Article

TGD Variant of Twistor Story

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Abstract

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D imbedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure. The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the imbedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD and classical TGD defined by the extremals of Kähler action. In the following I summarize the background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams.

There is also a very closely analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework. The physical interpretation is totally different in TGD. The landscape is replaced with twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings. The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach. Furthermore, one ends up to a formulation of the scattering amplitudes in terms of Yangian of the super-symplectic algebra relying on the idea that scattering amplitudes are sequences consisting of algebraic operations (product and co-product) having interpretation as vertices in the Yangian extension of super-symplectic algebra. These sequences connect given initial and final states and having minimal length. One can say that Universe performs calculations.

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1 Introduction

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D imbedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the imbedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD [36, 37, 43] and classical TGD [35] defined by the extremals of Kähler action.

In the following I summarize first the basic results and problems of the twistor approach. After that I describe some of the mathematical background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams having as lines space-time surfaces with Euclidian signature of induced metric and having wormhole contacts as basic building bricks.

There is also a very close analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds [1, 5] and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry [20]emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. Twistor space has space-time as base-space rather than forming with it Cartesian factors of a 10-D space-time. The Calabi-Yau landscape is replaced with the space of twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings [22]. The space of twistor spaces is the lift of the "world of classical worlds" (WCW) by adding the CP_1 fiber to the space-time surfaces so that the analog of landscape has beautiful geometrization.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach.

1. The notion of quaternion analyticity extending the notion of ordinary analyticity to 4-D context is highly attractive but has remained one of the long-standing ideas difficult to take quite seriously but equally difficult to throw to paper basked. Four-manifolds possess almost quaternion structure. In twistor space context the formulation of quaternion analyticity becomes possible and relies on an old notion of tri-holomorphy about which I had not been aware earlier. The natural formulation for the preferred extremal property is as a condition stating that various charges associated with generalized conformal algebras vanish for preferred extremals. This leads to ask whether Euclidian space-time regions could be quaternion-Kähler manifolds for which twistor spaces are so called Fano spaces. In Minkowskian regions so called Hamilton-Jacobi property would apply.

2. The generalization of Witten's twistor theory to TGD framework is a natural challenge and the 2-surfaces studied defining scattering amplitudes in Witten's theory could correspond to partonic 2-surfaces identified as algebraic surfaces characterized by degree and genus. Besides this also string world sheets are needed. String worlds have 1-D lines at the light-like orbits of partonic 2-surfaces as their boundaries serving as carriers of fermions. This leads to a rather detailed generalization of Witten's approach using the generalization of twistors to 8-D context.

3. The generalization of the twistor Grassmannian approach to 8-D context is second fascinating challenge. If one requires that the basic formulas relating twistors and four-momentum generalize

one must consider the situation in tangent space M^8 of imbedding space ($M^8 - H$ duality) and replace the usual sigma matrices having interpretation in terms of complexified quaternions with octonionic sigma matrices.

The condition that octonionic spinors are are equivalent with ordinary spinors has strong consequences. Induced spinors must be localized to 2-D string world sheets, which are (co-)commutative sub-manifolds of (co-)quaternionic space-time surface. Also the gauge fields should vanish since they induce a breaking of associativity even for quaternionic and complex surface so that CP_2 projection of string world sheet must be 1-D. If one requires also the vanishing of gauge potentials, the projection is geodesic circle of CP_2 so that string world sheets are restricted to Minkowskian space-time regions. Although the theory would be free in fermionic degrees of freedom, the scattering amplitudes are non-trivial since vertices correspond to partonic 2-surfaces at which partonic orbits are glued together along common ends. The classical light-like 8-momentum associated with the boundaries of string world sheets defines the gravitational dual for 4-D momentum and color quantum numbers associated with imbedding space spinor harmonics. This leads to a more detailed formulation of Equivalence Principle which would reduce to $M^8 - H$ duality basically.

Number theoretic interpretation of the positivity of Grassmannians is highly suggestive since the canonical identification maps p-adic numbers to non-negative real numbers. A possible generalization is obtained by replacing positive real axis with upper half plane defining hyperbolic space having key role in the theory of Riemann surfaces. The interpretation of scattering amplitudes as representations of permutations generalizes to interpretation as braidings at surfaces formed by the generalized Feynman diagrams having as lines the light-like orbits of partonic surfaces. This because 2-fermion vertex is the only interaction vertex and induced by the non-continuity of the induced Dirac operator at partonic 2-surfaces. OZI rule generalizes and implies an interpretation in terms of braiding consistent with the TGD as almost topological QFT vision. This suggests that non-planar twistor amplitudes are constructible as analogs of knot and braid invariants by a recursive procedure giving as an outcome planar amplitudes.

4. Yangian symmetry is associated with twistor amplitudes and emerges in TGD from completely different idea interpreting scattering amplitudes as representations of algebraic manipulation sequences of minimal length (preferred extremal instead of path integral over space-time surfaces) connecting given initial and final states at boundaries of causal diamond. The algebraic manipulations are carried out in Yangian using product and co-product defining the basic 3-vertices analogous to gauge boson absorption and emission. 3-surface representing elementary particle splits into two or vice versa such that second copy carries quantum numbers of gauge boson or its super counterpart. This would fix the scattering amplitude for given 3-surface and leave only the functional integral over 3-surfaces.

2 Background and motivations

In the following some background plus basic facts and definitions related to twistor spaces are summarized. Also reasons for why twistor are so relevant for TGD is considered at general level.

2.1 Basic results and problems of twistor approach

The author describes both the basic ideas and results of twistor approach as well as the problems.

2.1.1 Basic results

There are three deep results of twistor approach besides the impressive results which have emerged after the twistor resolution.

1. Massless fields of arbitrary helicity in 4-D Minkowski space are in 1-1 correspondence with elements of Dolbeault cohomology in the twistor space CP_3. This was already the discovery of Penrose..The connection comes from Penrose transform. The light-like geodesics of M^4 correspond to points of 5-D sub-manifold of CP_3 analogous to light-cone boundary. The points of M^4 correspond to complex lines (Riemann spheres) of the twistor space CP_3: one can imagine that the point of M^4 corresponds to all light-like geodesics emanating from it and thus to a 2-D surface (sphere) of CP_3. Twistor transform represents the value of a massless field at point of M^4 as a weighted average of its values at sphere of CP_3. This correspondence is formulated between open sets of M^4 and of CP_3. This fits very nicely with the needs of TGD since causal diamonds which can be regarded as open sets of M^4 are the basic objects in zero energy ontology (ZEO).

2. Self-dual instantons of non-Abelian gauge theories for SU(n) gauge group are in one-one correspondence with holomorphic rank-N vector bundles in twistor space satisfying some additional conditions. This generalizes the correspondence of Penrose to the non-Abelian case. Instantons are also usually formulated using classical field theory at four-sphere S^4 having Euclidian signature.

3. Non-linear gravitons having self-dual geometry are in one-one correspondence with spaces obtained as complex deformations of twistor space satisfying certain additional conditions. This is a generalization of Penrose's discovery to the gravitational sector.

Complexification of M^4 emerges unavoidably in twistorial approach and Minkowski space identified as a particular real slice of complexified M^4 corresponds to the 5-D subspace of twistor space in which the quadratic form defined by the SU(2,2) invariant metric of the 8-dimensional space giving twistor space as its projectivization vanishes. The quadratic form has also positive and negative values with its sign defining a projective invariant, and this correspond to complex continuations of M^4 in which positive/negative energy parts of fields approach to zero for large values of imaginary part of M^4 time coordinate.

Interestgingly, this complexification of M^4 is also unavoidable in the number theoretic approach to TGD: what one must do is to replace 4-D Minkowski space with a 4-D slice of 8-D complexified quaternions. What is interesting is that real M^4 appears as a projective invariant consisting of light-like projective vectors of C^4 with metric signature (4,4). Equivalently, the points of M^4 represented as linear combinations of sigma matrices define hermitian matrices.

2.1.2 Basic problems of twistor approach

The best manner to learn something essential about a new idea is to learn about its problems. Difficulties are often put under the rug but the thesis is however an exception in this respect. It starts directly from the problems of twistor approach. There are two basic challenges.

1. Twistor approach works as such only in the case of Minkowski space. The basic condition for its applicability is that the Weyl tensor is self-dual. For Minkowskian signature this leaves only Minkowski space under consideration. For Euclidian signature the conditions are not quite so restrictive. This looks a fatal restriction if one wants to generalize the result of Penrose to a general space-time geometry. This difficulty is known as "googly" problem.

