Article

Maxwell, Lanczos & Weyl Spinors

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Abstract

The use of an arbitrary null tetrad generates a simple method to obtain the associated spinors to Maxwell, Lanczos and Weyl tensors.

Keywords: Electromagnetic spinor, Lanczos spinor, Newman-Penrose formalism, conformal spinor.

1. Introduction

This work is continuation of [1] with the same notation and conventions. Here we consider the spinorial aspects of an arbitrary second order symmetric tensor without trace:

$$E_{\mu\nu} = E_{\nu\mu} \, , \qquad\qquad E^{\nu}{}_{\nu} = 0, \tag{1}$$

with special emphasis in the Maxwell tensor $T_{\mu\nu}$ because it verifies (1) by the equivalence between matter and energy, and the null mass of the photon [2, 3].

Besides we deduce the associated spinor with the Lanczos potential $K_{\mu\nu\alpha}$ verifying the symmetries [4]:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha} \, , \qquad K_{\mu}{}^{\nu}{}_{\nu} = 0 \, , \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0 \, , \tag{2}$$

whose existence in all spacetimes was proved by Bampi-Caviglia [5] and Illge [6].

The conformal tensor has the properties [7]:

$$C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\nu\beta\alpha} \, , \qquad C_{\mu}{}^{\nu}{}_{\nu\beta} = 0 \, , \qquad C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0 \, , \tag{3}$$

and the Lanczos spintensor is its generator [8]; here we determine the corresponding Weyl spinor.

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The present analysis is based in an arbitrary null tetrad of Newman-Penrose (NP) [7, 9, 10] at an event of the spacetime, hence our study is algebraic. The differential aspects of the Maxwell, Lanczos, and Weyl spinors will be considered in another paper with the intimate relationship between $K_{\mu\nu\alpha}$ and the conformal tensor.

We note that a better understanding of the Lanczos potential permits to know more about the Liénard-Wiechert field, for example, to obtain the physical meaning of the Weert generator [11, 12] and to construct [13] a Petrov classification [7, 14] for the electromagnetic field produced by a point charge in arbitrary motion. The Lanczos spintensor is known for arbitrary types O, N and III 4-spaces [15], Kerr geometry [16], Gödel cosmological model [16, 17], plane gravitational waves [16, 18], and several spacetimes [19-21] of interest in general relativity. The deduction of $K_{\mu\nu\alpha}$ for arbitrary types I, II and D is an open problem.

2. Second order symmetric tensor without trace

Here we consider an arbitrary real tensor $E_{\mu\nu}$ verifying (1). The real orthonormal tetrad $e_{(a)}{}^{\mu}$ [1] permits to introduce the symmetric tensors:

$$Q_{(j)\mu\nu} = e_{(j)\mu}\, e_{(j)\nu}\,,\; j = 0, \dots, 3, \qquad Q_{(4)\mu\nu} = e_{(0)\mu} * e_{(1)\nu}\,, \qquad Q_{(5)\mu\nu} = e_{(0)\mu} * e_{(2)\nu}\,, \tag{4}$$

$$Q_{(6)\mu\nu} = e_{(0)\mu} * e_{(3)\nu}\,, \quad Q_{(7)\mu\nu} = e_{(1)\mu} * e_{(2)\nu}\,, \quad Q_{(8)\mu\nu} = e_{(1)\mu} * e_{(3)\nu}\,, \quad Q_{(9)\mu\nu} = e_{(2)\mu} * e_{(3)\nu}\,,$$

with the notation:

$$A_\mu * B_\nu \equiv A_\mu B_\nu + A_\nu B_\mu\,, \tag{5}$$

hence:

$$E_{\mu\nu} = 2 \sum_{j=0}^{9} q_j\, Q_{(j)\mu\nu}\,. \tag{6}$$

The condition of null trace implies:

$$q_0 = q_1 + q_2 + q_3\,, \tag{7}$$

therefore in four dimensions $E_{\mu\nu}$ has nine independent components.

