words: Euclidean space and time; pseudo-Euclidean space-time; pseudo-Riemanian space-time

1. Introduction: Geometry and Physic in the GR Equation

The Einstein-Hilbert field equations may be written in the form:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi T_{\mu\nu}\,, \tag{1.1}$$

The practical side of the Einstein-Hilbert theory (Tonnelat, 1965/1966) is following:
All the predictions of general relativity follow from the field equations:

$$G_{\mu\nu}\left(g_{\alpha\beta}, \partial_\rho g_{\alpha\beta}, \partial^2_{\rho\sigma} g_{\alpha\beta}\right) = \chi T_{\mu\nu}(m, \vec{u}) \to g_{\alpha\beta}\ , \tag{1.2}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, $\chi = \frac{8\pi G}{c^4}$, $R_{\mu\nu} = \frac{\Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - \frac{\Gamma^\nu_{\mu\lambda}}{\partial x^\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\alpha} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}$ is the Ricci curvature tensor, $\Gamma^\lambda_{\mu\nu}$ *are the Christoffel symbols*, R is the scalar curvature, γ_N is Newton's gravitational constant, c is the speed of light in vacuum and $T_{\mu\nu}$ is the stress–energy tensor.
and $g_{\mu\nu}$ *is the metric tensor of Riemannian space, and*
2) The law of motion (geodesic equation) for a massless body (photon):

$$\delta \int ds = 0\,, \tag{1.3}$$

or the Hamilton-Jacobi equation for a massive body (Landau and Lifshitz, 1973):

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$$g^{ik}\left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial y^k}\right)+m^2c^2=0\,, \tag{1.4}$$

The equation (1.3.1) allows to determine $g_{\mu\nu}$ and to put this value in (1.2).

Since the metric tensor is contained in the square of interval of Riemannian space:

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu\,, \tag{1.5}$$

it is often said that the purpose of solution of equation (1.1) is to find the interval (1.5).

Depending on the type of energy-momentum tensor the solutions of (1.1) can be divided into several types. The most important of them are the vacuum solutions, since it was possible to verify some of them experimentally. Such solutions can be obtained from the equation (1.1), if the energy-momentum tensor vanishes: $T_{\mu\nu}=0$.

These solutions describe the empty space-time around a massive compact source of the gravitational field, down to its surface or singularities. These include the Schwarzschild metric, the Lense-Thirring, Kerr, Reissner - Nordstrom, Kerr - Newman and others metrics.

In general relativity, a vacuum solutions are a Lorentzian manifold, i.e., they relate to asymptotically flat space-time. A Lorentzian manifold is an important special case of a pseudo-Riemannian manifold in which metric is called Lorentzian metric or the pseudo-Euclidian metric of special relativity.

In the vacuum equations of general relativity only the left side is used - purely geometric part of this equation. At the same time, the clarification of the physical meaning of significant elements of the metric tensor (MT) takes place on the basis of a comparison with Newton's theory of gravitation. Hitherto, the question, why a purely geometrical functions produce physical results, has not clarified. In other words, we do not know, how the MT is associated with physics.

The basis for the introduction and use of MT is the interval (often they are identical). Then the question can be reformulated in a different way: how interval and MT in this composition relates to physics?

It is often said that interval in SRT is a generalization of interval of Euclidean geometry on pseudo-Euclidean geometry. In turn, the interval in general relativity is a generalization of interval of pseudo-Euclidean geometry on pseudo-Riemannian geometry. But it is easy to make sure, that the introduction of interval in SRT and GTR is a postulates rather than a logical conclusion. The intervals in SRT and GRT are a generalization of interval of Euclidean geometry. And the reason for the introduction of these new intervals is not geometry, but physics. Then what was postulated and on what basis did it take place in each of these cases?

2. Geometry and Physics of Newtonian Mechanics (Euclidean Space)

Let us begin from the relation of the Euclidean interval with physics. To do this, we need to recall the meaning of the geometrical point and the material point, as well as of the geometric and physical trajectory, as the line of motion of a material point.

The line in geometry is an independent geometric object, almost not related to physics. Not strictly speaking, the line is a continuum (continuous sequence) of dots, for each adjacent pair of which the same relationship is set. If this relationship can always be reduced to a constant number, a line is called Euclidean; if this relationship is a function of the position on the line, the line is called Riemannian.

The line in geometry is defined (described) by specifying coordinates, i.e., some of the numbered lines, which are specified by the location of the material points (objects) of the real world.

In physics, the line is a continuum of points, which a material point passes successively while moving by inertia or under the influence of forces. And this line is determined by the law of motion of a material point with respect to the others, outsider material points which allows to establish a base coordinate system of lines. Namely here, geometry comes in contact with physics.

The interval in Euclidean geometry is a generalized description of Pythagoras theorem for an infinitesimal segment of line (arc): *square of the length of any line segment is equal to the sum of the squares of the projections of the segment on the three coordinate lines*. The objectives of the geometry, which requires the use of this law, has no connection with physics. But for the trajectory of a material point the theorem of Pythagoras is some condition - restrictive law, which must take place in any problem of the motion of material body.

Conditionally speaking, the law of Pythagoras must be contained in the law of motion. Obviously, this one-to-one relationship should allow to restore the movement law by means of the known interval. Approximately in this manner the problem is set on the theory of gravitation of Hilbert and Einstein.