   According to the thesis MHV construction of tree amplitudes of $\mathcal{N} = 4$ SYM based on topological twistor strings in CP_3 meant a breakthrough and one can indeed understand also have analogs of non-self-dual amplitudes. The problem is however that the gravitational theory assignable to topological twistor strings is conformal gravity, which is generally regarded as non-physical. There have been several attempts to construct the on-shell scattering amplitudes of Einstein's gravity theory as subset of amplitudes of conformal gravity and also thesis considers this problem.

2. The construction of quantum theory based on twistor approach represents second challenge. In this respect the development of twistor approach to $\mathcal{N}=4$ SYM meant a revolution and one can indeed construct twistorial scattering amplitudes in M^4.

2.2 Results about twistors relevant for TGD

First some background.

1. The twistors originally introduced by Penrose (1967) have made breakthrough during last decade. First came the twistor string theory of Edward Witten [22] proposed twistor string theory and the work of Nima-Arkani Hamed and collaborators [12] led to a revolution in the understanding of the scattering amplitudes of scattering amplitudes of gauge theories [15, 21, 13]. Twistors do not only provide an extremely effective calculational method giving even hopes about explicit formulas for the scattering amplitudes of $\mathcal{N}=4$ supersymmetric gauge theories but also lead to an identification of a new symmetry: Yangian symmetry [2], [17, 16], which can be seen as multilocal generalization of local symmetries.

   This approach, if suitably generalized, is tailor-made also for the needs of TGD. This is why I got seriously interested on whether and how the twistor approach in empty Minkowski space M^4 could generalize to the case of $H=M^4\times CP_2$. The twistor space associated with H should be just the cartesian product of those associated with its Cartesian factors. Can one assign a twistor space with CP_2?

2. First a general result [3] deserves to be mentioned: any oriented manifold X with Riemann metric allows 6-dimensional twistor space Z as an almost complex space. If this structure is integrable, Z becomes a complex manifold, whose geometry describes the conformal geometry of X. In general relativity framework the problem is that field equations do not imply conformal geometry and twistor Grassmann approach certainly requires conformal structure.

3. One can consider also a stronger condition: what if the twistor space allows also Kähler structure? The twistor space of empty Minkowski space M^4 (and its Euclidian counterpart S^4 is the Minkowskian variant of $P_3=SU(2,2)/SU(2,1)\times U(1)$ of 3-D complex projective space $CP_3=SU(4)/SU(3)\times U(1)$ and indeed allows Kähler structure.

   Rather remarkably, there are *no other space-times* with Minkowski signature allowing twistor space with Kähler structure. Does this mean that the empty Minkowski space of special relativity is much more than a limit at which space-time is empty?

   This also means a problem for GRT. Twistor space with Kähler structure seems to be needed but general relativity does not allow it. Besides twistor problem GRT also has energy problem: matter makes space-time curved and the conservation laws and even the definition of energy and momentum are lost since the underlying symmetries giving rise to the conservation laws through Noether's theorem are lost. GRT has therefore two bad mathematical problems which might explain why the quantization of GRT fails. This would not be surprising since quantum theory is to high extent representation theory for symmetries and symmetries are lost. Twistors would extend these symmetries to Yangian symmetry but GRT does not allow them.

4. What about twistor structure in CP_2? CP_2 allows complex structure (Weyl tensor is self-dual), Kähler structure plus accompanying symplectic structure, and also quaternion structure. One of the really big personal surprises of the last years has been that CP_2 twistor space indeed allows Kähler structure meaning the existence of antisymmetric tensor representing imaginary unit whose tensor square is the negative of metric in turn representing real unit.

   The article by Nigel Hitchin, a famous mathematical physicist, describes a detailed argument identifying S^4 and CP_2 as the only compact Riemann manifolds allowing Kählerian twistor space [3].

Hitchin sent his discovery for publication 1979. An amusing co-incidence is that I discovered CP_2 just this year after having worked with S^2 and found that it does not really allow to understand standard model quantum numbers and gauge fields. It is difficult to avoid thinking that maybe synchrony indeed a real phenomenon as TGD inspired theory of consciousness predicts to be possible but its creator cannot quite believe. Brains at different side of globe discover simultaneously something closely related to what some conscious self at the higher level of hierarchy using us as instruments of thinking just as we use nerve cells is intensely pondering.

Although 4-sphere S^4 allows twistor space with Kähler structure, it does not allow Kähler structure and cannot serve as candidate for S in $H = M^4 \times S$. As a matter of fact, S^4 can be seen as a Wick rotation of M^4 and indeed its twistor space is CP_3.

In TGD framework a slightly different interpretation suggests itself. The Cartesian products of the intersections of future and past light-cones - causal diamonds (CDs) - with CP_2 - play a key role in zero energy ontology (ZEO) [23]. Sectors of "world of classical worlds" (WCW) [32, 28] correspond to 4-surfaces inside $CD \times CP_2$ defining a the region about which conscious observer can gain conscious information: state function reductions - quantum measurements - take place at its light-like boundaries in accordance with holography. To be more precise, wave functions in the moduli space of CDs are involved and in state function reductions come as sequences taking place at a given fixed boundary. This kind of sequence is identifiable as self and give rise to the experience about flow of time. When one replaces Minkowski metric with Euclidian metric, the light-like boundaries of CD are contracted to a point and one obtains topology of 4-sphere S^4.

5. Another really big personal surprise was that there are *no other* compact 4-manifolds with Euclidian signature of metric allowing twistor space with Kähler structure! The imbedding space $H = M^4 \times CP_2$ is not only physically unique since it predicts the quantum number spectrum and classical gauge potentials consistent with standard model but also mathematically unique!

   After this I dared to predict that TGD will be the theory next to GRT since TGD generalizes string model by bringing in 4-D space-time. The reasons are many-fold: TGD is the only known solution to the two big problems of GRT: energy problem and twistor problem. TGD is consistent with standard model physics and leads to a revolution concerning the identification of space-time at microscopic level: at macroscopic level it leads to GRT but explains some of its anomalies for which there is empirical evidence (for instance, the observation that neutrinos arrived from SN1987A at two different speeds different from light velocity [**?**]SN1987A has natural explanation in terms of many-sheeted space-time). TGD avoids the landscape problem of M-theory and anthropic nonsense. I could continue the list but I think that this is enough.

6. The twistor space of CP_2 is 3-complex dimensional flag manifold $F_3 = SU(3)/U(1) \times U(1)$ having interpretation as the space for the choices of quantization axes for the color hypercharge and isospin. This choice is made in quantum measurement of these quantum numbers and a means localization to single point in F_3. The localization in F_3 could be higher level measurement leading to the choice of quantizations for the measurement of color quantum numbers.

   F_3 is symmetric space meaning that besides being a coset space with $SU(3)$ invariant metric it also has involutions acting as a reflection at geodesics through a point remaining fixed under the involution. As a symmetric space with Fubini-Study metric F_3 is positive constant curvature space having thus positive constant sectional curvatures. This implies Einstein space property. This also conforms with the fact that F_3 is CP_1 bundle over CP_2 as base space (for more details see `http://www.cirget.uqam.ca/~apostolo/papers/AGAG1.pdf`).

7. Analogous interpretation could make sense for M^4 twistors represented as points of P_3. Twistor corresponds to a light-like line going through some point of M^4 being labelled by 4 position coordinates and 2 direction angles: what higher level quantum measurement could involve a choice of light-like line going through a point of M^4? Could the associated spatial direction specify spin quantization

axes? Could the associated time direction specify preferred rest frame? Does the choice of position mean localization in the measurement of position? Do momentum twistors relate to the localization in momentum space? These questions remain fascinating open questions and I hope that they will lead to a considerable progress in the understanding of quantum TGD.

8. It must be added that the twistor space of CP_2 popped up much earlier in a rather unexpected context [31]: I did not of course realize that it was twistor space. Topologist Barbara Shipman [7] has proposed a model for the honeybee dance leading to the emergence of F_3. The model led her to propose that quarks and gluons might have something to do with biology. Because of her position and specialization the proposal was forgiven and forgotten by community. TGD however suggests both dark matter hierarchies and p-adic hierarchies of physics [30, 45]. For dark hierarchies the masses of particles would be the standard ones but the Compton scales would be scaled up by $h_{eff}/h = n$ [45]. Below the Compton scale one would have effectively massless gauge boson: this could mean free quarks and massless gluons even in cell length scales. For p-adic hierarchy mass scales would be scaled up or down from their standard values depending on the value of the p-adic prime.