The orthonormal tetrad has connection with the null tetrad of Newman-Penrose (NP) [1, 7, 9, 10, 20-24]:

$$\sqrt{2}\, e_{(0)\mu} = l_\mu + n_\mu\,, \quad \sqrt{2}\, e_{(1)\mu} = m_\mu + \overline{m}_\mu\,, \quad \sqrt{2}\, e_{(2)\mu} = i\,(m_\mu - \overline{m}_\mu), \quad \sqrt{2}\, e_{(3)\mu} = l_\mu - n_\mu\,, \tag{8}$$

thus (6) adopts the form:

$$E_{\mu\nu} = (q_0 + q_3 + 2q_6)l_\mu l_\nu + (q_0 + q_3 - 2q_6)n_\mu n_\nu + (q_1 - q_2 + 2iq_7)m_\mu m_\nu + (q_1 - q_2 - 2iq_7)\overline{m}_\mu \overline{m}_\nu +$$

$$+(q_0 - q_3)l_\mu * n_\nu + (q_1 + q_2)m_\mu * \overline{m}_\nu + [q_4 + q_8 + i\,(q_5 + q_9)]l_\mu * m_\nu + \quad (9)$$

$$+[q_4 + q_8 - i(q_5 + q_9)]l_\mu * \overline{m}_\nu + [q_4 - q_8 + i(q_5 - q_9)]n_\mu * m_\nu + [q_4 - q_8 - i(q_5 - q_9)]n_\mu * \overline{m}_\nu \,.$$

If we establish the following order for the NP's tetrad:

$$\left(Z_{(a)\mu}\right) = \left(l_\mu \,, n_\mu \,, m_\mu \,, \overline{m}_\mu\right), \quad a = 1, \ldots, 4 \,, \quad (10)$$

then the projections of $E_{\mu\nu}$ onto null tetrad can be written in a compact manner in according to:

$$E_{(a)(b)} = E_{\mu\nu}\, Z_{(a)}{}^\mu\, Z_{(b)}{}^\nu \,, \quad (11)$$

hence from (9):

$$-2\,\Phi_{00} = E_{(1)(1)} = q_0 + q_3 - 2q_6 \,, \qquad -2\,\Phi_{01} = -2\,\overline{\Phi}_{10} = E_{(1)(3)} = -q_4 + q_8 + i(q_5 - q_9) \,,$$

$$-2\,\Phi_{02} = -2\,\overline{\Phi}_{20} = E_{(3)(3)} = q_1 - q_2 - 2iq_7 \,, \quad -2\,\Phi_{11} = E_{(1)(2)} = E_{(3)(4)} = q_0 - q_3 = q_1 + q_2 \,, \quad (12)$$

$$-2\,\Phi_{22} = E_{(2)(2)} = q_0 + q_3 + 2q_6, \qquad -2\,\Phi_{12} = -2\,\overline{\Phi}_{21} = E_{(2)(3)} = -q_4 - q_8 + i(q_5 + q_9) \,,$$

therefore:

$$E_{\mu\nu} = 2[\left(\overline{\Phi}_{12}m_\mu + \Phi_{12}\overline{m}_\mu\right) * l_\nu + \left(\overline{\Phi}_{01}m_\mu + \Phi_{01}\overline{m}_\mu\right) * n_\nu - \Phi_{11}\left(l_\mu * n_\nu + m_\mu * \overline{m}_\nu\right) -$$

$$-\Phi_{22}l_\mu l_\nu - \Phi_{00}n_\mu n_\nu - \overline{\Phi}_{02}m_\mu m_\nu - \Phi_{02}\overline{m}_\mu \overline{m}_\nu] \,, \quad (13)$$

where is simple to check the properties (1).

