Article

Rindler Frames with $\frac{dg}{dt} > 0$, Unruh Radiation and RG Flow

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Abstract

The variation in acceleration of a frame is examined in order to compute some elementary results with Unruh radiation. This is a model for the variation in black hole mass due to Hawking radiation and the adjustment of an accelerated frame closer to the event horizon. The increase in detected radiation on this frame is a form of IR to UV renormalization group flow. The group flow leads to topological phase of gauge charge carriers, or charged electrons in the $U(1)$ electromagnetic case, on a stretched horizon.

1 Introduction

This paper is an exploration of the physics on an increasing accelerated frame. This is equivalent to slowly lowering a frame to just above the event horizon of a black hole. Physically we expect the temperature of Unruh radiation to become enormous [1]. This is examined by looking at a probe field that is increasingly accelerated and approaches the Planck acceleration. This study reveals a number of interesting relationships between the topological phase of a quantum system and spacetime.

The paper first examines the a time variation in acceleration within special relativity. The basic results for the parameterization of position, time and four velocity are worked. This is then used to look at the role of Unruh radiation. As the acceleration increases the Unruh temperature will increase and eventually approach the Planck temperature. This is illustrated in elementary terms. The next step is to describe the dynamics of a probe field in this accelerated frame. The statistics of this interaction reproduces the Shannon result which intertwines information and entropy. These results with accelerated frames also hold for black hole physics. Finally the thermodynamics of this system is found to have topological symmetry or phase. The probe field approaching the horizon interacts with particles that are anyonic and where the gauge field interaction, or electromagnetic field, interacts with these charge carriers by quantum Hall physics [2].

This paper is organized by starting with elementary physics of special relativity and building to understand deeper foundations. The underlying physics of this is just quantum fields that interact with a geodesic on a parametrized set of Lorentz boosts. The parametrization corresponding to an increasing acceleration is then a renormalization group flow for a gauge theory which conserves charges for the field. This examination of $\frac{dg}{dt} > 0$ and the RG flow is a potential method for addressing the relationship between gauge field and the fermion fields which carry their charges and spacetime. While this is not addressed in this paper this method with topological symmetry and phase is potentially a way of looking at the change of phase with gravitation.

2 The Rindler Frame with $\frac{dg}{dt} > 0$

The accelerated observer is an elementary approach for the argument of the holographic principle. A detector on the accelerated frame measures a thermal response from the vacuum with $T \simeq g/2\pi$[1]. If the acceleration were increased it makes physical sense that the Unruh temperature would proportionately increase. A physical situation where this acceleration increases, or where the gravitational acceleration

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of an evaporating black hole increases would then suggest that the energy of quanta emitted from the Rindler horizon would increase.

A particle moves from rest according to an inertial frame F along the x-axis with an acceleration g. After a time t this particle is now moving with velocity v along with an other inertial frame F'. For the particle then moving at velocity u' in F' the acceleration is $g = \frac{du'}{dt'}$. The Rindler frame is easily seen from the velocity addition formula of special relativity $u' = \frac{u - v}{1 - uv}$[3]. With $t' = \gamma t$ and knowing the accelerated particle is at this time in F' with $v = u$ it is easy to see that

$$g = \frac{du'}{dt'} = \gamma^3\frac{du}{dt}. \tag{2.1}$$

The acceleration may then be expressed as $gdt = d(\gamma u)$, which when integrated from $u = 0$ at $t = 0$ results in

$$gt = \gamma u = \frac{u}{\sqrt{1 - u^2}}. \tag{2.2}$$

By elementary algebra is is easy to see that $u = gt/(1 - (gt)^2)$ and this is

$$u = \frac{dx}{dt} = g^{-1}\frac{d}{dt}\sqrt{1 + (gt)^2}, \tag{2.3}$$

and we may integrate $dx = g^{-1}d\sqrt{1 + (gt)^2}$ to find the hyperbolic condition on the path $x^2 - t^2 = g^{-2}$. Clearly this means $x = g^{-1}cosh(g\tau)$ and $t = g^{-1}sinh(g\tau)$, with the further consequence $\gamma = cosh(g\tau)$ and $u = tanh(g\tau)$. The position and time asymptote to the split horizon given by $x = \pm t$.

For a variation in the acceleration we write the acceleration as

$$g = \frac{du'}{dt'} + t'\frac{d^2u'}{dt'2} + \dots, \tag{2.4}$$

where this expansion is only carried out to second order. This means we only consider a constant change to the acceleration. The generalization of equation 2 is

$$gt = \gamma u + t\gamma^2\frac{du}{dt}, \tag{2.5}$$

To solve this equation within the series, the acceleration is written as $g = g_1 + t'g_2 + O(t'^2)$ with $g_1 = \gamma u_1$ and $g_2 = \gamma^2\frac{d^2u_2}{dt^2}$. The two velocities are computed and the relativistic addition formula of velocities is used. As before $u_1 = tanh(g\tau)$, and u_2 is found from

$$\gamma\frac{du_2}{dt} = g_2 t \rightarrow u_2 = \frac{(1 - u_1)e^{g_2t^2} - (1 + u_1)e^C}{e^{g_2t^2} + e^C}. \tag{2.6}$$