Actually, the interval at any point of the trajectory of motion of a point must be mutually uniquely associated with the solution of the dynamic (physical) problem. Otherwise the decision will be wrong, i.e., the trajectory will not be one that is dictated by the law of motion. But this bond can not be associated with a coordinate system, since the latter is not related to the physical problem, and it can be chosen in many ways. This bond must occur in any coordinate system, in which the law of Pythagoras acts. In this case, the introduction and the choice of the coordinate system is a agreement, required for a quantitative calculation of the physical problem.

Let us demonstrate the correctness of our conclusion in the framework of non-relativistic and then relativistic (i.e., the Lorentz-invariant) mechanics.

2.1. Cartesian system of coordinates

Subject of mechanics (see (Webster, 1912)) is study of motion in space and time of the matter particle or system of particles, as solid body, under the action of forces.

Since the motion description of a material point involves four variables x, y, z, t, kinematics was called by Larange "geometry of four dimensions".

Suppose that we have a system of *n* material points. If they are free to move, a single particle requires 3 coordinates x, y, z, and a system of particles require 3*n* coordinates: $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n$

If any particle j at x_j, y_j, z_j is displaced by a small amount, it has the coordinates $x_j + dx_j, y_j + dy_j, z_j + dz_j$

If a number of particles are displaced, we must take the sums like the above for all the particles.

The infinitesimal distance between two points

$$ds = \sqrt{dx^2 + dy^2 + dz^2}\,, \qquad (2.1)$$

is a scalar, whereas the geometrical difference in position of the two points is known only when we specify not merely the length, but also the direction of the line joining them. This is usually done by giving its length s and the cosines of the angles λ, μ, ν made by the line with the three rectangular axes, $\cos\lambda, \cos\mu, \cos\nu$, which in virtue of the relation

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1, \qquad (2.2)$$

leaves three independent data.

We may otherwise make the specification by giving the three projections of the line upon the coordinate axes:

$$ds_x = s\cos\lambda = dx,\ ds_y = s\cos\mu = dy,\ ds_z = s\cos\nu = dz\ , \qquad (2.3)$$

Squaring and adding we have in virtue of relation (2.2):

$$ds^2 = ds_x^2 + ds_y^2 + ds_z^2\ , \ (2.4)$$

The quantities $dx/ds,\ dy/ds,\ dz/ds$ are the direction cosines of the tangent to the arc ds.

The vector denned by the product of the scalar quantity mass by the vector quantity acceleration (vector quantity), whose components are

$$F_x = m\frac{d^2x}{dt^2},\ F_y = m\frac{d^2y}{dt^2},\ F_z = m\frac{d^2z}{dt^2}\,, \qquad (2.5)$$

is called the force acting upon the body, and is the applied force of the Newton second law. The second and third laws taken together accordingly give us a complete definition and mode of measurement of force.

It is customary to characterize the product of the mass by the vector velocity as the momentum of the body, a vector whose components are

$$p_x = m\frac{dx}{dt} = m\upsilon_x,\;\; p_y = m\frac{dy}{dt} = m\upsilon_y,\;\; p_z = m\frac{dz}{dt} = m\upsilon_z, \tag{2.6}$$

This is the *momentum* whose rate of change measures the force, so that equations (2.5) may be written

$$\frac{dp_x}{dt} = F_x,\;\; \frac{dp_y}{dt} = F_y,\;\; \frac{dp_{zx}}{dt} = F_z, \tag{2.7}$$

These equations are a generalization of equation (2.5), since they may be applied in the case when mass m changes, for example, in the case the engine of the rocket is running.

$$T = \frac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \frac{1}{2}m\left(\upsilon_x{}^2 + \upsilon_y{}^2 + \upsilon_z{}^2\right), \tag{2.8}$$

the half-sum of the products of the mass of particle by the square of its velocity, is called the *kinetic energy* of the particle *T*.

If we have a system of *n* material points then:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left\{\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2\right\} = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\upsilon_{x_i}{}^2 + \upsilon_{y_i}{}^2 + \upsilon_{z_i}{}^2\right), \tag{2.9}$$

The kinetic energy may be written, bearing in mind the definition of momentum, as:

$$\begin{aligned} T &= \frac{1}{2}m\left(\upsilon_x{}^2 + \upsilon_y{}^2 + \upsilon_z{}^2\right) = \frac{1}{2}\left(p_x\upsilon_x + p_y\upsilon_y + p_z\upsilon_z\right) = \\ &= \frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right) = \frac{1}{2m}\vec{p}^{\,2} \end{aligned}, \tag{2.10}$$

It is easily to see:

$$\frac{d}{dt}T = \frac{d}{dt}\left(\frac{1}{2}m\sum_i \upsilon_i^2\right) = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt},$$

whence

$$dT = F_x dx + F_y dy + F_z dz$$

is the work done upon the particle at relocation it on infinitesimal distance.

The equation

$$T_{t_1} - T_{t_0} = \int_{t_0}^{t_1}\left(F_x dx + F_y dy + F_z dz\right), \tag{2.11}$$

is called the equation of energy, and states that the gain of kinetic energy is equal to the work done by the forces during the motion.