From [1] we know the associated spinors with (10):

$$l^\mu \leftrightarrow o^A o^B \,, \qquad n^\mu \leftrightarrow \iota^A \iota^B \,, \qquad m^\mu \leftrightarrow o^A \iota^B \,, \qquad \overline{m}^\mu \leftrightarrow \iota^A o^B \,, \quad (14)$$

with $o_A \iota^A = -o^A \iota_A = 1$; then it is easy to obtain the spinorial version of the symmetric tensors into (13):

$$\bar{m}_\mu * l_\nu : \ (o_A * \iota_C) o_{\dot{B}} o_{\dot{D}} , \qquad m_\mu * l_\nu : \ o_A o_C (o_{\dot{B}} * \iota_{\dot{D}}) , \qquad \bar{m}_\mu * n_\nu : \ \iota_A \iota_C (o_{\dot{B}} * \iota_{\dot{D}}) ,$$
$$m_\mu * n_\nu : \ (o_A * \iota_C) \iota_{\dot{B}} \iota_{\dot{D}} , \qquad l_\mu l_\nu : \ o_A o_C o_{\dot{B}} o_{\dot{D}} , \qquad n_\mu n_\nu : \ \iota_A \iota_C \iota_{\dot{B}} \iota_{\dot{D}} , \qquad m_\mu m_\nu : \ o_A o_C \iota_{\dot{B}} \iota_{\dot{D}} , \tag{15}$$

$$l_\mu * n_\nu + m_\mu * \bar{m}_\nu : \ (o_A * \iota_C)(o_{\dot{B}} * \iota_{\dot{D}}) , \qquad \bar{m}_\mu \bar{m}_\nu : \ \iota_A \iota_C o_{\dot{B}} o_{\dot{D}} ,$$

hence [20]:

$$E_{AC\dot{B}\dot{D}} = -2 \, \Phi_{AC\dot{B}\dot{D}} , \tag{16}$$

such that:

$$\Phi_{AC\dot{B}\dot{D}} = \iota_A \iota_C [\Phi_{00} \iota_{\dot{B}} \iota_{\dot{D}} - \Phi_{01} (o_{\dot{B}} * \iota_{\dot{D}}) + \Phi_{02} o_{\dot{B}} o_{\dot{D}}] + o_A o_C [\Phi_{20} \iota_{\dot{B}} \iota_{\dot{D}} - \Phi_{21} (o_{\dot{B}} * \iota_{\dot{D}}) + \Phi_{22} o_{\dot{B}} o_{\dot{D}}] -$$
$$-(o_A * \iota_C)[\Phi_{10} \iota_{\dot{B}} \iota_{\dot{D}} - \Phi_{11} (o_{\dot{B}} * \iota_{\dot{D}}) + \Phi_{12} o_{\dot{B}} o_{\dot{D}}] , \tag{17}$$

which permits to prove the symmetries [25]:

$$\Phi_{AC\dot{B}\dot{D}} = \Phi_{CA\dot{B}\dot{D}} = \Phi_{AC\dot{D}\dot{B}} , \qquad \overline{\Phi_{AC\dot{B}\dot{D}}} = \Phi_{BD\dot{A}\dot{C}} , \qquad \Phi_A{}^A{}_{\dot{B}\dot{D}} = 0 , \qquad \Phi_{AC\dot{B}}{}^{\dot{B}} = 0 . \tag{18}$$

From (12), (14) and (16) the NP components of $E_{\mu\nu}$ acquire the form [20]:

$$\Phi_{00} = \Phi_{AC\dot{B}\dot{D}} o^A o^C o^{\dot{B}} o^{\dot{D}} , \qquad \Phi_{01} = \Phi_{AC\dot{B}\dot{D}} o^A o^C o^{\dot{B}} \iota^{\dot{D}} , \qquad \Phi_{02} = \Phi_{AC\dot{B}\dot{D}} o^A o^C \iota^{\dot{B}} \iota^{\dot{D}} , \tag{19}$$
$$\Phi_{11} = \Phi_{AC\dot{B}\dot{D}} o^A \iota^C o^{\dot{B}} \iota^{\dot{D}} , \qquad \Phi_{12} = \Phi_{AC\dot{B}\dot{D}} \iota^A o^C \iota^{\dot{B}} \iota^{\dot{D}} , \qquad \Phi_{22} = \Phi_{AC\dot{B}\dot{D}} \iota^A \iota^C \iota^{\dot{B}} \iota^{\dot{D}} .$$