The constant is $C = g_2t_0^2$ where at $u_1 = 0$ we set $t_0 = 0$ so that $u_2 = tanh(g_2t)$. The coordinate time t is related to the proper time τ by $t = g^{-1}sinh(g\tau)$ and this results in the velocity

$$u = \frac{tanh(g_1\tau) + tanh\left(\frac{g_2}{g_1^2}sinh^2(g\tau)\right)}{1 - tanh(g_1\tau)tanh\left(\frac{g_2}{g_1^2}sinh^2(g\tau)\right)}. \tag{2.7}$$

Given that $g = g_1 + t'g_2$, this might seem to require further analysis, but the tanh function for large arguments is close to unity and we may then approximate this by substituting $g \rightarrow g_1$ with little loss of information. The velocity addition formula is a simple trigonometric identity which gives the final approximate result

$$u \simeq tanh\left(g_1\tau + \frac{g_2}{g_1^2}sinh^2(g_1\tau)\right). \tag{2.8}$$

The position and time variables are easily seen to be $x = \int^\tau d\tau' sinh(g_1\tau' + \frac{g_2}{g_1^2}sinh^2(g_1\tau'))$ and $t = \int^\tau d\tau' cosh(g_1\tau' + \frac{g_2}{g_1^2}sinh^2(g_1\tau'))$, and the hyperbolic condition on the path is evident.

3 Unruh Radiation

An observer on this frame will detect radiation that is emerges from the horizon at $x' = \pm t\prime$. We introduce the null coordinates $u = t + x$, $v = t - x$ for the advanced and retarded propagation of fields. Parametrized by the proper time these coordinates are $u = exp\left(g_1\tau + \frac{g_2}{g_1^2}sinh^2(g_1\tau)\right)$ and $v = -exp\left(-g_1\tau - \frac{g_2}{g_1^2}sinh^2(g_1\tau)\right)$. The hyperbolic condition on the coordinates of the accelerated frame $X^\mu X_\mu = -1$ means there is a parametrized set of Lorentz boosts generated by a Killing field with $K^\mu K_\mu = -1$. The $(u,\ v)$ Killing vector is [1]

$$K^\mu = -g\left(e^{-\Gamma\tau}\frac{\partial}{\partial u} + e^{\Gamma\tau}\frac{\partial}{\partial v}\right), \tag{3.1}$$

for $\Gamma = \Gamma(\tau) = g_1 + \tau^{-1}\frac{g_2}{g_1}sinh^2(g_1\tau)$. The Killing vector is normal to the horizon along v, with

$$K^\mu K_\mu = g\left(e^{-\Gamma(\tau)}\frac{\partial}{\partial u}e^{\Gamma\tau} + e^{\Gamma\tau}\frac{\partial}{\partial v}e^{-\Gamma\tau}\right) = 0. \tag{3.2}$$

On the future horizon the relationship between the inertial time v and the Killing time $\theta = g^{-1}ln|v|$. This may be seen with the Killing vector such that $K\cdot\theta = gv\frac{\partial\theta}{\partial v} = 1$. Similarly we have $\phi = g^{-1}ln|u|$.

A wave function with frequency ω restricted to the horizon with $v > 0$ is [1]

$$\psi_\omega(\theta,\ x,\ y) = f(x,\ y)e^{-i\omega\theta}, \tag{3.3}$$

that under a Fourier transform is

$$\mathcal{F}\psi_\omega(\theta,\ x,\ y) = \psi(\Omega,\ x,\ y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\Omega v}\psi_\omega(\theta,\ x,\ y) \tag{3.4}$$

$$= \frac{1}{\sqrt{2\pi}}f(x,\ y)\int_{-\infty}^{\infty}exp\left[i\left(\Omega v - \frac{\omega}{\Gamma}lnv\right)\right]dv. \tag{3.5}$$

A $\pi/2$ rotation of the axis of integration with $v = iy$, and $ln(iy) = lny + i\pi/2$ gives the Fourier transform

$$\int_{-\infty}^{\infty}exp\left[i\left(\Omega v - \frac{\omega}{\Gamma}lnv\right)\right]dv = i\ exp\left(\frac{\pi\omega}{2\Gamma}\right)\int_{-\infty}^{\infty}exp\left[-\left(\Omega y - \frac{\omega}{\Gamma}lnv\right)\right]dv. \tag{3.6}$$

A similar wave function for $v < 0$ across the horizon

$$\psi_\omega(\theta,\ x,\ y) = f(x,\ y)exp\left(-i\frac{i\omega}{\Gamma}\right), \tag{3.7}$$

and analysis with the reflection leads to the wave function Ψ connecting the wave ψ_I in the Rindler wedge region I to ψ_{II} across the horizon II as[1]

$$\Psi = \psi_I + exp\left(-\frac{\pi\omega}{\Gamma}\right)\psi_{II}. \tag{3.8}$$

The Boltzmann factor $exp\left(-\frac{\pi\omega}{\Gamma}\right)$ then appears as

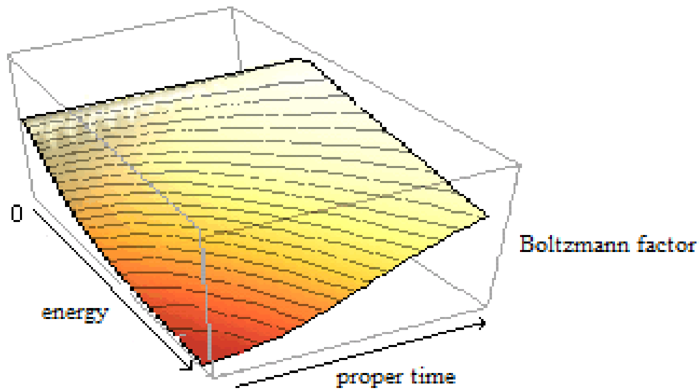

Figure 1: Surface graph of how the Boltzmann factor changes with the acceleration dependent temperature.