2.3 Basic definitions related to twistor spaces

One can find from web several articles explaining the basic notions related to twistor spaces and Calabi-Yau manifolds. At the first look the notions of twistor as it appears in the writings of physicists and mathematicians don't seem to have much common with each other and it requires effort to build the bridge between these views. The bridge comes from the association of points of Minkowski space with the spheres of twistor space: this clearly corresponds to a bundle projection from the fiber to the base space, now Minkowski space. The connection of the mathematician's formulation with spinors remains still somewhat unclear to me although one can understand CP_1 as projective space associated with spinors with 2 complex components. Minkowski signature poses additional challenges. In the following I try my best to summarize the mathematician's view, which is very natural in classical TGD.

There are many variants of the notion of twistor depending on whether how powerful assumptions one is willing to make. The weakest definition of twistor space is as CP_1 bundle of almost complex structures in the tangent spaces of an orientable 4-manifold. Complex structure at given point means selection of antisymmetric form J whose natural action on vector rotates a vector in the plane defined by it by $\pi/2$ and thus represents the action of imaginary unit. One must perform this kind of choice also in normal plane and the direct sum of the two choices defines the full J. If one choses J to be self-dual or anti-self-dual (eigenstate of Hodge star operation), one can fix J uniquely. Orientability makes possible the Hodge start operation involving 4-dimensional permutation tensor.

The condition $i^1 = -1$ is translated to the condition that the tensor square of J equals to $J^2 = -g$. The possible choices of J span sphere S^2 defining the fiber of the twistor spaces. This is not quite the complex sphere CP_1, which can be thought of as a projective space of spinors with two complex components. Complexification must be performed in both the tangent space of X^4 and of S^2. Note that in the standard approach to twistors the entire 6-D space is projective space P_3 associated with the C^8 having interpretation in terms of spinors with 4 complex components.

One can introduce almost complex structure also to the twistor space itself by extending the almost complex structure in the 6-D tangent space obtained by a preferred choices of J by identifiying it as a point of S^2 and acting in other points of S^2 identified as antisymmetric tensors. If these points are interpreted as imaginary quaternion units, the action is commutator action divided by 2. The existence of quaternion structure of space-time surfaces in the sense as I have proposed in TGD framework might be closely related to the twistor structure.

Twistor structure as bundle of almost complex structures having itself almost complex structure is characterized by a hermitian Kähler form ω defining the almost complex structure of the twistor space. Three basic objects are involved: the hermitian form h, metric g and Kähler form ω satisfying $h = g + i\omega$,

$g(X,Y) = \omega(X, JY)$.

In the base space the metric of twistor space is the metric of the base space and in the tangent space of fibre the natural metric in the space of antisymmetric tensors induced by the metric of the base space. Hence the properties of the twistor structure depend on the metric of the base space.

The relationship to the spinors requires clarification. For 2-spinors one has natural Lorentz invariant antisymmetric bilinear form and this seems to be the counterpart for J?

One can consider various additional conditions on the definition of twistor space.

1. Kähler form ω is not closed in general. If it is, it defines symplectic structure and Kähler structure. S^4 and CP_2 are the only compact spaces allowing twistor space with Kähler structure.

2. Almost complex structure is not integrable in general. In the general case integrability requires that each point of space belongs to an open set in which vector fields of type (1, 0) or (0, 1) having basis ∂/∂_{z^k} and $\partial/\partial_{\overline{z}^k}$ expressible as linear combinations of real vector fields with complex coefficients commute to vector fields of same type. This is non-trivial conditions since the leading names for the vector field for the partial derivatives does not yet guarantee these conditions.

   This necessary condition is also enough for integrability as Newlander and Nirenberg have demonstrated. An explicit formulation for the integrability is as the vanishing of Nijenhuis tensor associated with the antisymmetric form J (see (`http://insti.physics.sunysb.edu/conf/simonsworkII/talks/LeBrun.pdf` and `http://en.wikipedia.org/wiki/Almost_complex_manifold#Integrable_almost_complex_structures`). Nijenhuis tensor characterizes Nijenhuis bracket generalizing ordinary Lie bracket of vector fields (for detailed formula see `http://en.wikipedia.org/wiki/FrlicherNijenhuis_bracket`).

3. In the case of twistor spaces there is an alternative formulation for the integrability. Curvature tensor maps in a natural manner 2-forms to 2-forms and one can decompose the Weyl tensor W identified as the traceless part of the curvature tensor to self-dual and anti-self-dual parts W^+ and W^-, whose actions are restricted to self-dual resp. antiself-dual forms (self-dual and anti-self-dual parts correspond to eigenvalue + 1 and -1 under the action of Hodge * operation: for more details see `http://www.math.ucla.edu/~greene/YauTwister(8-9).pdf`). If W^+ or W^- vanishes - in other worlds W is self-dual or anti-self-dual - the assumption that J is self-dual or anti-self-dual guarantees integrability. One says that the metric is anti-self-dual (ASD). Note that the vanishing of Weyl tensor implies local conformal flatness (M^4 and sphere are obviously conformally flat). One might think that ASD condition guarantees that the parallel translation leaves J invariant.

   ASD property has a nice implication: the metric is balanced. In other words one has $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$.

4. If the existence of complex structure is taken as a part of definition of twistor structure, one encounters difficulties in general relativity. The failure of spin structure to exist is similar difficulty: for CP_2 one must indeed generalize the spin structure by coupling Kähler gauge potential to the spinors suitably so that one obtains gauge group of electroweak interactions.

5. One could also give up the global existence of complex structure and require symplectic structure globally: this would give $d\omega = 0$. A general result is that hyperbolic 4-manifolds allow symplectic structure and ASD manifolds allow complex structure and hence balanced metric.

2.4 Why twistor spaces with Kähler structure?

I have not yet even tried to answer an obvious question. Why the fact that M^4 and CP_2 have twistor spaces with Kähler structure could be so important that it could fix the entire physics? Let us consider a less general question. Why they would be so important for the classical TGD - exact part of quantum TGD - defined by the extremals of Kähler action [25] ?

1. Properly generalized conformal symmetries are crucial for the mathematical structure of TGD [28, 40, 27]. Twistor spaces have almost complex structure and in these two special cases also complex, Kähler, and symplectic structures (note that the integrability of the almost complex structure to complex structure requires the self-duality of the Weyl tensor of the 4-D manifold).

   The Cartesian product $CP_3 \times F_3$ of the two twistor spaces with Kähler structure is expected to be fundamental for TGD. The obvious wishful thought is that this space makes possible the construction of the extremals of Kähler action in terms of holomorphic surfaces defining 6-D twistor sub-spaces of $CP_3 \times F_3$ allowing to circumvent the technical problems due to the signature of M^4 encountered at the level of $M^4 \times CP_2$. It would also make the the magnificent machinery of the algebraic geometry so powerful in string theories a tool of TGD. For years ago I considered the possibility that complex 3-manifolds of $CP_3 \times CP_3$ could have the structure of S^2 fiber space and have space-time surfaces as base space. I did not realize that this spaces could be twistor spaces nor did I realize that CP_2 allows twistor space with Kähler structure so that $CP_3 \times F_3$ is a more plausible choice.

2. Every 4-D orientable Riemann manifold allows a twistor space as 6-D bundle with CP_1 as fiber and possessing almost complex structure. Metric and various gauge potentials are obtained by inducing the corresponding bundle structures. Hence the natural guess is that the twistor structure of space-time surface defined by the induced metric is obtained by induction from that for $CP_3 \times F_3$ by restricting its twistor structure to a 6-D (in real sense) surface of $CP_3 \times F_3$ with a structure of twistor space having at least almost complex structure with CP_1 as a fiber. If so then one can indeed identify the base space as 4-D space-time surface in $M^4 \times SCP_2$ using bundle projections in the factors CP_3 and F_3.

3. There might be also a connection to the number theoretic vision about the extremals of Kähler action. At space-time level however complexified quaternions and octonions could allow alternative formulation. I have indeed proposed that space-time surfaces have associative of co-associative meaning that the tangent space or normal space at a given point belongs to quaternionic subspace of complexified octonions.

3 About the identification of 6-D twistor spaces as sub-manifolds of $CP_3 \times F_3$

How to identify the 6-D sub-manifolds with the structure of twistor space? Is this property all that is needed? Can one find a simple solution to this condition? What is the relationship of twistor spaces to the Calabi-Yau manifolds of suyper string models? In the following intuitive considerations of a simple minded physicist. Mathematician could probably make much more interesting comments.