In the case that the forces acting on the particles depend only on the positions of the particles, and that the components may be represented by the partial derivatives of a single function of the coordinates $U(x,y,z)$ so that

$$F_x = \frac{dU}{dx}, F_y = \frac{dU}{dy}, F_{zx} = \frac{dU}{dz}, \tag{2.12}$$

the equation of energy then is

$$T_{t_1} - T_{t_0} = U_{t_1} - U_{t_0}, \tag{2.13}$$

The function *U* is called the force -function, and its negative *W*= —*U* is called the *potential energy* of the system. Inserting *W* in (2.13) we have

$$T_{t_1} + W_{t_1} = T_{t_0} + W_{t_0}, \tag{2.14}$$

the *principle of conservation of energy*.

Suppose that the particle instead of being free is constrained to lie on a given surface. The path described must then be an arc of a shortest or geodesic line of the surface. The calculus of variations enables us to find the differential equations of such a line.

The principle of least action says that in the natural or unconstrained motion it will go from P to Q along the shortest path, that is, an arc of a great circle.

2.2. Generalized system of coordinates

As was shown by Beltrami (Beltrami, 1869), and worked out in detail by Hertz, that the properties of Laginterval's equations have to do with a quadratic form, of exactly the sort that represents the *arc of a curve* in geometry.

For instance if a particle is constrained to move on the surface of a sphere of radius r, we may specify its position by giving its longitude φ and colatitude ϑ. These are two independent variables.

The potential energy depending only on position will be expressed in terms of cp and #. The kinetic energy will depend upon the expression for the length of the arc of the path in terms of ϑ and φ:

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Dividing by dt^2 and writing $\dot{\theta} = d\theta/dt, \dot{\varphi} = d\varphi/dt$, we have

$$T = \frac{1}{2}mr^2\left(d\dot{\theta}^2 + \sin^2\theta\, d\dot{\varphi}^2\right), \tag{2.15}$$

The parameters ϑ and φ are coordinates of the point, since when they are known the position of the point is fully specified. Their time -derivatives $\dot{\vartheta}$ and $\dot{\varphi}$ being time-rates of change of coordinates may be termed velocities, and when they together with ϑ and φ are known, the velocity of the particle may be calculated. The kinetic energy in this case involves both the coordinates ϑ and φ and the velocities $\dot{\vartheta}$ and $\dot{\varphi}$. Inasmuch as the particle in any given position may have any given velocity, the variables ϑ, φ, $\dot{\vartheta}$, $\dot{\varphi}$ are to be considered in this sense as independent, although in any given actual motion they will all be functions of a single variable t.

The form of the function T is worthy of attention. It is a homogeneous quadratic function of the velocities $\dot{\vartheta}$ and $\dot{\varphi}$, the coefficients of their squares being functions of the coordinates ϑ and φ, the product term in $\dot{\vartheta}$ and $\dot{\varphi}$ being absent in this case. We may prove that if a point moves on any surface the kinetic energy is always of this form. We may prove that if a point moves on any surface the kinetic energy is always of this form.

In the geometry of surfaces it is convenient to express the coordinates of a point in terms of two parameters q_1 and q_2. Suppose

$$x = f_1(q_1, q_2), y = f_{21}(q_1, q_2), z = f_3(q_1, q_2),$$

from these three equations we can eliminate the two parameters q_1, q_2, obtaining a single equation between *x, y, z,* the equation of the surface. The parameters q_1 and q_2 may be called the coordinates of a point

We may obtain the length of the infinitesimal arc of any curve in terms of q_1 and q_2. We have

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2,\ dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2,\ dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2, \tag{2.16}$$

Squaring and adding,

$$ds^2 = dx^2 + dy^2 + dz^2 = Edq_1^{\ 2} + 2Fdq_1 q_2 + Gdq_2^{\ 2}, \tag{2.17}$$

where

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2 \\ F &= \frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2}, \\ G &= \left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2 \end{aligned} \tag{2.18}$$

Thus the square of the length of any infinitesimal arc is a homogeneous quadratic function of the differentials of the coordinates q_1 and q_2, the coefficients *E*, *F*, *G* being functions of the coordinates q_1, q_2 themselves.

If the coordinate lines cut each other everywhere at right angles we shall have

$$ds^2 = Edq_1^{\ 2} + Gdq_2^{\ 2}, \tag{2.19}$$

The coordinates q_1, q_2 are then said to be orthogonal *curvilinear* coordinates.