As proper time increases the argument of the exponent becomes very small and the Boltzmann factor $Z = \sum_i exp(-\frac{\pi\omega_i}{\Gamma})$ approaches unity. This is interpreted as an increase in temperature $T = T_0(g_1 + \frac{g_2}{g_1^2}sinh^2(g_1\tau)/\tau)$. The expectation of the energy $\langle E\rangle = -\partial Z/\partial\Gamma^{-1}$ for a continuum of energy levels is

$$\langle E\rangle = -\int_0^\infty exp\left(-\frac{\pi\omega}{\Gamma}\right)d\omega = \frac{\Gamma}{\pi}, \tag{3.9}$$

which expands with the increase in acceleration $\langle E\rangle \simeq g_1 + (g_2\tau^{-1}/g_1^2)e^{2g_1\tau}$. To compute the ratio $r = \langle E\rangle/E_0$ the proper time is given by the Lambert W-function

$$\tau = -\frac{1}{2\pi g_1}W\left(-\frac{\pi g_1 rE_0}{2g_2}\right) = \frac{1}{2\pi g_1}\left(\left(\frac{\pi g_1 rE_0}{2g_2}\right) - \left(\frac{\pi g_1 rE_0}{2g_2}\right)^2 + \right.$$
$$\left.\frac{3}{2}\left(\frac{\pi g_1 rE_0}{2g_2}\right)^3 - \frac{8}{3}\left(\frac{\pi g_1 rE_0}{2g_2}\right)^4 + \dots\right. \tag{3.10}$$

4 A Quantum Test Field

A quantum field on an accelerating frame will interact with the Unruh radiation. The energy scale of the interaction, or transverse momentum, with the accelerated particle is determined by the magnitude of the acceleration. In the case of an increasing acceleration the scale of these interactions will scale from the IR to UV domain. The particle is considered by be as a sort of detector. The Lagrangian chosen is $\mathcal{L} = \mu \int_\gamma \phi d\tau$, which for a constant ϕ and $\mu = m$ is the standard Lagrangian for a particle in spacetime. The Euler-Lagrange equation of motion for the particle is

$$\frac{d}{d\tau}\left(\mu\phi g_{\mu\nu}U^\nu\right) - \mu\left(\frac{\partial\phi}{\partial x^\mu} + \frac{1}{2}\phi\frac{\partial g_{\alpha\beta}}{\partial x^\mu}U^\alpha U^\beta\right) = 0. \tag{4.1}$$

This results in two equations

$$-\frac{d\phi}{d\tau}cosh(\Gamma\tau) - \phi sinh(\Gamma\tau)\left(g_1 + \frac{2g_2}{g_1}sinh(\Gamma\tau)cosh(\Gamma\tau)\right) - \mu\frac{\partial\phi}{\partial t} = 0$$
$$\frac{d\phi}{d\tau}sinh(\Gamma\tau) + \phi cosh(\Gamma\tau)\left(g_1 + \frac{2g_2}{g_1}sinh(\Gamma\tau)cosh(\Gamma\tau)\right) - \mu\frac{\partial\phi}{\partial x} = 0. \tag{4.2}$$

For large enough τ we have $\tanh(\Gamma\tau) \simeq 1$ and we expand

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial\tau} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\tau} = \frac{\partial\phi}{\partial t}cosh(\Gamma\tau) + \frac{\partial\phi}{\partial x}sinh(\Gamma\tau)$$

in order to write the approximate equation

$$\left(cosh(\gamma\tau) + sinh(\Gamma\tau)\right)\frac{\partial\phi}{\partial t} + 2\phi\left(g_1 + \frac{2g_2}{g_1}sinh(\Gamma\tau)cosh(\Gamma\tau)\right) \simeq 0 \tag{4.3}$$

For large τ this is further approximated with $cosh(\gamma\tau) + sinh(\Gamma\tau) = 2cosh(\Gamma\tau)$ so that $(cosh(\gamma\tau) + sinh(\Gamma\tau))\frac{\partial\phi}{\partial t} \simeq \frac{d\phi}{d\tau}$. Further approximations permits us to write the differential equation as

$$\frac{d\phi}{d\tau} \simeq \left(g_1 + \frac{2g_2}{g_1}e^{2g_1\tau}\right)\phi. \tag{4.4}$$