Now, as a special case, we accept that $E_{\mu\nu}$ is the Maxwell tensor of the electromagnetic field [3]:

$$T_{\mu\nu} = T_{\nu\mu} = -F_\mu{}^\alpha F_{\nu\alpha} + \tfrac{I_1}{4} \, g_{\mu\nu} , \qquad I_1 = F_{\alpha\beta} F^{\alpha\beta} , \tag{20}$$

whose spinorial version gives:

$$T_{AC\dot{B}\dot{D}} = \tfrac{I_1}{4} \, \varepsilon_{AC} \varepsilon_{\dot{B}\dot{D}} - F_{AL\dot{B}\dot{W}} F_C{}^L{}_{\dot{D}}{}^{\dot{W}} , \tag{21}$$

but from [1] we have the relations:

$$F_{AC\dot{B}\dot{D}} = \varphi_{AC} \varepsilon_{\dot{B}\dot{D}} + \varepsilon_{AC} \varphi_{\dot{B}\dot{D}} , \qquad \varepsilon_{AL} \varepsilon_C{}^L = \varepsilon_{AC} , \qquad \varepsilon_{\dot{B}\dot{W}} \varepsilon_{\dot{D}}{}^{\dot{W}} = \varepsilon_{\dot{B}\dot{D}} , \qquad I_2 = {}^*F_{\mu\nu} F^{\mu\nu} , \tag{22}$$
$$\varphi_{AB} \varphi_C{}^B = \tfrac{1}{2} (\varphi^{ED} \varphi_{ED}) \varepsilon_{AC} , \qquad \varphi^{ED} \varphi_{ED} = 2[\Phi_0 \Phi_2 - (\Phi_1)^2] = \tfrac{1}{4} (I_1 + i \, I_2) ,$$

where φ_{AC} is the Maxwell spinor [20]. Then (21) implies:

$$T_{AC\dot{B}\dot{D}} = 2\varphi_{AC}\,\bar{\varphi}_{\dot{B}\dot{D}}\,, \tag{23}$$

verifying the properties (18).

If into (23) we employ the expression [1]:

$$\varphi_{AC} = \Phi_0\,\iota_A\iota_C - \Phi_1(o_A * \iota_C) + \Phi_2\,o_A o_C\,, \tag{24}$$

we obtain (16) and (17) with:

$$\Phi_{ab} = -\Phi_a\bar{\Phi}_b\,, \tag{25}$$

for any Maxwell field.

Now we study two algebraic situations for an arbitrary electromagnetic field:

Non-null case

We take to l^{μ} and n^{μ} as the two principal null directions of the Faraday tensor [3, 22, 26, 27], hence [1, 27, 28]:

$$\Phi_0 = \Phi_2 = 0, \qquad \Phi_1 = \tfrac{1}{2}(-\lambda + i\,\tau)\,, \qquad \Phi_1\bar{\Phi}_1 = \tfrac{1}{8}\sqrt{I_1^2 + I_2^2}\,, \qquad \lambda = \tfrac{1}{2}(-I_1 + 8\,|\Phi_1|^2)^{\frac{1}{2}} \geq 0\,,$$
$$\tau = \tfrac{\epsilon}{2}(I_1 + 8\,|\Phi_1|^2)^{\frac{1}{2}}, \quad \epsilon I_2 \geq 0\,, \quad \epsilon = \pm 1\,, \quad \varphi_{AC} = -\Phi_1 o_A * \iota_C\,, \quad \varphi_{AC}\,o^C = \Phi_1 o_A\,, \quad \varphi_{AC}\,\iota^C = -\Phi_1\iota_A\,, \tag{26}$$

and from (23):

$$T_{AC\dot{B}\dot{D}} = 2\,\Phi_1\bar{\Phi}_1\,(o_A * \iota_C)\,(\bar{o}_{\dot{B}} * \bar{\iota}_{\dot{D}})\,, \tag{27}$$

whose tensorial version is immediate if we apply (15), in fact:

$$T_{\mu\nu} = 2\,|\Phi_1|^2(l_\mu * n_\nu + m_\mu * \bar{m}_\nu) = \tfrac{1}{2}\sqrt{I_1^2 + I_2^2}\,\left(l_\mu * n_\nu - \tfrac{1}{2}g_{\mu\nu}\right), \tag{28}$$

because $g_{\mu\nu} = l_\mu * n_\nu - m_\mu * \bar{m}_\nu$. It is clear that:

$$T_{\mu\nu}l^{\nu} = 2\,|\Phi_1|^2\,l_\mu\,, \qquad T_{\mu\nu}n^{\nu} = 2\,|\Phi_1|^2\,n_\mu\,, \tag{29}$$

therefore l_μ and n_μ also are proper vectors of the Maxwell tensor.