In general we have the equations of change of coordinates,

$$x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi,\ \ z = r\cos\theta, \tag{2.20}$$

from which

$$\frac{\partial x}{\partial\theta} = r\cos\theta\cos\varphi,\ \frac{\partial y}{\partial\theta} = r\cos\theta\sin\varphi,\ \frac{\partial z}{\partial\theta} = -r\sin\theta,$$
$$\frac{\partial x}{\partial\varphi} = r\sin\theta\sin\varphi,\ \frac{\partial y}{\partial\theta} = r\sin\theta\cos\varphi,\ \frac{\partial z}{\partial\theta} = 0$$

and

$$E = r^2,\ F = 0,\ G = r^2\sin^2\theta$$

Employing the expression (2.17) for the length of the arc, dividing by dt^2 and writing

$$\frac{dq_1}{dt} = \dot{q}_1, \frac{dq_2}{dt} = \dot{q}_2$$

we find for the kinetic energy,

$$T = \frac{1}{2}m\left(Ed\dot{q}_1{}^2 + 2Fd\dot{q}_1\dot{q}_2 + Gd\dot{q}_2{}^2\right), \tag{2.21}$$

This is a typical example of the employment of the *generalised coordinates* introduced by Lagrange interval, q_1 and q_2 being the coordinates, $\dot{q}_1$, $\dot{q}_2$, the velocities corresponding, and T being a homogeneous quadratic function or quadratic form in the velocities q[, q2 , the coefficients of the squares and products of the velocities being functions of the coordinates alone. We shall show that this is a characteristic property of the kinetic energy for any system depending upon any number of variables.

In the case of a single free particle we may express the coordinates x, y, z in terms of three parameters q_1, q_2, q_3, and we shall then have as in (2.16) and (2.17)

$$ds^2 = E_{11}dq_1^2 + E_{22}dq_2^2 + E_{33}dq_3^2 + 2E_{12}q_1q_2 + 2E_{12}q_1q_2 + 2E_{31}dq_3dq_1, \tag{2.22}$$

where

$$E_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}, \tag{2.23}$$

(here $i, j = 1,2,3$)

Proceeding now to the general case of any number of particles, whether constrained or not, let us express all the coordinates as functions of m independent parameters, $q_1, q_2, ...q_m$, the generalized coordinates of the system,

$$x_k = x_r(q_1, q_2, ...q_m),\ y_k = y_r(q_1, q_2, ...q_m),\ z_k = z_r(q_1, q_2, ...q_m), \tag{2.24}$$

where $k = 1,2,3,...,n$

Differentiating, squaring and adding, we obtain

$$ds_k^2 = E_{11}^{(k)}dq_1^2 + E_{22}^{(k)}dq_2^2 + ... + E_{mn}^{(k)}dq_m^2 + 2E_{12}^{(k)}dq_1dq_2 + 2E_{13}^{(k)}dq_1dq_3 + ..., \tag{2.25}$$

where

$$E_{ij}^{(k)} = \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} + \frac{\partial y_k}{\partial q_i}\frac{\partial y_k}{\partial q_j} + \frac{\partial z_k}{\partial q_i}\frac{\partial z_k}{\partial q_j} \qquad (2.26)$$

Thus the square of each infinitesimal arc is a quadratic form in the differentials of all the coordinates *q*. Dividing by dt^2, multiplying by $m_k/2$ and taking the sum for all the particles, we obtain

$$T = \frac{1}{2} m_k \left(E_{11}^{(k)} d\dot{q}_1^2 + E_{22}^{(k)} d\dot{q}_2^2 + ... + E_{mn}^{(k)} d\dot{q}_m^2 + 2E_{12}^{(k)} d\dot{q}_1 d\dot{q}_2 + 2E_{13}^{(k)} d\dot{q}_1 d\dot{q}_3 + ... \right), \qquad (2.27)$$

Thus the kinetic energy possesses the characteristic property mentioned above of being a quadratic form in the *generalized velocities* $\dot{q}$, the coefficients E_{ij} being functions of only the generalized coordinates q. They must satisfy the conditions necessary, in order that for all assignable values of the q's T shall be positive.

It is sometimes convenient to employ the language of multidimensional geometry. This signifies nothing more than that when we speak of a point as being in n dimensional space we mean that it requires n parameters to determine its position.

Inasmuch as in motion along a curve, that is in a space of *one dimension* we have for the length of arc

$$ds^2 = \left(\frac{ds}{dq}\right)^2 dq^2$$

on a surface, that is in a space of *two dimensions*,

$$ds^2 = Edq_1{}^2 + 2Fdq_1 q_2 + Gdq_2{}^2,$$

and in space of *three dimensions*

$$ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} F_{ij} dq_i q_j$$

so by analogy , in space of n *dimensions*,

$$ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij} dq_i q_j \quad , \qquad (2.28)$$

That is to say a quadratic form in n differentials may be interpreted as the square of an arc in n dimensional space. Thus we may assimilate our system depending upon m coordinates to a single point moving in space of n dimensions.

To each possible position of this point corresponds a possible configuration of our system. No matter what be taken as the mass of the point, n, its kinetic energy, $T = (m/2)(ds/dt)^2$ is equal to the kinetic energy of our system, the coefficients in the quadratic form for ds and T being proportional.

The advantage of this mode of speaking (*for it is no more*) may easily be seen from the many analogies that arise, connecting the dynamical theory of least action with the purely geometrical theory of geodesic lines.

This method is adopted by Hertz in his book (Hertz, 1894). The ideas involved were first set forth by Beltrami. (Beltrami, 1869).