The solution to this equation is

$$\phi = \phi_0 e^{-g_1\tau}\, exp\left(\frac{-g_2}{g_1}e^{2g_1\tau}\right)$$

The Euclidean time is $t = \hbar/kT$ from the temperature $T = T_0\left(g_1 + (g_2/g_1^2)sinh^2(g_1\tau)/\tau\right)$. For large proper time $cosh(\Gamma\tau) \simeq \frac{1}{2}e^{\Gamma\tau}$ and $sinh^2(g_1\tau)/\tau \simeq \frac{1}{4}e^{g_1\tau}/\tau$. The coordinate time $t \simeq 4\frac{g_1^2}{g_2}\tau e^{-g_1\tau}$ gives the proper time

$$\tau = -\frac{1}{g_1}W\left(-\frac{g_1^2}{2g_2}\frac{t}{t_0}\right). \tag{4.5}$$

For the temperature T_0 small we have the time t_0 large, which is proportional to uncertainty in time Δt given the low energy $\Delta E \simeq T\Delta S$ near zero. The asymptotic expansion for the Lambert W-function for the argument ≥ 3 is used $W(z) = ln(z) - ln\, ln(z) + \dots$, with

$$g_1\tau \simeq -\left[ln\left(-\frac{g_1^2}{2g_2}\frac{t}{t_0}\right) - ln\, ln\left(-\frac{g_1^2}{2g_2}\frac{t}{t_0}\right)\right] \tag{4.6}$$

A simple approximation with just the first term permits us to write the solution

$$\phi = -\phi_0 z\, exp(-g_2 z/g_1),\ z = \frac{g_1^2 t}{2g_2 t_0} \tag{4.7}$$

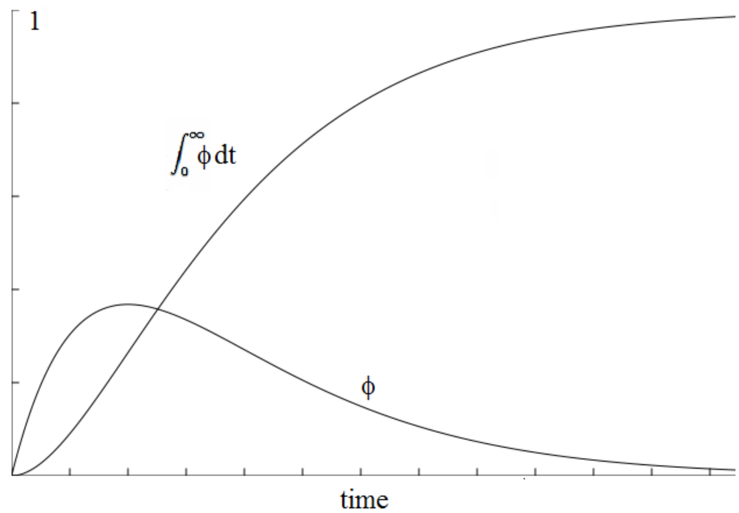

Figure 2: The probe field $\phi(t)$ and its integration over time.

This solution and its integration with time is shown below

The time parameter may be written as $g_2 z/g_1 = ln\ p$ so the field is $\phi = -(g_1/g_2)p\ ln\ p$, which gives a form of the Shannon formula for information when integrated over time or equivalently summed over p. The function is seen in the above graph where the integration is $\int \phi dt = \sum_i p_i\ ln(p_i)$. The amount of information saturates. This means the quantum probe will only encounter a finite amount of information and once the acceleration is extreme the information flux decreases to zero. Now consider the physical ansatz that the quantum field defines von Neumann entropy. The quantum test field starts out in a pure state, reaches a maximum entropy that then decreases to a pure state. The integration of this excitation by the Unruh vacuum, divided the net time interval of the integration, describes the thermal entropy. The difference of between these two will be proportional to the information available to the probe. The drop in the entropy of the probe physically corresponds to the situation where the horizon at $d = c^2/g$ approaches the Planck scale. As the Planck scale is approached the amount of information available decreases unit not more can be accessed at ℓ_p.

There is a remarkable similarity to the theory of superfluidity. The condition for superfluidity was found by Feynman to occur when the wave function of the Helium atoms were in a condition of maximal smoothness corresponding to large separation of the atoms where they do not overlap[6]. This results in certain quantum states of the atoms in rotons which are elementary quantum states of vortices. This is a simple coherent state description, which has similar physics as above. Here we see with equation 25 a field description that is similar to the type of wave function employed in Feynmans theory of superfluidity. This then suggests there is some type of phase transition which we may expect in any field theory or string theory of black hole holography.

5 Black Holes

The Rindler wedge is an approximation to the case of a stationary observer near a black hole horizon. The Reissner-Nordstrom (RN) metric is

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2}dt^2 - \frac{r^2}{(r - r_+)(r - r_-)}dr^2 - r^2 d\Omega^2, \tag{5.1}$$

for $r_\pm = m \pm \sqrt{m^2 - q^2}$ the outer and inner horizons. We consider the near horizon condition with $r = r_+ + \rho$ and define $r_+ - r_- = \epsilon$ so that $\epsilon >> \rho$. The RN metric is for g_{tt} to $O(\rho)$ approximately

$$ds^2 = \frac{\epsilon\rho}{r_+^2}dt^2 - \frac{r_+^2}{\epsilon\rho}d\rho^2 - r^2 d\Omega^2. \tag{5.2}$$