Null case

We select to l^μ as the 2-degenerate principal direction of $F_{\mu\nu}$, then [3, 27, 28]:

$$I_1 = I_2 = 0, \qquad \Phi_0 = \Phi_1 = 0, \qquad \varphi_{AC}\, o^C = 0, \qquad \varphi_{AC} = \Phi_2\, o_A o_C \,, \tag{30}$$

and (23) implies:

$$T_{AC\dot{B}\dot{D}} = 2\,|\Phi_2|^2 o_A o_C\, o_{\dot{B}} o_{\dot{D}} \,, \tag{31}$$

hence from (15) its tensorial version has the structure [3, 29]:

$$T_{\mu\nu} = 2\,|\Phi_2|^2\, l_\mu l_\nu \,. \tag{32}$$

Our analysis has immediate application to the Ricci tensor without trace, that is, $E_{\mu\nu} = R_{\mu\nu} - \frac{R}{4} g_{\mu\nu}$, for the study of the Einstein's field equations.

3. Lanczos spinor

The Weyl tensor is generated by the Lanczos potential [4, 8] satisfying the symmetries (2), therefore $K_{\mu\nu\alpha}$ has 16 independent components in four dimensions. In analogy with the Faraday's antisymmetric tensor, we introduce the dual tensor [1]:

$$^*K_{\mu\nu\alpha} \equiv \frac{1}{2}\eta_{\mu\nu\beta\gamma}\, K^{\beta\gamma}{}_{\alpha} = -\, ^*K_{\nu\mu\alpha} \,, \tag{33}$$

and it is simple to show that:

$$K_\mu{}^\nu{}_\nu = 0 \qquad \Leftrightarrow \qquad ^*K_{\mu\nu\alpha} + \, ^*K_{\nu\alpha\mu} + \, ^*K_{\alpha\mu\nu} = 0 \,, \tag{34}$$
$$^*K_\mu{}^\nu{}_\nu = 0 \qquad \Leftrightarrow \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0 \,,$$

which suggests to work with the complex Lanczos tensor [24]:

$$S_{\mu\nu\alpha} \equiv K_{\mu\nu\alpha} + i\, ^*K_{\mu\nu\alpha} = -S_{\nu\mu\alpha} \,; \tag{35}$$

with the auto-duality:

$$^*S_{\mu\nu\alpha} = -i\, S_{\mu\nu\alpha} \,, \tag{36}$$

similar to the antisymmetric tensors [1, 7, 23]:

$$V_{\mu\nu} = l_\mu \text{ x } m_\nu \,, \qquad U_{\mu\nu} = \bar{m}_\mu \text{ x } n_\nu \,, \qquad M_{\mu\nu} = n_\mu \text{ x } l_\nu + m_\mu \text{ x } \bar{m}_\nu \,, \tag{37}$$

where was used the Lowry's notation [1, 30]:

$$A_\mu \text{ x } B_\nu \equiv A_\mu B_\nu - A_\nu B_\mu \,, \tag{38}$$

because:

$${}^*V_{\mu\nu} = -i\, V_{\mu\nu} \,, \qquad {}^*U_{\mu\nu} = -i\, U_{\mu\nu} \,, \qquad {}^*M_{\mu\nu} = -i\, M_{\mu\nu} \,; \tag{39}$$

hence the properties (2) are equivalent to:

$$S_\mu{}^\nu{}_\nu = 0 \,. \tag{40}$$

The tensor (35) can be generated via the expression:

$$\tfrac{1}{2} S_{\mu\nu\alpha} = V_{\mu\nu}(\Omega_7 l_\alpha + \Omega_2 n_\alpha - \Omega_3 m_\alpha - \Omega_6 \bar{m}_\alpha) + M_{\mu\nu}(\Omega_8 l_\alpha + \Omega_1 n_\alpha + \Omega_9 m_\alpha - \Omega_5 \bar{m}_\alpha) + \\ + U_{\mu\nu}(\Omega_{10} l_\alpha + \Omega_0 n_\alpha + \Omega_{11} m_\alpha - \Omega_4 \bar{m}_\alpha), \tag{41}$$

then the condition (40) gives:

$$\Omega_8 = \Omega_6 \,, \qquad \Omega_9 = -\Omega_2 \,, \qquad \Omega_{10} = \Omega_5 \,, \qquad \Omega_{11} = -\Omega_1 \,, \tag{42}$$

and (41) adopts the structure:

$$S_{\mu\nu\alpha} = 2\,[\Omega_0 U_{\mu\nu} n_\alpha + \Omega_1(M_{\mu\nu} n_\alpha - U_{\mu\nu} m_\alpha) + \Omega_2(V_{\mu\nu} n_\alpha - M_{\mu\nu} m_\alpha) - \Omega_3 V_{\mu\nu} m_\alpha - \\ \Omega_4 U_{\mu\nu} \bar{m}_\alpha + \\ + \Omega_5\,(U_{\mu\nu}\, l_\alpha - M_{\mu\nu}\, m^*{}_\alpha) + \Omega_6(M_{\mu\nu} l_\alpha - V_{\mu\nu} \bar{m}_\alpha) + \Omega_7 V_{\mu\nu} l_\alpha]\,, \tag{43}$$

such that [20, 31]:

$$2\,\Omega_0 = S_{(1)(3)(1)} \,, \qquad 2\,\Omega_1 = S_{(1)(3)(4)} \,, \qquad 2\,\Omega_2 = S_{(4)(2)(1)} \,, \qquad 2\,\Omega_3 = S_{(4)(2)(4)} \,, \tag{44}$$

$$2\,\Omega_4 = S_{(1)(3)(3)} \,, \qquad 2\,\Omega_5 = S_{(1)(3)(2)} \,, \qquad 2\,\Omega_6 = S_{(4)(2)(3)} \,, \qquad 2\,\Omega_7 = S_{(4)(2)(2)} \,.$$

From [1] we know the spinorial transcription of (37):

$$V_{\mu\nu} \leftrightarrow o_A o_C\, \varepsilon_{\dot{B}\dot{D}} \,, \qquad U_{\mu\nu} \leftrightarrow \iota_A \iota_C\, \varepsilon_{\dot{B}\dot{D}} \,, \qquad M_{\mu\nu} \leftrightarrow (o_A * \iota_C)\, \varepsilon_{\dot{B}\dot{D}} \,, \tag{45}$$

whose application in (43) implies:

$$S_{ACE\dot{B}\dot{D}\dot{F}} = 2\, L_{ACE\dot{F}}\, \varepsilon_{\dot{B}\dot{D}}\,, \tag{46}$$

with the Lanczos spinor [20, 25, 32-34]:

$$L_{ACE\dot{F}} = [\Omega_0\, \iota_A \iota_C \iota_E - \Omega_1(\iota_A \iota_C o_E + (o_A * \iota_C)\, \iota_E) + \Omega_2(o_A o_C \iota_E + (o_A * \iota_C)\, o_E) - \Omega_3\, o_A o_C o_E]\, \iota_{\dot{F}} + \tag{47}$$

$$+[-\Omega_4\, \iota_A \iota_C \iota_E + \Omega_5(\iota_A \iota_C o_E + (o_A * \iota_C)\, \iota_E) - \Omega_6(o_A o_C \iota_E + (o_A * \iota_C)\, o_E) + \Omega_7\, o_A o_C o_E]\, o_{\dot{F}}\,,$$

verifying the symmetries:

$$L_{ACE\dot{F}} = L_{CAE\dot{F}} = L_{AEC\dot{F}}\,, \qquad L_A{}^A{}_{C\dot{F}} = 0\,. \tag{48}$$