3.1 Conditions for twistor spaces as sub-manifolds

Consider the conditions that must be satisfied using local trivializations of the twistor spaces. Before continuing let us introduce complex coordinates $z_i = x_i + iy_i$ resp. $w_i = u_i + iv_i$ for CP_3 resp. F_3.

1. 6 conditions are required and they must give rise by bundle projection to 4 conditions relating the coordinates in the Cartesian product of the base spaces of the two bundles involved and thus defining 4-D surface in the Cartesian product of compactified M^4 and CP_2.

2. One has Cartesian product of two fiber spaces with fiber CP_1 giving fiber space with fiber $CP_1^1 \times CP_1^2$. For the 6-D surface the fiber must be CP_1. It seems that one must identify the two spheres CP_1^i. Since holomorphy is essential, holomorphic identification $w_1 = f(z_1)$ or $z_1 = f(w_1)$ is the first guess. A stronger condition is that the function f is meromorphic having thus only finite numbers

of poles and zeros of finite order so that a given point of CP_1^i is covered by CP_1^{i+1}. Even stronger and very natural condition is that the identification is bijection so that only Möbius transformations parametrized by SL(2, C) are possible.

3. Could the Möbius transformation $f : CP_1^1 \rightarrow CP_1^2$ depend parametrically on the coordinates z_2, z_3 so that one would have $w_1 = f_1(z_1, z_2, z_3)$, where the complex parameters a, b, c, d $(ad - bc = 1)$ of Möbius transformation depend on z_2 and z_3 holomorphically? Does this mean the analog of local SL(2, C) gauge invariance posing additional conditions? Does this mean that the twistor space as surface is determined up to SL(2, C) gauge transformation?

   What conditions can one pose on the dependence of the parameters a, b, c, d of the Möbius transformation on (z_2, z_3)? The spheres CP_1 defined by the conditions $w_1 = f(z_1, z_2, z_3)$ and $z_1 = g(w_1, w_2, w_3)$ must be identical. Inverting the first condition one obtains $z_1 = f^{-1}(w_1, z_2, z_3)$. If one requires that his allows an expression as $z_1 = g(w_1, w_2, w_3)$, one must assume that z_2 and z_3 can be expressed as holomorphic functions of (w_2, w_3): $z_i = f_i(w_k)$, $i = 2, 3$, $k = 2, 3$. Of course, non-holomorphic correspondence cannot be excluded.

4. Further conditions are obtained by demanding that the known extremals - at least non-vacuum extremals - are allowed. The known extremals [25] can be classified into CP_2 type vacuum extremals with 1-D light-like curve as M^4 projection, to vacuum extremals with CP_2 projection, which is Lagrangian sub-manifold and thus at most 2-dimensional, to massless extremals with 2-D CP_2 projection such that CP_2 coordinates depend on arbitrary manner on light-like coordinate defining local propagation direction and space-like coordinate defining a local polarization direction, and to string like objects with string world sheet as M^4 projection (minimal surface) and 2-D complex sub-manifold of CP_2 as CP_2 projection, . There are certainly also other extremals such as magnetic flux tubes resulting as deformations of string like objects. Number theoretic vision relying on classical number fields suggest a very general construction based on the notion of associativity of tangent space or co-tangent space.

5. The conditions coming from these extremals reduce to 4 conditions expressible in the holomorphic case in terms of the base space coordinates (z_2, z_3) and (w_2, w_3) and in the more general case in terms of the corresponding real coordinates. It seems that holomorphic ansatz is not consistent with the existence of vacuum extremals, which however give vanishing contribution to transition amplitudes since WCW ("world of classical worlds") metric is completely degenerate for them.

   The mere condition that one has CP_1 fiber bundle structure does not force field equations since it leaves the dependence between real coordinates of the base spaces free. Of course, CP_1 bundle structure alone does not imply twistor space structure. One can ask whether non-vacuum extremals could correspond to holomorphic constraints between (z_2, z_3) and (w_2, w_3).

6. The metric of twistor space is not Kähler in the general case. However, if it allows complex structure there is a Hermitian form ω, which defines what is called balanced Kähler form [8] satisfying $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$: ordinary Kähler form satisfying $d\omega = 0$ is special case about this. The natural metric of compact 6-dimensional twistor space is therefore balanced. Clearly, mere CP_1 bundle structure is not enough for the twistor structure. If the the Kähler and symplectic forms are induced from those of $CP_3 \times Y_3$, highly non-trivial conditions are obtained for the imbedding of the twistor space, and one might hope that they are equivalent with those implied by Kähler action at the level of base space.

7. Pessimist could argue that field equations are additional conditions completely independent of the conditions realizing the bundle structure! One cannot exclude this possibility. Mathematician could easily answer the question about whether the proposed CP_1 bundle structure with some added conditions is enough to produce twistor space or not and whether field equations could be the additional condition and realized using the holomorphic ansatz.

3.2 Twistor spaces by adding CP_1 fiber to space-time surfaces

The physical picture behind TGD is the safest starting point in an attempt to gain some idea about what the twistor spaces look like.

1. Canonical imbeddings of M^4 and CP_2 and their disjoint unions are certainly the natural starting point and correspond to canonical imbeddings of CP_3 and F_3 to $CP_3 \times F_3$.

2. Deformations of M^4 correspond to space-time sheets with Minkowskian signature of the induced metric and those of CP_2 to the lines of generalized Feynman diagrams. The simplest deformations of M^4 are vacuum extremals with CP_2 projection which is Lagrangian manifold.

   Massless extremals represent non-vacuum deformations with 2-D CP_2 projection. CP_2 coordinates depend on local light-like direction defining the analog of wave vector and local polarization direction orthogonal to it.

   The simplest deformations of CP_2 are CP_2 type extremals with light-like curve as M^4 projection and have same Kähler form and metric as CP_2. These space-time regions have Euclidian signature of metric and light-like 3-surfaces separating Euclidian and Minkowskian regions define parton orbits.

   String like objects are extremals of type $X^2 \times Y^2$, X^2 minimal surface in M^4 and Y^2 a complex sub-manifold of CP_2. Magnetic flux tubes carrying monopole flux are deformations of these.

   Elementary particles are important piece of picture. They have as building bricks wormhole contacts connecting space-time sheets and the contacts carry monopole flux. This requires at least two wormhole contacts connected by flux tubes with opposite flux at the parallel sheets.

3. Space-time surfaces are constructed using as building bricks space-time sheets, in particular massless exrremals, deformed pieces of CP_2 defining lines of generalized Feynman diagrams as orbits of wormhole contacts, and magnetic flux tubes connecting the lines. Space-time surfaces have in the generic case discrete set of self intersections and it is natural to remove them by connected sum operation. Same applies to twistor spaces as sub-manifolds of $CP_3 \times F_3$ and this leads to a construction analogous to that used to remove singularities of Calabi-Yau spaces [8].

Physical intuition suggests that it is possible to find twistor spaces associated with the basic building bricks and to lift this engineering procedure to the level of twistor space in the sense that the twistor projections of twistor spaces would give these structure. Lifting would essentially mean assigning CP_1 fiber to the space-time surfaces.

1. Twistor spaces should decompose to regions for which the metric induced from the $CP_3 \times F_3$ metric has different signature. In particular, light-like 5-surfaces should replace the light-like 3-surfaces as causal horizons. The signature of the Hermitian metric of 4-D (in complex sense) twistor space is (1, 1, -1, -1). Minkowskian variant of CP_3 is defined as projective space $SU(2,2)/SU(2,1) \times U(1)$. The causal diamond (CD) (intersection of future and past directed light-cones) is the key geometric object in zero energy ontology (ZEO) and the generalization to the intersection of twistorial light-cones is suggestive.

2. Projective twistor space has regions of positive and negative projective norm, which are 3-D complex manifolds. It has also a 5-dimensional sub-space consisting of null twistors analogous to light-cone and has one null direction in the induced metric. This light-cone has conic singularity analogous to the tip of the light-cone of M^4.

   These conic singularities are important in the mathematical theory of Calabi-You manifolds since topology change of Calabi-Yau manifolds via the elimination of the singularity can be associated with them. The S^2 bundle character implies the structure of S^2 bundle for the base of the singularity (analogous to the base of the ordinary cone).