The proper distance for $dt = 0 = d\Omega$ is $\sigma = 2r_+\sqrt{\frac{\rho}{\epsilon}}$ and is used in the reduced metric with $d\Omega = 0$ to derive the Rindler wedge metric, based on [5],

$$ds^2 = \frac{4\epsilon^2\sigma^2}{r_+^4}dt^2 - d\sigma^2. \tag{5.3}$$

For $r_+ = \epsilon = 2m$ this reduces to the Rindler metric for the close horizon case with the Schwarzschild metric. The Christoffel symbols are

$$\Gamma^\sigma_{tt} = -\frac{4\epsilon^2\sigma}{r_+^4},\ \Gamma^t_{t\sigma} = \frac{1}{\sigma}, \tag{5.4}$$

and the σ geodesic equation for $\frac{d\sigma}{dt} = 0$ is

$$\frac{d^2\sigma}{dt^2} = \frac{4\epsilon^2\sigma}{r_+^4}. \tag{5.5}$$

The potential is then $V(\sigma) = -\frac{\epsilon^2\sigma^2}{8r_+^4}$ and the force is $F = -\partial_\sigma V(\sigma)$.

A scalar wave in the Rindler wedge, where we restore the other coordinates as x and y has the Lagrangian

$$\mathcal{L} = \left(\frac{\partial\phi}{\partial\tau}\right)^2 - \sigma^2\left(\frac{\partial\phi}{\partial\sigma}\right)^2 - \sigma^2\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right], \tag{5.6}$$

for $\tau = t/4m$. The transverse directions have the eigenvalue $\partial_i\phi - i\lambda\phi$. The Lagrangian is then $\mathcal{L} = (\partial_\tau\phi)^2 - \sigma^2(\partial_\sigma\phi)^2 - \lambda^2\sigma^2\phi^2$. The potential $V(\phi) = \lambda^2\sigma^2\phi^2$ is a harmonic oscillator potential, and a small region near the horizon the probe state is of this form. The distance is $\sigma = g^{-1}$, and in line with the above this acceleration is defined as $g = g_1 + \tau g_2$. The changing acceleration corresponds to a time dependent decrease to the distance to the horizon $\sigma = 1/(g_1 + \tau g_2)$. The increased acceleration corresponds to a reduced σ, and just as waves in a cavity will become shorter in wavelength as the cavity dimensions are reduced so too we have k increasing. The Lagrangian used in equation 19 is

$$(-g)^{-1/2}\partial_\mu\left((-g)^{1/2}g^{\mu\nu}\partial_\nu\phi\right) - \partial_\phi V = 0, \tag{5.7}$$

which is the scalar wave equation 31. The field ϕ rises and declines as it responds to the vacuum in the increasingly accelerated frame. While $k \propto g_1 + \tau g_2$ and $dk/d\tau > 0$ the amplitude of the probe field declines to zero with time. This prevents a divergence of the probe field in the limit it approaches the horizon.

This approach of a particle to the horizon on an accelerated frame increases the acceleration which reaches a maximum at the stretched horizon at $d = \ell_s = g_s^{-1/4}\ell_p$ gives the acceleration $g = g_s^{-1/4}5.6\times 10^{54}cm/s^2$. At this extreme acceleration the probe wave function for the particle is finite and the probe has reached the stretched horizon. The radial dimension at this limit is of no consequence to the wave/field dynamics of a particle or probe. The field is reduced to three dimensions of 2-space plus time. In this setting the field is reduced to anyon statistics. If there are two fields with some charge if their positions are permuted on a path γ there is an overall phase change given by their vector potential $\theta = \int_\gamma \mathbf{A}$ so that an interchange is $|\phi\phi\prime\rangle = e^{i\theta}|\phi\prime\phi\rangle$. This interchange may be interpreted according to the Jones polynomial, braid groups and knots. Anyons may also be a route to superconductivity [7], where it is also known that the event horizon of a black hole has a conductivity that permits the surface charge density on the stretched horizon to exponentially dilute across the horizon. The stretched horizon is known to be a classical conductor of charge. In the limit this probe is lowered to the stretched horizon from an accelerated frame this conductor may become superconducting. The increase in the acceleration needed to lower a probe to the stretched horizon is then an RG flow which leads to a phase change in the system.

It is worth noting that up to this point no new physics has been introduced. Everything from the changing acceleration of a frame to the black hole is based on standard accepted physics. The conclusions do however indicate there may be new physics, in particular with respect to a phase transition on the horizon boundary.

6 Topological Symmetry of Phase

The approach to the horizon of the black hole $d \to 0$, or equivalently the divergent increase in acceleration $g = c^2/d$, increases the number of degrees of freedom available to the probe up to a saturation point which then declines. As the probe becomes close to the event horizon at some point the number of modes available decreases until there are only string or Planck length modes available. The physics has entered into some for phase change. At a distance d far from the horizon with $d >> \lambda_{IR}$, for λ_{IR} the long wavelength cut off of the theory, there are many modes available with the UV modes suppressed. As the probe approaches the BH more UV modes are available until they appear to the exclusion of longer IR wavelength modes. The physics transitions from a Boltzmann type of statistics to a coherent state system.

The temperature of this system approaches the Hagedorn or Planck temperature and the number of modes available declines and the system becomes coherent. These modes are entangled with the black hole, and this entanglement is strongest at shorter distances. These states are in a quantum entanglement with the black hole at a short range, and where these states exhibit symmetry of spacetime/gauge fields, in the sense of a holography. These states are a large ground state degeneracy for a quantized non-Abelian gauge field with geometric phases for the degenerate states. This topological order occurs for a zero entropy situation, which for spacetime physics corresponds to the Planck temperature, instead of zero temperature. If the symmetry of the holonomy or geometric phase is corresponds to a particular phase of the system and topological order this is similar to a symmetry protected topological order[8].