The relation (46) is similar to $S_{AC\dot{B}\dot{D}} = 2\, \varphi_{AC}\, \varepsilon_{\dot{B}\dot{D}}$ for the Maxwell spinor [1, 28]. From (44), (46) and (47):

$$\Omega_0 = L_{ABC\dot{D}}\, o^A o^B o^C o^{\dot{D}}, \quad \Omega_1 = L_{ABC\dot{D}}\, o^A o^B \iota^C o^{\dot{D}}, \quad \Omega_2 = L_{ABC\dot{D}}\, o^A \iota^B \iota^C o^{\dot{D}}, \quad \Omega_3 = L_{ABC\dot{D}}\, \iota^A \iota^B \iota^C o^{\dot{D}}, \tag{49}$$

$$\Omega_4 = L_{ABC\dot{D}}\, o^A o^B o^C \iota^{\dot{D}}, \quad \Omega_5 = L_{ABC\dot{D}}\, o^A o^B \iota^C \iota^{\dot{D}}, \quad \Omega_6 = L_{ABC\dot{D}}\, o^A \iota^B \iota^C \iota^{\dot{D}}, \quad \Omega_7 = L_{ABC\dot{D}}\, \iota^A \iota^B \iota^C \iota^{\dot{D}}.$$

We can employ (46) to obtain the spinor association:

$$\overline{S_{\mu\nu\alpha}} \quad \leftrightarrow \quad 2\, \varepsilon_{AC}\, L_{\dot{B}\dot{D}\dot{F}E}\,, \tag{50}$$

with the notation:

$$\overline{L_{BDFE}} \equiv L_{\dot{B}\dot{D}\dot{F}E} = L_{\dot{D}\dot{B}\dot{F}E} = L_{\dot{B}\dot{F}\dot{D}E}\,. \tag{51}$$

The expression (35) gives the Lanczos potential:

$$K_{\mu\nu\alpha} = \tfrac{1}{2}\left(S_{\mu\nu\alpha} + \overline{S_{\mu\nu\alpha}}\right), \tag{52}$$

and its spinor version is immediate from (50) [20]:

$$K_{ACE\dot{B}\dot{D}\dot{F}} = L_{ACE\dot{F}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{AC} L_{\dot{B}\dot{D}\dot{F}E} = -K_{CAE\dot{D}\dot{B}\dot{F}}\,, \tag{53}$$

such that:

$$L_{ACE\dot{F}} = \tfrac{1}{2} K_{ACE\dot{B}}{}^{\dot{B}}{}_{\dot{F}} = \tfrac{1}{4} S_{ACE\dot{B}}{}^{\dot{B}}{}_{\dot{F}}\,. \tag{54}$$

The Weyl-Lanczos equations [15, 20, 31, 33] permit to deduce the NP quantities (44) for a given geometry, then the Lanczos generator is determined via (43) and (52). It is interesting to note that in many spacetimes the components $\Omega_r,\ r = 0, \ldots, 7$ have relationship with the spin-coefficients [15, 19, 21] associated to the corresponding Debever-Penrose vectors [7, 35].

4. Conformal spinor

The Weyl tensor $C_{\mu\nu\alpha\beta}$ verifies the symmetries (3), therefore it has ten independent components in four dimensions. In analogy with (33), we shall work with its simple dual [23]:

$$^*C_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\eta_{\mu\nu\lambda\tau}\, C^{\lambda\tau}{}_{\alpha\beta} = -\ ^*C_{\nu\mu\alpha\beta} = -\ ^*C_{\mu\nu\beta\alpha}\,, \tag{55}$$

such that

$$^*C_{\mu}{}^{\nu}{}_{\nu\beta} = 0 \qquad \Leftrightarrow \qquad C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0\,, \tag{56}$$
$$C_{\mu}{}^{\nu}{}_{\nu\beta} = 0 \qquad \Leftrightarrow \qquad ^*C_{\mu\nu\alpha\beta} +\ ^*C_{\mu\alpha\beta\nu} +\ ^*C_{\mu\beta\nu\alpha} = 0\,.$$