Analogies that arise, connecting the dynamical theory of least action with the purely geometrical theory of geodesic lines были развиты далее благодаря principle of varying action of Hamilton

Hamilton showed that the function S , which is named *action*

$$S = \int_{t_0}^{t} L dt \,, \tag{2.29}$$

where $L = T - W$ is Lagrange function, satisfies a certain partial differential equation, a solution of which being obtained, the whole problem is solved:

$$\frac{\partial S}{\partial t} + H\left(t, q_1,...q_m, \frac{\partial S}{\partial q_1},..., \frac{\partial S}{\partial q_m} \right) = 0, \tag{2.30}$$

where $H = T + W$ is the Hamilton function

The equation is of the first order since only first derivatives of S appear, and, from the way in which T contains the momenta, is of the second degree in the derivatives. Since S appears only through its derivatives an arbitrary constant may be added to it.

Hamilton's equation (2.30) assumes a somewhat simpler form when the force-function and consequently H are independent of the time that is when the system is conservative. We may then advantageously replace the principal function S by another function called by Hamilton the *characteristic function*, which represents the action *A*. Making use of the equation of energy, $T + W = h$, to eliminate W , we have

$$A = \int_{t_0}^{t} 2T dt = S + h(t - t_0), \tag{2.31}$$

and the above partial differential equation (2.30) becomes merely

$$H\left(q_1,...q_m, \frac{\partial A}{\partial q_1},... \frac{\partial A}{\partial q_m} \right) = h \,, \tag{2.32}$$

In these system a new variational principle will work; this principle was obtained in 1837 by Jacobi (Encyclpedia of matematics, 2011).
The kinetic energy of a system may be expressed in generalized coordinates q_i as follows:

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j \,, \tag{2.33}$$

The metric of the coordinate space is given by the formula

$$ds^2 = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} q_i q_j \tag{2.34}$$

The initial and final positions r_0 and r_1 of the system in some actual motion are also given.

Jacobi's principle of stationary action: *if the initial and final positions of a holonomic conservative system are given, then the following equation is valid for the actual motion*:

$$\delta \int_{r_0}^{r_1} 2(h - W)ds = 0, \tag{2.35}$$

as compared to all other infinitely near motions between identical initial and final positions and for the same constant value of the energy h as in the actual motion.
Jacobi's principle reduces the study of the motion of a holonomic conservative system to the geometric problem of finding the extremals of the variational problem (2.35) in a Riemannian space with the metric (2.34) which represents the real trajectories of the system. Jacobi's principle reveals the close connection between the motions of a holonomic conservative system and the geometry of Riemannian spaces.

If the motion of the system takes place in the absence of applied forces, i.e.,$U = 0$, the system moves along a geodesic line of the coordinate space $(q_1,...q_n)$ at a constant rate. This fact is a generalization of Galilei's law of inertia. If $U \neq 0$, determining the motion of a holonomic conservative system is also reduced to the task of determining the geodesics in a Riemannian space with the metric

$$ds_1^{\;2} = 2(U + h)ds^2 = \frac{1}{2}\sum_{i,j=1}^{n} b_{ij} q_i q_j \tag{2.36}$$

In the case of a single material point, when the line element ds is the element of three-dimensional Euclidean space, Jacobi's principle is the mechanical analogue of Fermat's principle in optics.

These results prove that in a Riemannian form we can write all classical potential fields, not just gravity field.

3. Geometry and Physics of Theory of Elementary Particles (Pseudo-Euclidean or Lorentz-invariant Space)

Let us now consider the connection of interval with physics in the case of the pseudo-Euclidean geometry.

A study of the literature shows that the pseudo-Euclidean coordinates and interval of the four-dimensional space-time are introduced into physics by analogy with the interval of Euclidean geometry (Landau and Lifshitz, 1973): "*It is frequently useful for reasons of presentation to use a fictitious four-dimensional space, on the axes of which are marked three space coordinates and the time*".

3.1. Interval and square of 4-distance differential

In Cartesian coordinate system of the Euclidean geometry an interval is the distance s between two points on a straight line in space, which is calculated according to the Pythagorean theorem. Since in physics trajectories are often curved lines, the Pythagorean theorem in this case is valid only for the infinitelisemal distances. Therefore, an interval is defined here according to (2.1) as the square root of the square of the distance differential in Euclidean space.

In the pseudo-Euclidean geometry an interval is defined as the square root of the square of the 4-distance differential and is given by the sum (taking into account the summation of Einstein)

$$ds = \sqrt{dx_\mu dx_\mu} \,,$$

where $\mu = 0,1,2,3$ $dx_0 = icdt$.
The square of the interval looks like:

$$(ds)^2 = (ic\,dt)^2 + (d\vec{r})^2 = -c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

Note that currently the imaginary time coordinate is rarely used (although it is by no means a mistake and has certain advantages), and is written as:

$$(ds)^2 = (c\,dt)^2 - (d\vec{r})^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \,, \qquad (3.1)$$

$$(ds)^2 = dx_\mu dx_\mu \,, \qquad (3.1')$$

where $\mu = 1,2,3,4$, and $dx_4 = cdt$. In addition, the squares of differentials are often written without parentheses: ds^2, dx^2, instead of $(ds)^2$, $(dx)^2$, etc ..

Thus, the use of characteristics of the 3-dimensional space in the case of 4-dimensional space –time is a postulate, i.e., some chosen mathematical expression, which is necessary for the construction of special relativity by Minkowski . It also follows from the fact that in nature the length of the arc in the 4- space-time is not measurable.

Therefore the question of the physical meaning of the 4-interval arises. Let's try to answer it.