The black hole or thermal event horizon describes the Unruh-Hawking vacuum for each value of the acceleration parameter g such that $\Gamma = g_1 + \frac{g_2}{g_1}sinh^2(g_1\tau)/\tau$ corresponds to a set of vacua. Each of these vacua are accompanied by a thermal bath of bosons. These bosons are often identified as gauge bosons, such as photons, and they obey some gauge symmetry. The set of vacua defined by Γ for this gauge field have the same symmetry and the set of states for each vacua are then continuously deformed into each other. The states in one vacua are then adiabatically adjusted into another by small changes in the acceleration parameter, which maintains the same gauge symmetry in each vacua. The statistics derived from the probe field do indicate however that the quantum statistics of the thermal bath of bosons changes as the probe is lowered very closely to the black hole horizon. Hence the continuous change invacua results in a phase transition of the system. The phase transition is similar to a Bose-Einstein condensate, or for a massless gauge boson with the transistion to overcomplete coherent states of a laser.

The probe field could only preserve its phase if the boson field is of another symmetry close to the horizon. The constriction in the number of degrees of freedom as $d \rightarrow 0$ available to the probe can only be reversed if there is a manner in which degrees of freedom can be combined to define new ones. The approach of the probe field to the horizon is a selection process which reduces the number of degrees of freedom available to the probe as this distance approaches the string or Planck scale. The phase of the system is preserved as the probe selects a set of states with a short range entanglement with the black hole. These states are in a particular quantum phase associated with this entanglement. Short ranged entangled states have trivial topological orders and the symmetry then protects the topological order of the system. If the number of degrees of freedom remains large near the horizon this means that the degrees of freedom the probe field measures near the horizon in the trivial topological order are transformed into an additional set of degrees of freedom. The symmetry of the system must be broken.

The restriction of this system to near the horizon is a highly boosted system. The dynamics is then largely in the space normal to the direction of the boost. The probe field is in each instance of time in a high Lorentz γ frame. The four momentum $P = (p,\ p_z,\ E)$ with p_z in the direction of the boost, is then

$$P^2 = m^2 = -p_z^2 - p^2 + E^2 \tag{6.1}$$

so the energy is

$$E = \sqrt{p_z^2 + p^2 + - m^2} = p_z\sqrt{1 + \frac{p^2}{p_z^2} + \frac{m^2}{p_z^2}} \simeq p_z + \frac{p^2}{2p_z} + \frac{m^2}{2p_z}, \tag{6.2}$$

where the last step is permitted because $p_z >> p$. This leads to the $2-1$ spacetime Hamiltonian $H\prime = \frac{1}{2}p^2 + \frac{1}{2}m^2$. The states are then in this reduced spacetime and non-trivial $2+1D$ SPT states carry non-trival statistics [9]and fractional quantum numbers [10] of the symmetry group.

SPT states have short range entanglements, which mean there is by their nature some scaling associated with them. This is something found in a correspondence between entangled states and the occurrence of event horizons [11]. This is a variant of gauge/gravity duality or holography. The equivalence between a conformal field theory and the asymptotic anti-de Sitter spacetime for large N is inidcates how gravitation may be quantized in a nonperturbative manner according to a quantum field theory [12]. The emergence of gravity may then be argued to emerge from entangled states or degrees of freedom of a quantum system, or a gauge theory, which is dual to an AdS spacetime. Raamsdonk illlustrated how state entanglement decreased by a variable parameter equivalent to an increasing proper distance between different regions of spacetime.

Suppose there exist two HIlbert spaces that compose a system with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with states on $\mathcal{H}$ of the form $|\psi\rangle = \sum_i |c_i|^2|\psi_i^1\rangle \otimes |\psi_i^2\rangle$. Each of these Hilbert spaces is an identical copy of the QFT. The density matrix

$$\rho_{ij} = |\psi\rangle\langle\psi| = \sum_i |c_i|^2|c_j|^2|\psi_i^1\rangle \otimes |\psi_i^2\rangle\langle\psi_i^1| \otimes \langle\psi_i^2| \tag{6.3}$$

for $|c_i|^2 = e^{-E_i\beta/2}$ gives a thermal density matrix with a trace over one Hilbert space. Quantum mutual information of operators in region $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by $I(1,\ 2) \sim \langle\mathcal{O}_1\mathcal{O}_2\rangle$. The entanglement may be reduced with the introduction of a distance measure $d(x_1,\ x_2)$ such that $\langle\mathcal{O}_1\mathcal{O}_2\rangle \sim e^{-d(x_1,x_2)}$. The tuning of the entanglement is then dependent upon this metric element which an easily be interpreted as a metric for a manifold. This connection is made firmer when it is noted that the thermal density matrix is identical to one for the AdS black hole.