3. Null twistor space corresponds at the level of M^4 to the light-cone boundary (causal diamond has two light-like boundaries). What about the light-like orbits of partonic 2-surfaces whose light-likeness is due to the presence of CP_2 contribution in the induced metric? For them the determinant of induced 4-metric vanishes so that they are genuine singularities in metric sense. The deformations for the canonical imbeddings of this sub-space (F_3 coordinates constant) leaving its metric degenerate should define the lifts of the light-like orbits of partonic 2-surface. The singularity in this case separates regions of different signature of induced metric.

   It would seem that if partonic 2-surface begins at the boundary of CD, conical singularity is not necessary. On the other hand the vertices of generalized Feynman diagrams are 3-surfaces at which 3-lines of generalized Feynman digram are glued together. This singularity is completely analogous to that of ordinary vertex of Feynman diagram. These singularities should correspond to gluing together 3 deformed F_3 along their ends.

4. These considerations suggest that the construction of twistor spaces is a lift of construction space-time surfaces and generalized Feynman diagrammatics should generalize to the level of twistor spaces. What is added is CP_1 fiber so that the correspondence would rather concrete.

5. For instance, elementary particles consisting of pairs of monopole throats connected buy flux tubes at the two space-time sheets involved should allow lifting to the twistor level. This means double connected sum and this double connected sum should appear also for deformations of F_3 associated with the lines of generalized Feynman diagrams. Lifts for the deformations of magnetic flux tubes to which one can assign CP_3 in turn would connect the two F_3s.

6. A natural conjecture inspired by number theoretic vision is that Minkowskian and Euclidian space-time regions correspond to associative and co-associative space-time regions. At the level of twistor space these two kinds of regions would correspond to deformations of CP_3 and F_3. The signature of the twistor norm would be different in this regions just as the signature of induced metric is different in corresponding space-time regions.

   These two regions of space-time surface should correspond to deformations for disjoint unions of CP_3s and F_3s and multiple connected sum form them should project to multiple connected sum (wormhole contacts with Euclidian signature of induced metric) for deformed CP_3s. Wormhole contacts could have deformed pieces of F_3 as counterparts.

There are interesting questions related to the detailed realization of the twistor spaces of space-time surfaces.

1. In the case of CP_2 J would naturally correspond to the Kähler form of CP_2. Could one identify J for the twistor space associated with space-time surface as the projection of J? For deformations of CP_2 type vacuum extremals the normalization of J would allow to satisfy the condition $J^2 = -g$. For general extremals this is not possible. Should one be ready to modify the notion of twistor space by allowing this?

2. Or could the associativity/co-associativity condition realized in terms of quaternionicity of the tangent or normal space of the space-time surface guaranteeing the existence of quaternion units solve the problem and J could be identified as a representation of unit quaternion? In this case J would be replaced with vielbein vector and the decomposition 1+3 of the tangent space implied by the quaternion structure allows to use 3-dimensional permutation symbol to assign antisymmetric tensors to the vielbein vectors. Also the triviality of the tangent bundle of 3-D space allowing global choices of the 3 imaginary units could be essential.

3. Does associativity/co-associativity imply twistor space property or could it provide alternative manner to realize this notion? Or could one see quaternionic structure as an extension of almost complex

structure. Instead of single J three orthogonal J: s (3 almost complex structures) are introduced and obey the multiplication table of quaternionic units? Instead of S^2 the fiber of the bundle would be $SO(3) = S^3$. This option is not attractive. A manifold with quaternionic tangent space with metric representing the real unit is known as quaternionic Riemann manifold and CP_2 with holonomy U(2) is example of it. A more restrictive condition is that all quaternion units define closed forms: one has quaternion Kähler manifold, which is Ricci flat and has in 4-D case Sp(1)=SU(2) holonomy. (see `http://www.encyclopediaofmath.org/index.php/Quaternionic_structure`).

4. Anti-self-dual property (ASD) of metric guaranteeing the integrability of almost complex structure of the twistor space implies the condition $\omega \wedge d\omega = 0$ for the twistor space. What does this condition mean physically for the twistor spaces associated with the extremals of Kähler action? For the 4-D base space this property is of course identically true. ASD property need of course not be realized.

3.3 Twistor spaces as analogs of Calabi-Yau spaces of super string models

CP_3 is also a Calabi-Yau manifold in the strong sense that it allows Kähler structure and complex structure. Witten's twistor string theory considers 2-D (in real sense) complex surfaces in twistor space CP_3. This inspires some questions.

1. Could TGD in twistor space formulation be seen as a generalization of this theory?

2. General twistor space is not Calabi-Yau manifold because it does does not have Kähler structure. Do twistor spaces replace Calabi-Yaus in TGD framework?

3. Could twistor spaces be Calabi-Yau manifolds in some weaker sense so that one would have a closer connection with super string models.

Consider the last question.

1. One can indeed define non-Kähler Calabi-Yau manifolds by keeping the hermitian metric and giving up symplectic structure or by keeping the symplectic structure and giving up hermitian metric (almost complex structure is enough). Construction recipes for non-Kähler Calabi-Yau manifold are discussed in [8]. It is shown that these two manners to give up Kähler structure correspond to duals under so called mirror symmetry [20] which maps complex and symplectic structures to each other. This construction applies also to the twistor spaces.

2. For the modification giving up symplectic structure, one starts from a smooth Kähler Calabi-Yau 3-fold Y, such as CP_3. One assumes a discrete set of disjoint rational curves diffeomorphic to CP_1. In TGD framework work they would correspond to special fibers of twistor space.

   One has singularities in which some rational curves are contracted to point - in twistorial case the fiber of twistor space would contract to a point - this produces double point singularity which one can visualize as the vertex at which two cones meet (sundial should give an idea about what is involved). One deforms the singularity to a smooth complex manifold. One could interpret this as throwing away the common point and replacing it with connected sum contact: a tube connecting the holes drilled to the vertices of the two cones. In TGD one would talk about wormhole contact.

3. Suppose the topology looks locally like $S^3 \times S^2 \times R_\pm$ near the singularity, such that two copies analogous to the two halves of a cone (sundial) meet at single point defining double point singularity. In the recent case S^2 would correspond to the fiber of the twistor space. S^3 would correspond to 3-surface and $R_\pm$ would correspond to time coordinate in past/future direction. S^3 could be replaced with something else.

The copies of $S^3 \times S^2$ contract to a point at the common end of R_+ and R_- so that both the based and fiber contracts to a point. Space-time surface would look like the pair of future and past directed light-cones meeting at their tips.

For the first modification giving up symplectic structure only the fiber S^2 is contracted to a point and $S^2 \times D$ is therefore replaced with the smooth "bottom" of S^3. Instead of sundial one has two balls touching. Drill small holes two the two S^3s and connect them by connected sum contact (wormhole contact). Locally one obtains $S^3 \times S^3$ with k connected sum contacts.

For the modification giving up Hermitian structure one contracts only S^3 to a point instead of S^2. In this case one has locally two CP_3: s touching (one can think that CP_n is obtained by replacing the points of C^n at infinity with the sphere CP_1). Again one drills holes and connects them by a connected sum contact to get k-connected sum of CP_3.

For k CP_1s the outcome looks locally like to a k-connected sum of $S^3 \times S^3$ or CP_3 with $k \geq 2$. In the first case one loses symplectic structure and in the second case hermitian structure. The conjecture is that the two manifolds form a mirror pair.

The general conjecture is that all Calabi-Yau manifolds are obtained using these two modifications. One can ask whether this conjecture could apply also the construction of twistor spaces representable as surfaces in $CP_3 \times F_3$ so that it would give mirror pairs of twistor spaces.

4. This smoothing out procedures isa actually unavoidable in TGD because twistor space is submanifold. The 6-D twistor spaces in 12-D $CP_3 \times F_3$ have in the generic case self intersections consisting of discrete points. Since the fibers CP_1 cannot intersect and since the intersection is point, it seems that the fibers must contract to a point. In the similar manner the 4-D base spaces should have local foliation by spheres or some other 3-D objects with contract to a point. One has just the situation described above.

   One can remove these singularities by drilling small holes around the shared point at the two sheets of the twistor space and connected the resulting boundaries by connected sum contact. The preservation of fiber structure might force to perform the process in such a manner that local modification of the topology contracts either the 3-D base (S^3 in previous example or fiber CP_1 to a point.