3.2. Derivation of pseudo-Euclidean interval from the physical equations

The vectors of the Lorentz-invariant (i.e., relativistic) theories necessarily depend on the 4-coordinate: one time coordinate and three space coordinates. In other words, these equations are "working" in a 4-dimensional space-time. Does this theory contain the equations, which have a sum of terms, each of which is associated with one of the four coordinates, like the square of the interval?

As we know, in the first time such equations in classical electrodynamics appear, and then in quantum field theory. The wave equations of these theories include a sum of terms, each of which is associated with one of the variables t, x, y, z. It would be logical, to seek the cause and the meaning of the appearance of 4-interval in them, instead of introducing them artificially, as did Minkowski.

Recall that our study of the gravitational field is based on an inhomogeneous wave equation of the so-called "massive photon" (which in mathematical notation is similar to the Klein-Gordon equation). It is an equation for the two vectors of the electric and magnetic fields that give this photon a mass. From this equation follows the well-known equation of conservation of energy and momentum for massive particles, which is easy to obtain also from the definitions of 4-vectors of momentum and energy (see above). (Landau and Lifshitz, 1973)

From (3.1) we can easily obtain:

$$(ds)^2 = c^2(dt)^2 - (d\vec{r})^2 = c^2(dt)^2\left(1-\frac{(dr/dt)^2}{c^2}\right) = c^2(dt)^2\left(1-\frac{\upsilon^2}{c^2}\right), \tag{3.2}$$

At the same time interval is associated with proper time $d\tau$ by relation:

$$ds = c\sqrt{1-\upsilon^2/c^2}\,dt = cd\tau\,, \tag{3.3}$$

For a free material point the concept of the 4-momentum is introduced:

$$p_\mu = mcu_\mu \quad \text{or} \quad p_\mu = (p_0, p_i), \tag{3.4}$$

where $p_0 = i\dfrac{\varepsilon}{c} = \dfrac{mc}{\sqrt{1-\upsilon_i^2/c^2}}$, $p_i = \dfrac{m\upsilon_i}{\sqrt{1-\upsilon_i^2/c^2}}$, $\varepsilon == \dfrac{mc^2}{\sqrt{1-\upsilon^2/c^2}}$; u_μ is the 4-velocity.

From this:

$$\frac{\varepsilon^2}{c^2} - p_i^2 = m^2c^2, \tag{3.5}$$

where the energy and momentum is rewritten for convenience as follows: $p_0 \equiv \varepsilon = mc^2\gamma_L$, $p_i = m\upsilon_i\gamma_L = m(dx_i/dt)\gamma_L$ (where $\gamma_L = 1/\sqrt{1-\upsilon^2/c^2}$ and $\gamma_L^{-1} = \sqrt{1-\upsilon^2/c^2}$ are the Lorentz factor and antifactor, respectively). Hence, in the Cartesian coordinate system:

$$\frac{\varepsilon^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = m^2c^2\,, \tag{3.5'}$$

Since $p_i = m\upsilon_i\gamma_L = m(dx_i/dt)\gamma_L$, a $\varepsilon = mc^2\gamma_L$, this relation can be rewritten as:

$$\gamma_L^2c^2(dt)^2 - \gamma_L^2(dx)^2 - \gamma_L^2(dy)^2 - \gamma_L^2(dz)^2 = c^2(dt)^2\,, \tag{3.6}$$

Multiplying it by γ_L^{-2} , we get:

$$c^2(dt)^2\gamma_L^{-2} = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2, \tag{3.7}$$

Since (see above (3.2)) we got $c^2(dt)^2\gamma_L^{-2} = c^2(dt)^2(1-\upsilon^2/c^2) = (ds)^2$, the expression (3.7) can be written as square of a 4-interval:

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2, \tag{3.1'}$$

In general case of use in Euclidean space of any other, than the Cartesian, coordinate system for recording of the relation (3.5'), particularly, the orthogonal curvilinear coordinates, this interval takes the form:

$$(ds)^2 = g^{\mu\nu}dx_\mu dx_\mu\,, \tag{3.8}$$

where $g^{\mu\nu}$ is a so-called metric tensor, whose elements take into account the changes in the projections of the segments of the trajectory of the body on the coordinate axes, at the transition from the Cartesian coordinate system to any other. In a Cartesian system, all elements $g^{\mu\nu}$ are equal to unities.

Obviously, if we go in the opposite direction, we can obtain the equation (3.5') from the square of the interval. This implies, firstly, that these equations - (3.1) and (3.5') - closely bind the massive elementary particles physics and geometry. Secondly, the equation of "massive photon" is derived from Maxwell's equations of a massless photon as a result of his self-interaction of fields (Kyriakos, 2014a).

This non-linearity of a self-acting fields of the "massive photon" does not mean transition from Euclidean to some new geometry. From this it follows that (3.1) is not a metric of pseudo-Euclidean geometry, but it is a metric of Euclidean geometry that describes the Lorentz-invariant field equations. The only change in the geometry, which we can observe in this case is the transition from rectilinear to curvilinear geometry.