The distance dependency on this entanglement has the properties of SPT states. The probe field that approaches the horizon will detect states in a different quantum phase as the local entanglement between states near the horizon and the black hole are "turned on.
rq' The probe field further from the horizon detects a thermal distribution of bosons that are statistically uncorrelated, but which have the same density matrix. The entangled state density matrix $|\psi\rangle\langle\psi|$ under the trace

$$\sum e^{-E\beta}|\psi_1\rangle\langle\psi_1| \ \rightarrow\ Schwarzschild\ BH \tag{6.4}$$

so the horizon and black hole state is ultimately constructed from entanglements, and where the symmetry of the underlying interaction is what preserves the topological order of the system.

The SPT is an energy gapped phase at zero temperature, which in our case is near the Hagedorn temperature, where the heat capacity of event horizons is negative. The SPT states with a symmetry given by a group $\mathcal{G}$ have topological orders given by the cohomology $H^2(\mathcal{G}.\ U(1))$. For the case $d\ =\ 2$ the 2-space plus ime model with the Euclideanized $\mathcal{G}\ =\ SO(3)$ gives the $2\ +\ 1$ spin Hall effect and the time revese symmetry group $\mathbb{Z}$. The projective representation of this group is found by the quotient with the normalizer of that group $P\mathcal{G}\ =\ \mathcal{G}/N$ with the map $\pi : \mathcal{G}\ =\ P\mathcal{G}$ that defines a bundle of lift elements. These elements λ obey for $g\ \in\ \mathcal{G}$ the rule $\lambda(g, g\prime)\ =\ \sigma(g, g\prime)\lambda(g)\lambda(g)$, where $\sigma(g, g\prime)$ is a Schur multiplier. The projective representation of the group "mods out
rq' the action of this normalizer. This is a cocycle in the cohomology $H^2(\mathcal{G},\ N)$ with the normalizer $N\ =\ U(1)$.

The Schur multiplier is a way in which the projective realization of a group can be represented according to the second cohomology that is an abelian group that defines a covering or line bundle. The projective representation is a homomorphism into the projective linear group $PGL(n,\ \mathbb{K})$,specifically for $\mathbb{K}\ =\ \mathbb{R}$ or $\mathbb{C}$, and for $n\ =\ 2$. A field F is mapped into $\mathbb{K}$, the group C and B have the sequence $1\ \rightarrow\ F\ \rightarrow C\ \rightarrow\ G\ \rightarrow\ 1$ for C the centeral of G and this sequence is the central extension of the group G. The following diagram ensues

$$\begin{array}{ccccc} F & \rightarrow & C & \rightarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{K} & \rightarrow & GL(n,\ \mathbb{K}) & \rightarrow & PGL(n,\ \mathbb{K}) \end{array} \tag{6.5}$$

The group $SO(2,\ 1)\ \sim\ SU(1,\ 1)$ is homomorphic to $SL(2,\ \mathbb{R}) \times SL(2,\ \mathbb{R})/\mathbb{Z}_2$. The linear fractional group $SL(2,\ \mathbb{Z})$ the gives the braid group as

$$\begin{array}{ccc} B_3 & \rightarrow & PSL(2,\ \mathbb{Z}) \\ \downarrow & & \downarrow \\ SL(2,\ \mathbb{R}) & \rightarrow & PSL(2,\ \mathbb{R}) \end{array} \tag{6.6}$$

The braid group is then the central extension of the linear fractional or modular group $PSL(2,\ \mathbb{Z})$. The braid group B_3 then contains a center which under this map gives $PSL(2,\ \mathbb{Z})\ \simeq\ B_3/C$. This center is under the correspondence between C and $G\ =\ B_3$ with the normalizer $N\ =\ U(1)$ projective realization $PSL(2,\ \mathbb{R})\ =\ SL(2,\ \mathbb{R})/U(1)$ a set of elements that defines the modular group as a projective realization of the braid group. This is easily seen. For the elementary braid group elements σ_1 and σ_2, $\sigma_i^2\ =\ 1$, we may define $x\ =\ \sigma_1\sigma_2\sigma_1$ and $y\ =\ \sigma_1\sigma_2$, with $z\ =\ x^2\ =\ y^3$, $z\ \in\ C$ is an element of the center with $\sigma_i z\sigma_i^{-1}\ =\ z$. The center of the braid group is given by correspondence between $\mathbb{R}$ and $\mathbb{Z}$ and the special linear group and the modular group of factional transformations.

The discrete group structure is a Kleinian system. With the AdS spacetime this discrete group constructs a conformal completion on the manifold [13]. The action constructed by the discrete system is in the continuum limit the Wess-Zumino-Witten Lagrangian [7][14] as well. A theory of this variety should be completely compatible with supergravity theories and the gravity/gauge duality.

7 Physical Quantum Gravity

Physically this is connected with the quantum mechanics of event horizons. The range of entanglement is near the event horizon and its "strength" is determined by distance from the event horizon. The symmetry protected topological order occurs when the horizon is not a "sharp" geometric object. The event horizon is quantum uncertain.