The interpretation of twistor spaces is of course totally different from the interpretation of Calabi-Yaus in superstring models. The landscape problem of superstring models is avoided and the multiverse of string models is replaced with generalized Feynman diagrams! Different twistor spaces correspond to different space-time surfaces and one can interpret them in terms of generalized Feynman diagrams since bundle projection gives the space-time picture. Mirror symmetry means that there are two different Calabi-Yaus giving the same physics. Also now twistor space for a given space-time surface can have several imbeddings - perhaps mirror pairs define this kind of imbeddings.

To sum up, the construction of space-times as surfaces of H lifted to that of (almost) complex submanifolds in $CP_3 \times F_3$ with induced twistor structure shares the spirit of the vision that induction procedure is the key element of classical and quantum TGD. It also gives deep connection with the mathematical methods applied in super string models and these methods should be of direct use in TGD.

3.4 Are Euclidian regions of preferred extremals quaternion-Kähler manifolds?

In blog comments Anonymous gave a link to an article about construction of 4-D quaternion-Kähler metrics with an isometry: they are determined by so called $SU(\infty)$ Toda equation. I tried to see whether quaternion-Kähler manifolds could be relevant for TGD.

From Wikipedia one can learn that QK is characterized by its holonomy, which is a subgroup of $Sp(n) \times Sp(1)$: $Sp(n)$ acts as linear symplectic transformations of $2n$-dimensional space (now real). In

4-D case tangent space contains 3-D sub-manifold identifiable as imaginary quaternions. CP_2 is one example of QK manifold for which the subgroup in question is $SU(2)\times U(1)$ and which has non-vanishing constant curvature: the components of Weyl tensor represent the quaternionic imaginary units. QKs are Einstein manifolds: Einstein tensor is proportional to metric.

What is really interesting from TGD point of view is that twistorial considerations show that one can assign to QK a special kind of twistor space (twistor space in the mildest sense requires only orientability). Wiki tells that if Ricci curvature is positive, this (6-D) twistor space is what is known as projective Fano manifold with a holomorphic contact structure. Fano variety has the nice property that as (complex) line bundle (the twistor space property) it has enough sections to define the imbedding of its base space to a projective variety. Fano variety is also complete: this is algebraic geometric analogy of topological property known as compactness.

3.4.1 QK manifolds and twistorial formulation of TGD

How the QKs could relate to the twistorial formulation of TGD?

1. In the twistor formulation of TGD [39] the space-time surfaces are 4-D base spaces of 6-D twistor spaces in the Cartesian product of 6-D twistor spaces of M^4 and CP_2 - the only twistor spaces with Kähler structure. In TGD framework space-time regions can have either Euclidian or Minkowskian signature of induced metric. The lines of generalized Feynman diagrams are Euclidian.

2. Could the twistor spaces associated with the lines of generalized Feynman diagrams be projective Fano manifolds? Could QK structure characterize Euclidian regions of preferred extremals of Kähler action? Could a generalization to Minkowskian regions exist.

   I have proposed that so called Hamilton-Jacobi structure [40] characterizes preferred extremals in Minkowskian regions. It could be the natural Minkowskian counterpart for the quaternion Kähler structure, which involves only imaginary quaternions and could make sense also in Minkowski signature. Note that unit sphere of imaginary quaternions defines the sphere serving as fiber of the twistor bundle.

Why it would be natural to have QK that is corresponding twistor space, which is projective contact Fano manifold?

1. QK property looks very strong condition but might be true for the preferred extremals satisfying very strong conditions stating that the classical conformal charges associated with various conformal algebras extending the conformal algebras of string models [40], [**?**]variationalhamed. These conditions would be essentially classical gauge conditions stating that strong form of holography implies by strong form of General Coordinate Invariance (GCI) is realized: that is partonic 2-surfaces and their 4-D tangent space data code for quantum physics.

2. Kähler property makes sense for space-time regions of Euclidian signature and would be natural is these regions can be regarded as small deformations of CP_2 type vacuum extremals with light-like M^4 projection and having the same metric and Kähler form as CP_2 itself.

3. Fano property implies that the 4-D Euclidian space-time region representing line of the Feynman diagram can be imbedded as a sub-manifold to complex projective space CP_n. This would allow to use the powerful machinery of projective geometry in TGD framework. This could also be a space-time correlate for the fact that CP_ns emerge in twistor Grassmann approach expected to generalize to TGD framework.

4. CP_2 allows both projective (trivially) and contact (even symplectic) structures. $\delta M^4_+ \times CP_2$ allows contact structure - I call it loosely symplectic structure. Also 3-D light-like orbits of partonic 2-surfaces allow contact structure. Therefore holomorphic contact structure for the twistor space is natural.

5. Both the holomorphic contact structure and projectivity of CP_2 would be inherited if QK property is true. Contact structures at orbits of partonic 2-surfaces would extend to holomorphic contact structures in the Euclidian regions of space-time surface representing lines of generalized Feynman diagrams. Projectivity of Fano space would be also inherited from CP_2 or its twistor space $SU(3)/U(1) \times U(1)$ (flag manifold identifiable as the space of choices for quantization axes of color isospin and hypercharge).

The article considers a situation in which the QK manifold allows an isometry. Could the isometry (or possibly isometries) for QK be seen as a remnant of color symmetry or rotational symmetries of M^4 factor of imbedding space? The only remnant of color symmetry at the level of imbedding space spinors is anomalous color hyper charge (color is like orbital angular momentum and associated with spinor harmonic in CP_2 center of mass degrees of freedom). Could the isometry correspond to anomalous hypercharge?

3.4.2 How to choose the quaternionic imaginary units for the space-time surface?

Parallellizability is a very special property of 3-manifolds allowing to choose quaternionic imaginary units: global choice of one of them gives rise to twistor structure.

1. The selection of time coordinate defines a slicing of space-time surface by 3-surfaces. GCI would suggest that a generic slicing gives rise to 3 quaternionic units at each point each 3-surface? The parallelizability of 3-manifolds - a unique property of 3-manifolds - means the possibility to select global coordinate frame as section of the frame bundle: one has 3 sections of tangent bundle whose inner products give rose to the components of the metric (now induced metric) guarantees this. The tri-bein or its dual defined by two-forms obtained by contracting tri-bein vectors with permutation tensor gives the quanternionic imaginary units. The construction depends on 3-metric only and could be carried out also in GRT context. Note however that topology change for 3-manifold might cause some non-trivialities. The metric 2-dimensionality at the light-like orbits of partonic 2-surfaces should not be a problem for a slicing by space-like 3-surfaces. The construction makes sense also for the regions of Minkowskian signature.

2. In fact, any 4-manifold [4] allows almost quaternionic as the above slicing argument relying on parallelizibility of 3-manifolds strongly suggests.

3. In zero energy ontology (ZEO)- a purely TGD based feature - there are very natural special slicings. The first one is by linear time-like Minkowski coordinate defined by the direction of the line connecting the tips of the causal diamond (CD). Second one is defined by the light-cone proper time associated with either light-cone in the intersection of future and past directed light-cones defining CD. Neither slicing is global as it is easy to see.

3.4.3 The relationship to quaternionicity conjecture and $M^8 - H$ duality

One of the basic conjectures of TGD is that preferred extremals consist of quaternionic/ co-quaternionic (associative/co-associative) regions [38]. Second closely related conjecture is $M^8 - H$ duality allowing to map quaternionic/co-quaternionic surfaces of M^8 to those of $M^4 \times CP_2$. Are these conjectures consistent with QK in Euclidian regions and Hamilton-Jacobi property in Minkowskian regions? Consider first the definition of quaternionic and co-quaternionic space-time regions.

1. Quaternionic/associative space-time region (with Minkowskian signature) is defined in terms of induced octonion structure obtained by projecting octonion units defined by vielbein of $H = M^4 \times CP_2$ to space-time surface and demanding that the 4 projections generate quaternionic sub-algebra at each point of space-time.

If there is also unique complex sub-algebra associated with each point of space-time, one obtains one can assign to the tangent space-of space-time surface a point of CP_2. This allows to realize $M^8 - H$ duality [38] as the number theoretic analog of spontaneous compactification (but involving no compactification) by assigning to a point of $M^4 = M^4 \times CP_2$ a point of $M^4 \times CP_2$. If the image surface is also quaternionic, this assignment makes sense also for space-time surfaces in H so that $M^8 - H$ duality generalizes to $H - H$ duality allowing to assign to given preferred extremal a hierarchy of extremals by iterating this assignment. One obtains a category with morphisms identifiable as these duality maps.