In addition, another link between the interval (2.1) and the physical equation is detected. As we have shown in a previous article (Kyriakos, 2010; 2014b), using the Schroedinger definition of action ($p_\mu = \partial S/\partial x_\mu$), from the equation (3.5') it is easy obtain Lorentz-invariant Hamilton-Jacobi equation in general view. For this it is enough to write the equation (3.5') in a form, suitable for any of the Euclidean coordinate system:

$$g_{\mu\nu} p^\mu p^\nu = m^2 c^2 , \tag{3.9}$$

where, we recall, $g_{\mu\nu}$ is the metric tensor of geometrical space, but not of the gravitational space-time of general relativity (in other words, in this case the tensor $g_{\mu\nu}$ does not include the physical characteristics of the field). In this case the Hamilton-Jacobi equation of free particles obtains the form:

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial y^\nu}\right) - m^2 c^2 = 0, \tag{3.10}$$

Recall that the physical field (e.g., electromagnetic field) is included in Hamilton-Jacobi equation in the following way:

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - p_{\mu\ ex}\right)\left(\frac{\partial S}{\partial y^\nu} - p_{\nu\ ex}\right) - m^2 c^2 = 0 , \tag{3.11}$$

Thus, we conclude that the three equations: (3.1) (3.5') and (3.10) are bonded to each other one-to-one and, in fact, are equivalent. From this follows that the interval (3.1) within a relativistic physics is the physical law, and not a geometric relation.

Next, we consider how the 4-interval is introduced in the transition from Euclidean geometry to *Riemann geometry*.

4. Geometry and Physics in the Pseudo-Riemannian Space

In general relativity an interval similar to (3.8) is introduced (postulated), where $g^{\mu\nu}$ takes into account the peculiarities of the Riemann geometry. But the most important thing here is other: in general relativity, it is postulated that, due to the transition to the Riemann geometry, the metric tensor is a function of the gravitational field.

Whether this is proved by experiment, we do not know because all the experimental confirmation of general relativity are obtained for problems in the pseudo-Euclidean metric.

Another fact also raises the question about the significance of Riemann geometry in physics. As we know, all theories of physics, except the GTR, are built in a Euclidean space, although mathematically, relativistic theories can be constructed in the pseudo-Euclidean space. But there is no such theory, which needs the introduction of the Riemann geometry.

Let us write the interval GRT as follows:

$$(ds)^2 = g_{\mu\nu}^{GR} dx^\mu dx^\nu , \tag{4.1}$$

where the metric tensor $g_{\mu\nu}^{GR}$ contains the characteristics of the gravitational field.

In addition, instead of the equation for the external field (3.11) in general relativity the equation of external field of type (3.10) is taken, but with the appropriate metric tensor $g_{\mu\nu}^{GR}$:

$$g_{\mu\nu}^{GR}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial y^\nu}\right) - m^2c^2 = 0 , \tag{4.2}$$

The question is, why is there such a difference between (3.10) and (4.2), as well as between (3.8) and (4.1), and why is the external field in GTR inserted through the metric tensor?

To answer this question, we will try to find out the physical sense of the metric tensor. Let us turn first to Euclidean geometry.

4.1. The physical sense of the metric tensor of curvilinear coordinates' system of the Euclidean geometry

Recall the generalized coordinate system and particularly, curvilinear coordinates. (Korn and Korn, 1968)
Let us introduce a new set of coordinates q_1, q_2, q_3, so that among x, y, z and q_1, q_2, q_3 there are some relations

$$x = x(q_1, q_2, .q_3),\ y = y(q_1, q_2, .q_3),\ z = z(q_1, q_2, q_3) , \tag{4.3}$$

The differentials are then

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 , \tag{4.4}$$

and the same for dy and dz.

In Cartesian coordinates the measure of distance, or metric, in a given coordinate system is the arc length ds, which is defined by

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{4.5}$$

In general, taking into account (4.4), from (4.5) we obtain

$$ds^2 = g_{11}dq_1{}^2 + g_{12}dq_1dq_2 + ... = \sum_{ij} g_{ij}dq_idq_j, \tag{4.6}$$

where g_{ij} is the metric tensor. Thus in orthogonal system we can write

$$ds^2 = (H_1dq_1)^2 + (H_2dq_2)^2 + (H_3dq_3)^2, \tag{4.7}$$

where the H_i's где H_i

$$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}, \tag{4.8}$$

are called Lame coefficients or scale factors, and are 1 for Cartesian coordinates.

Thus, the Riemann metric tensor, recorded in coordinates q_i, is a diagonal matrix whose diagonal contains the squares of Lame coefficients:

For example, in the case of spherical coordinates, the bond of spherical coordinates with Cartesian is given by (2.20).

The Lame coefficients in this case are equal to: $H_r = 1,\ H_\theta = r,\ H_\varphi = r\sin\theta$, and the square of the differential of arc (interval) is:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Since the metric tensor is determined by means of Lame coefficients, let us recall the geometric meaning of the latter: the Lame coefficients show how many units of length are contained in the unit of length of coordinates of the given point, and used to transform vectors when transition from one system to another takes place.

This means that the metric tensor in Euclidean geometry defines rescaling of three coordinates r,θ,φ, and in the pseudo-Euclidean or pseudo-Riemannian geometry it determines rescaling of four coordinates t,r,θ,φ.