The uncertainty with the event horizon goes back to the Bohr-Einstein debate[15]. Einstein argued that a box on a weight scale could be used to circumvent the uncertainty principle. The argument assumes there is a box with a hole in one wall covered by a shutter which may be opened and closed by a clock mechanism inside the box. Radiation in the box would add to the weight of the box. The box would be weighed and at a given moment the clock would open the shutter allowing a single photon of radiation to escape. The box could then be re-weighed, the difference between the two weights telling us the amount of energy that escaped using the formula $E = mc^2$. Under the uncertainty principle it is not possible to obtain an exact measurement of the energy of the released photon and the time at which it was released. Einstein's experiment was designed to show such exact measurements were possible, the clock measuring the time of release of the energy and the weighing of the box disclosing the amount of energy involved. Bohr however realized that the gravity field is involved. A change in potential $\Delta\Phi = g\Delta h$ is such there is an uncertainty in the height of the box on the scale and $\Delta h\Delta p = \hbar/2$ for $\Delta p \sim mc$. This defeated Einstein's attempt to work around the uncertainty principle.

The black hole may serve as the box and the uncertainty is quantum tunneling of particles out of the black hole. The Schwarzschild metric element $g_{tt} = 1 - 2\Phi/c^2$, for $\Phi = GM/r$, is defined for a mass contained in a radius $R_s = 2GM/c^2$. The uncertainty in the mass of the black hole is then a fluctuation in the metric element

$$\Delta g_{tt} = \frac{2G\Delta M}{rc^2} = \frac{\Delta R_s}{r} \tag{7.1}$$

The gravity potential fluctuates with an uncertainty $\Delta\Phi\Delta t = \hbar/2$, and for $r = R_s$ we have that

$$\Delta R\Delta t = G$$

$\overline{\hbar}$ $\overline{c^2},(7.2)$ which gives a space-time complementary fluctuation of the radius (horizon radius) and time. This is the radius uncertainty as seen by an exterior observer associated with an uncertainty in time. The term $G\hbar/c^2$ is a space-time measure of the area of the event horizon Planck units.

The fuzzing out of the event horizon by quantum fluctuations according to a noncommutative geometry is similar to the sort of null geometry twistor theory provides. A set of null geodesics satisfy a coincidence condition. The pair of spinors ω^A and $\pi^{A\prime}$ define a point in Minkowsi spacetime with $\omega(x)^a = \omega^A + ix^{AA\prime}\pi_{A\prime}$, for $x^\mu = \sigma^\mu_{AA\prime}x^{AA\prime}$. The twistor $Z^\alpha = (\omega^A,\ \pi_{A\prime})$ define a norm $2s = Z^\alpha\bar{Z}_\alpha = \omega^a\bar{\pi}_A + \bar{\omega}^{A\prime}\pi_{A\prime}$, where for $s = 0$ the twistor is null. The twistor norm is an invariant of the group $SU(2,\ 2) \rightarrow^{4-1}_{cover} C(3,\ 1)$, where $C(3,\ 1)$ is the conformal group of compactified Minkowski spacetimes[16]. The set of rays defines the space $\mathbb{CP}^3$, which is projective twistor space and $\mathbb{CT} = \mathbb{C}^4$. The projective twistor space is the identified with complexified Minkowski spacetime in a double fibration

$$\mathbb{CT} \leftarrow_\mu F^{1,2}(\mathbb{T})\ {}_\nu\!\rightarrow\ M \tag{7.3}$$

where the construction is with flag manifolds $F^1(\mathbb{T}) = \mathbb{CP}^3$ and $F^2(\mathbb{T}) = \mathbb{G}_{4,2}(\mathbb{C})$, where the latter is the compact complex Minkowki spacetime. The symmetry group $SU(2,\ 2) \simeq SO(4,\ 2)$ is the isometry group for the anti-de Sitter spacetime AdS_5. The boundary of the AdS_5 is the Einstein space $\partial AdS_5 = E_4$, or in the compact complex form is $\mathbb{G}_{4,2}(\mathbb{C})$. This data is contained in the double fibration of the flag manifold.

The coincidence equation $\omega^A = x^{AA\prime}\pi^{A\prime}$ contains the spinor in matrix form with components

$$x^\mu \mapsto x^{AA\prime} = \frac{1}{\sqrt{2}}\begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix} \tag{7.4}$$

The frame is boosted in the case above so the momentum along the direction of boost is removed. By the Lorentz contraction of lengths along that direction, along z, we may then consider $z \rightarrow 0$ and interchange the meaning of time and the z-direction with

$$x_3^{AA\prime} = \frac{1}{\sqrt{2}}\begin{pmatrix} t & x + iy \\ x - iy & -t \end{pmatrix}. \tag{7.5}$$

The spinors define the $SU(2) \rightarrow SU(1,\ 1)$, with the hyperbolic form occurring with $t \rightarrow it$.

This reduction from $SU(2,\ 2) \rightarrow SU(1,\ 1)$ reduces the anti-de Sitter isometry group to $SL(2,\ \mathbb{R})$ for AdS_2. The reduction from four to three dimensions with Minkowski spacetime induces the map $AdS_5 \rightarrow AdS_2$. The correspondence between the braid group B_3 and $SL(2,\ \mathbb{Z})$ in the lower dimensional case similarly carries in the AdS_5 case. In the higher dimensional case the discrete structure will be more complex.

The STP approach to quantum gravity leads into a homotopy realizations of quantum physics[17]. There have been recent developments along these lines by Isham [18] and others that connect category theory with a deeper foundation to physics and cosmology. This may mean that topology as the basis of physics is in its greatest generality is a category theory of types. The additional prospect exists that that foundations of mathematics is a homotopy type theory (HOTT) with connections to category theory[19].

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