2. Co-quaternionic/co-associative structure is conjectured for space-time regions of Euclidian signature and 4-D CP_2 projection. In this case normal space of space-time surface is quaternionic/associative. A multiplication of the basis by preferred unit of basis gives rise to a quaternionic tangent space basis so that one can speak of quaternionic structure also in this case.

3. Quaternionicity in this sense requires unique identification of a preferred time coordinate as imbedding space coordinate and corresponding slicing by 3-surfaces and is possible only in TGD context. The preferred time direction would correspond to real quaternionic unit. Preferred time coordinate implies that quaternionic structure in TGD sense is more specific than the QK structure in Euclidian regions.

4. The basis of induced octonionic imaginary unit allows to identify quaternionic imaginary units linearly related to the corresponding units defined by tri-bein vectors. Note that the multiplication of octonionic units is replaced with multiplication of antisymmetric tensors representing them when one assigns to the quaternionic structure potential QK structure. Quaternionic structure does not require Kähler structure and makes sense for both signatures of the induced metric. Hence a consistency with QK and its possible analog in Minkowskian regions is possible.

5. The selection of the preferred imaginary quaternion unit is necessary for $M^8 - H$ correspondence. This selection would also define the twistor structure. For quaternion-Kähler manifold this unit would be covariantly constant and define Kähler form - maybe as the induced Kähler form.

6. Also in Minkowskian regions twistor structure requires a selection of a preferred imaginary quaternion unit. Could the induced Kähler form define the preferred imaginary unit also now? Is the Hamilton-Jacobi structure consistent with this?

   Hamilton-Jacobi structure involves a selection of 2-D complex plane at each point of space-time surface. Could induced Kähler magnetic form for each 3-slice define this plane? It is not necessary to require that 3-D Kähler form is covariantly constant for Minkowskian regions. Indeed, massless extremals representing analogs of photons are characterized by local polarization and momentum direction and carry time-dependent Kähler-electric and -magnetic fields. One can however ask whether monopole flux tubes carry covariantly constant Kähler magnetic field: they are indeed deformations of what I call cosmic strings [25, 29] for which this condition holds true?

3.5 Could quaternion analyticity make sense for the preferred extremals?

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals.

3.5.1 Basic idea

One can argue that quaternion analyticity is the more natural option in the sense that the local octonionic imbedding space coordinate (or at least M^8 or E^8 coordinate, which is enough if $M^8 - H$ duality holds true) would for preferred extremals be expressible in the form

$$o(q) = u(q) + v(q) \times I \ .$$

Here q is quaternion serving as a coordinate of a quaternionic sub-space of octonions, and I is octonion unit belonging to the complement of the quaternionic sub-space, and multiplies $v(q)$ from *right* so that quaternions and qiaternionic differential operators acting from left do not notice these coefficients at all. A stronger condition would be that the coefficients are real. $u(q)$ and $v(q)$ would be quaternionic Taylor- of even Laurent series with coefficients multiplying powers of q from right for the same reason.

The signature of M^4 metric is a problem. I have proposed complexification of M^8 and M^4 to get rid of the problem by assuming that the imbedding space corresponds to surfaces in the space M^8 identified as octonions of form $o_8 = Re(o)+iIm(o)$, where o is imaginary part of ordinary octonion and i is commuting imaginary unit. M^4 would correspond to quaternions of form $q_4 = Re(q)+iIm(q)$. What is important is that powers of q_4 and o_8 belong to this sub-space (as follows from the vanishing of cross product term in the square of octonion/quaternion) so that powers of q_4 (o_8) has imaginary part proportional to $Im(q)$ ($Im(o)$)

I ended up to reconsider the idea of quaternion analyticity after having found two very interesting articles discussing the generalization of Cauchy-Riemann equations. The first article [4] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article [6] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

3.5.2 The first form of Cauchy-Rieman-Fueter conditions

Cauhy-Riemann-Fueter (CRF) conditions generalize Cauchy-Riemann conditions. These conditions are however not unique. Consider first the translationally invariant form of CRF conditions.

1. The translationally invariant form of CRF conditions is $\partial_{\overline{q}} f = 0$ or explicitly

$$\partial_{\overline{q}} f = (\partial_t - \partial_x I - \partial_y J - \partial_z K) f = 0 \ .$$

   This form does not allow quaternionic Taylor series. Note that the Taylor coefficients multiplying powers of the coordinate from right are arbitrary quaternions. What looks pathological is that even linear functions of q fail be solve this condition. What is however interesting that in flat space the equation is equivalent with Dirac equation for a pair of Majorana spinors [4].

2. The condition allows functions depending on complex coordinate z of some complex-plane only. It also allows functions satisfying two separate analyticity conditions, say

$$\partial_{\overline{u}} f = (\partial_t - \partial_x I) f = 0 \ ,$$

$$\partial_{\overline{v}} f = -(\partial_y J + \partial_z K) f = -J(\partial_y - \partial_z I) f = 0 \ .$$

   In the latter formula J multiplies from *left* ! One has good hopes of obtaining holomorphic functions of two complex coordinates. This might be enough to understand the preferred extremals of Kähler action as quaternion analytic mops.

   There are potential problems due to non-commutativity of $u = t \pm xI$ and $v = yJ \pm zK = (y \pm zI)J$ (note that J multiplies from *right* !) and ∂_u and ∂_v. A prescription for the ordering of the powers u and v in the polynomials of u and v appearing in the double Taylor series seems to be needed. For instance, powers of u can be taken to be at left and v or of a related variable at right.

By the linearity of $\partial_{\overline{v}}$ one can leave J to the left and commute only $(\partial_y - \partial_z I)$ through the u-dependent part of the series: this operation is trivial. The condition $\partial_v f = 0$ is satisfied if the polynomials of y and z are polynomials of $y + iz$ multiplied by J from right. The solution ansatz is thus product of Taylor series of monomials $f_{mn} = (x + iy)^m (y + iz)^n J$ with Taylor coefficients a_{mn}, which multiply the monomials from right and are arbitrary quaternions. Note that the monomials $(y + iz)^n$ do not reduce to polynomials of v and that the ordering of these powers is arbitrary. If the coefficients a_{mn} are real f maps 4-D quaternionic region to 2-D region spanned by J and K. Otherwise the image is 4-D.

3. By linearity the solutions obey linear superposition. They can be also multiplied if product is defined as ordered product in such a manner that only the powers $t + ix$ and $y + iz$ are multiplied together at left and coefficients a_{mn} are multiplied together at right. The analogy with quantum non-commutativity is obvious.

4. In Minkowskian signature one must multiply imaginary units I, J, K with an additional commuting imaginary unit i. This would give solutions as powers of (say) $t + ex$, $e = iI$ with $e^2 = 1$ representing imaginary unit of hyper-complex numbers. The natural interpretation would be as algebraic extension which is analogous to the extension of rational number by adding algebraic number, say $\sqrt{2}$ to get algebraically 2-dimensional structure but as real numbers 1-D structure. Only the non-commutativity with J and K distinguishes e from $e = \pm 1$ and if J and K do not appear in the function, one can replace e by ± 1 in $t + ex$ to get just $t \pm x$ appearing as argument for waves propagating with light velocity.

3.5.3 Second form of CRF conditions

Second form of CRF conditions proposed in [6] is tailored in order to realize the almost obvious manner to realize quaternion analyticity.

1. The ingenious idea is to replace preferred quaternionic imaginary unit by a imaginary unit which is in radial direction: $e_r = (xI + yJ + zK)/r$ and require analyticity with respect to the coordinate $t + er$. The solution to the condition is power series in $t + re_r = q$ so that one obtains quaternion analyticity.

2. The excplicit form of the conditions is

$$(\partial_t - e_r \partial_r) f = (\partial_t - \frac{e_r}{r} r \partial_r) f = 0 \ .$$

   This form allows both the desired quaternionic Taylor series and ordinary holomorphic functions of complex variable in one of the 3 complex coordinate planes as general solutions.

3. This form of CRF is neither Lorentz invariant nor translationally invariant but remains invariant under simultaneous scalings of t and r and under time translations. Under rotations of either coordinates or of imaginary units the spatial part transforms like vector so that quaternionic automorphism group $SO(3)$ serves as a moduli space for these operators.

4. The interpretation of the latter solutions inspired by ZEO would be that in Minkowskian regions r corresponds to the light-like radial coordinate of the either boundary of CD, which is part of $\delta M^4_\pm$. The radial scaling operator is that assigned with the light-like radial coordinate of the light-cone boundary. A slicing of CD by surfaces parallel to the $\delta M^4_\pm$ is assumed and implies that the line $r = 0$ connecting the tips of CD is in a special role. The line connecting the tips of CD defines coordinate line of time coordinate. The breaking of rotational invariance corresponds to the selection of a preferred quaternion unit defining the twistor structure and preferred complex sub-space.