As we have seen from the solution of the Kepler problem within LIGT (Kyriakos, 2014c), the relativistic corrections within LIGT correspond to changes of scales t and r, caused by the Lorentz-invariant effects (time dilation and Lorentz-Fitzgerald length contraction). In the next article, we will show that the same thing occurs in problems of a moving source.

Thus, we conclude that relationships (4.1) and (4.2) have metric tensor $g^{GR}_{\mu\nu}$ as a factor that takes into account the change of scales of time and distance due to relativistic effects associated with motion of bodies.

5. Consequences

From the foregoing analysis follows that by regular way the interval of a 4-space-time can be obtained only for the pseudo-Euclidean space, as a variant of the physical law of motion of elementary particles.

Since there is no other law of motion for massive particles, we can assume that the hypothesis of Einstein that the gravitational field is created by the curvature of space-time, which requires a transition to a pseudo-Riemannian geometry, needs considerable adjustment.

Following the theory of mass generation (Kyriakos, 2014a), we have to conclude that the gravitational field arises from the self-action of massless fields. It is indeed accompanied by the transformation of the linear movement of fields in curvilinear motion (mathematically, this is the transition from linear equations to nonlinear equations). But this has nothing to do with the Riemann geometry.

It can be assumed that the use of the Riemann geometry in GR is possible for the reason that math physics in the case of the Riemann geometry is very close to the math physics using generalized coordinates of Euclidean geometry. Formally, the coordinates of the Riemann geometry, can be considered as generalized coordinates of the set of n material points, or as one point in the n-dimensional space. This is evidenced by the form of squared length of arc (trajectory) element using generalized coordinates (2.25) with the values of the Gauss coefficients (2.26). In the transition to the Riemann geometry the Gaussian quadratic form coefficients E are replaced by elements of the metric tensor $g_{\mu\nu}$.

In this sense, the Riemann geometry should not be opposed to Euclidean geometry.

From a formal point of view (Bogorodskiy, 1971) the Riemann space can be determined, like the Euclidean multidimensional space, as a field of the metric tensor in the n-dimensional continuum in which the distance between the infinitely near points is using quadratic forms $ds^2 = g_{\mu\nu} x^\mu x^\nu$, and the angle between two linear elements - at $\cos\theta = g_{\mu\nu} x^\mu x^\nu \big/ ds\delta s$. Riemann geometry covers a wide class of spaces and includes Euclidean geometry as a simple special case.

Moreover, it is possible to choose the coordinates in Euclidean space in any way, that all g_{ij} and their first derivatives with respect to coordinates in Riemannian and Euclidean metrics were the same values in all points of the line. In this case, the Euclidean metric is in contact along a given curve with the Riemann metric. In an infinitely thin tube containing the curve, Euclidean space is Riemann space up to the second order. This is called the *contiguous* Euclidean space.

Another reason to believe that in the theory of gravity it is sufficient to use the pseudo-Euclidean metric is the possibility to present the Hilbert-Einstein equations in the form of generalized d'Alembert equation (Fock,.1964):

"*In the previous section we saw that, at least if* $\Gamma^{\nu} = 0$ *(*i.e., in harmonic coordinates*) Einstein's equations are of the type of the wave equation, because their main terms involve the d'Alamber operator*".

A similar result can be obtained based on the nonlinear theory of elementary particles. Since the equation (3.9) is a formal consequence of squaring of the Dirac electron equation, one might think that there is a connection between the Dirac matrices, and tensor metric space. Indeed, such a connection exists. And it gives the possibility to receive the general covariant form of the squared Dirac electron equation in a gravitational field (for details, see. (Schroedinger, 1932; Kyriakos, 2012).

The connection between the Dirac matrices and metric tensor is defined by relations

$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ and $S^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$, from which follows $\gamma^\mu\gamma^\nu = g^{\mu\nu} + S^{\mu\nu}$.

where γ_μ is the Dirac matrices

The seeking equation is the d'Alembert equation:

$$\frac{1}{\sqrt{g}}\nabla_\mu\sqrt{g}g^{\mu\nu}\nabla_\alpha - \frac{R}{4} - \frac{1}{2}f_{\mu\nu}S^{\mu\nu} = m^2, \tag{4.9}$$

where R is an invariant curvature $R = g^{\nu\mu}g^{\alpha\nu}R_{\nu\alpha,\mu\nu} = -\frac{1}{8}R_{\nu\alpha,\mu\nu}S^{\nu\alpha}S^{\mu\nu}$, and $R_{\nu\alpha,\mu\nu}$ is a symmetric Riemann tensor

In the first term of equation (4.9) is easy to find a regular operator of the Klein second order equation in the Riemann geometry. In the third term on the left is recognized well-known term associated with the spin magnetic and electric moments of the electron (tensor $S^{\mu\nu}$).

The second term provoked particular interest of Schrödinger: "*To me, the second term seems to be of considerable theoretical interest. To be sure, it is much too small by many powers of ten in order to replace, say, the term on the r.h.s. For* m *is the reciprocal Compton length, about* $10^{11}cm^{-1}$. *Yet it appears important that in the generalised theory a term is encountered at all which is equivalent to the enigmatic mass term*".

Nonlinear theory of elementary particles can explain the physical meaning of this term: it define the charge and mass of elementary particle.

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