In regions of Euclidian signature r could correspond to the radial Eguchi-Hanson coordinate of CP_2 and $r = 0$ corresponds to a fixed point of $U(2)$ subgroup under which CP_2 complex coordinates transform linearly.

5. Also in this case one can ask whether solutions depending on two complex local coordinates analogous to those for translationally invariant CRF condition are possible. The remain imaginary units would be associated with the surface of sphere allowing complex structure.

3.5.4 Generalization of CRF conditions?

Could the proposed forms of CRF conditions be special cases of much more general CRF conditions as CR conditions are?

1. Ordinary complex analysis suggests that there is an infinite number of choices of the quaternionic coordinates related by the above described quaternion-analytic maps with 4-D images. The form of of the CRF conditions would be different in each of these coordinate systems and would be obtained in a straightforward manner by chain rule.

2. One expects the existence of large number of different quaternion-conformal structures not related by quaternion-analytic transformations analogous to those allowed by higher genus Riemann surfaces and that these conformal equivalence classes of four-manifolds are characterized by a moduli space and the analogs of Teichmueller parameters depending on 3-topology. In TGD framework strong form of holography suggests that these conformal equivalence classes for preferred extremals could reduce to ordinary conformal classes for the partonic 2-surfaces. An attractive possibility is that by conformal gauge symmetries the functional integral over WCW reduces to the integral over the conformal equivalence classes.

3. The quaternion-conformal structures could be characterized by a standard choice of quaternionic coordinates reducing to the choice of a pair of complex coordinates. In these coordinates the general solution to quaternion-analyticity conditions would be of form described for the linear ansatz. The moduli space corresponds to that for complex or hyper-complex structures defined in the space-time region.

3.5.5 Geometric formulation of the CRF conditions

The previous naive generalization of CRF conditions treats imaginary units without trying to understand their geometric content. This leads to difficulties when when tries to formulate these conditions for maps between quaternionic and hyper-quaternionic spaces using purely algebraic representation of imaginary units since it is not clear how these units relate to each other.

In [4] the CRF conditions are formulated in terms of the antisymmetric (1, 1) type tensors representing the imaginary units: they exist for almost quaternionic structure and presumably also for almost hyper-quaternionic structure needed in Minkowskian signature.

The generalization of CRF conditions is proposed in terms of the Jacobian J of the map mapping tangent space TM to TN and antisymmetric tensors J_u and J_u representing the quaternionic imaginary units of N and M. The generalization of CRF conditions reads as

$$J - \sum_u J_u \circ J \circ j_u = 0 \quad .$$

For $N = M$ it reduces to the translationally invariant algebraic form of the conditions discussed above. These conditions seem to be well-defined also when one maps quaternionic to hyper-quaternionic space or vice versa. These conditions are not unique. One can perform an SO(3) rotation (quaternion automorphism) of the imaginary units mediated by matrix Λ^{uv} to obtain

$$J - \Lambda^{uv} J_u \circ J \circ j_v = 0 \ .$$

The matrix Λ can depend on point so that one has a kind of gauge symmetry. The most general triholomorphic map allows the presence of Λ Note that these conditions make sense on any coordinates and complex analytic maps generate new forms of these conditions.

Covariant forms of structure constant tensors reduce to octonionic structure constants and this allows to write the conditions explicitly. The index raising of the second index of the structure constants is however needed using the metrics of M and N. This complicates the situation and spoils linearity: in particular, for surfaces induced metric is needed. Whether local SO(3) rotation can eliminate the dependence on induced metric is an interesting question. Since M^4 imaginary units differ only by multiplication by i, Minkowskian structure constants differ only by sign from the Euclidian ones.

In the octonionic case the geometric generalization of CRF conditions does not seem to make sense. By non-associativity of octonion product it is not possible to have a matrix representation for the matrices so that a faithful representation of octonionic imaginary units as antisymmetric 1-1 forms does not make sense. If this representation exists it it must map octonionic associators to zero. Note however that CRF conditions do not involve products of three octonion units so that they make sense as algebraic conditions at least.

3.5.6 Does residue calculus generalize?

CRF conditions allow to generalize Cauchy formula allowing to express value of analytic function in terms of its boundary values [4]. This would give a concrete realization of the holography in the sense that the physical variables in the interior could be expressed in terms of the data at the light-like partonic orbits and and the ends of the space-time surface. Triholomorphic function satisfies d'Alembert/Laplace equations - in induced metric in TGD framework- so that the maximum modulus principle holds true. The general ansatz for a preferred extremals involving Hamilton-Jacobi structure leads to d'Alembert type equations for preferred extremals [40].

Could the analog of residue calculus exist? Line integral would become 3-D integral reducing to a sum over poles and possible cuts inside the 3-D contour. The space-like 3-surfaces at the ends of CDs could define natural integration contours, and the freedom to choose contour rather freely would reflect General Coordinate Invariance. A possible choice for the integration contour would be the closed 3-surface defined by the union of space-like surfaces at the ends of CD and by the light-like partonic orbits.

Poles and cuts would be in the interior of the space-time surface. Poles have co-dimension 2 and cuts co-dimension 1. Strong form of holography suggests that partonic 2-surfaces and perhaps also string world sheets serve as candidates for poles. Light-like 3-surfaces (partonic orbits) defining the boundaries between Euclidian and Minkowskian regions are singular objects and could serve as cuts. The discontinuity would be due to the change of the signature of the induced metric. There are CDs inside CDs and one can also consider the possibility that sub-CDs define cuts, which in turn reduce to cuts associated with sub-CDs.

3.5.7 Could one understand the preferred extremals in terms of quaternion- analyticity?

Could one understand the preferred extremals in terms of quaternion-analyticity or its possible generalization to an analytic representation for co-quaternionicity expected in space-time regions with Euclidian signature? What is the generalization of the CRF conditions for the counterparts of quaternion-analytic maps from hyper-quaternionic X^4 to quaternionic CP_2 and from quaternionic X^4 to hyper-quaternionic M^4? It has already become clear that this problem can be probably solved by using the the geometric representation for quaternionic imaginary units.

The best thing to do is to look whether this is possible for the known extremals: CP_2 type vacuum extremals, vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2, cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2. One can also check whether Hamilton-Jacobi structure of M^4

and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^2 and Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k \cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condense of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure. The non-geodesic motion of photons with light-velocity and variation of the polarization direction would be due to interactions with the space-time sheet to which it is topologically condensed. Note that light-likeness condition for the coordinate curve gives rise to Virasoro conditions. This observation led long time ago to the idea that 2-D conformal invariance must have a non-trivial generalization to 4-D case.

   Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t - x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

   The dependence on the real coordinate v is however problematic since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used of routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection are not expected to be preferred extremals for obvious reasons. One one can however try similar approach. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals the M^4 projection is random light-like curve. Now one expects co-quaternionicity and that quaternion-analyticity is not the correct manner to formulate the situation. "Co-" suggests that instead of expressing surface as graph one should perhaps express it in terms of conditions stating that some quaternionic analytic functions in H are vanish.

One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light-like coordinate would vary along random light-like curves providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined. This local choice must be integrable, which means that one has slicing by 2-D string world sheets and polarization planes orthogonal to them.

The problem is that the mapping of quaternionic CP_2 coordinate to hyper-quaternionic coordinates of M^4 (say $v = 0, w = 0$) in terms of quaternionic analyticity is not easy. "Co-" suggets that, one could formulate light-likeness condition using Hamilton-Jacobi structure as conditions $\overline{w} - constant = 0$ and $v - constant = 0$. Note that one has $\overline{u} = v$.

5. In the naive generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [6] both forms of the algebraic C-R-F conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

   Could octonion analytic maps of imbedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

3.5.8 Conclusions

To sum up, connections between different conjectures related to the preferred extremals - $M^8 - H$ duality, Hamilton-Jacobi structure, induced twistor space structure, quaternion-Kähler property and its Minkowskian counterpart, and even quaternion analyticity, are clearly emerging. The underlying reason is strong form of GCI forced by the construction of WCW geometry and implying strong from of holography posing extremely powerful quantization conditions on the extremals of Kähler action in ZEO. Without the conformal gauge conditions the mutual inconsistency of these conjectures looks rather infeasible.