Thus it is natural to introduce the complex Weyl tensor:

$$S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i\ ^*C_{\mu\nu\alpha\beta}\,, \qquad\qquad ^*S_{\mu\nu\alpha\beta} = -i\ S_{\mu\nu\alpha\beta}\,, \tag{57}$$

and (3) are equivalent to:

$$S_{\mu}{}^{\nu}{}_{\nu\beta} = 0\,. \tag{58}$$

The antisymmetric tensors (37) permit to write the expansion:

$$\frac{1}{2}S_{\mu\nu\alpha\beta} = V_{\mu\nu}\left(\psi_4 V_{\alpha\beta} + \psi_2\, U_{\alpha\beta} + \psi_3\, M_{\alpha\beta}\right) + U_{\mu\nu}\left(\psi_5 V_{\alpha\beta} + \psi_0\, U_{\alpha\beta} + \psi_1 M_{\alpha\beta}\right) + \tag{59}$$
$$+ M_{\mu\nu}\left(\psi_8\, V_{\alpha\beta} + \psi_6\, U_{\alpha\beta} + \psi_7\, M_{\alpha\beta}\right),$$

but the condition (58) implies:

$$\psi_5 = \psi_7 = \psi_2\,, \qquad \psi_6 = \psi_1\,, \qquad \psi_8 = \psi_3\,,$$

hence (59) takes the form [7]:

$$S_{\mu\nu\alpha\beta} = 2\,[\psi_0\, U_{\mu\nu} U_{\alpha\beta} + \psi_1\left(U_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu}\, U_{\alpha\beta}\right) + \psi_2\left(M_{\mu\nu} M_{\alpha\beta} + V_{\mu\nu}\, U_{\alpha\beta} + U_{\mu\nu}\, V_{\alpha\beta}\right) + \tag{60}$$
$$+ \psi_3\left(V_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu}\, V_{\alpha\beta}\right) + \psi_4 V_{\mu\nu} V_{\alpha\beta}]\,,$$

where [9, 10, 14, 20, 31, 36]:

$$\psi_0 = C_{(1)(3)(1)(3)}\,,\ \ \psi_1 = C_{(1)(3)(1)(2)}\,,\ \ \psi_2 = C_{(1)(3)(4)(2)}\,,\ \ \psi_3 = C_{(1)(2)(4)(2)}\,,\ \ \psi_4 = C_{(4)(2)(4)(2)}\,. \qquad (61)$$

The spinorial transcription of (60) is evident via (45):

$$S_{ACEG\dot{B}\dot{D}\dot{F}\dot{H}} = 2\,\psi_{ACEG}\,\varepsilon_{\dot{B}\dot{D}}\,\varepsilon_{\dot{F}\dot{H}}\,, \qquad (62)$$

with the Weyl spinor [7, 20, 25, 31, 37, 38]:

$$\psi_{ACEG} = \psi_0\iota_A\iota_C\iota_E\iota_G - \psi_1[\iota_A\iota_C(o_E*\iota_G) + (o_A*\iota_C)\iota_E\iota_G] + \psi_2[(o_A*\iota_C)(o_E*\iota_G) + o_Ao_C\iota_E\iota_G + +\iota_A\iota_C o_E o_G] - \psi_3[o_Ao_C(o_E*\iota_G) + (o_A*\iota_C)o_Eo_G] + \psi_4 o_Ao_Co_Eo_G\,, \qquad (63)$$

which is totally symmetric and without trace:

$$\psi_{ACEG} = \psi_{CAEG} = \psi_{ACGE} = \psi_{AECG}\,, \qquad \psi_{AC}{}^{C}{}_{G} = 0\,, \qquad (64)$$

besides [20]:

$$\psi_0 = \psi_{ACEG}\,o^Ao^Co^Eo^G\,, \quad \psi_1 = \psi_{ACEG}\,o^Ao^Co^E\iota^G\,, \quad \psi_2 = \psi_{ACEG}\,o^Ao^C\iota^E\iota^G\,, \quad \psi_3 = \psi_{ACEG}\,o^A\iota^C\iota^E\iota^G\,, \quad \psi_4 = \psi_{ACEG}\,\iota^A\iota^C\iota^E\iota^G\,. \qquad (65)$$

From (62) we have the spinor association:

$$\overline{S_{\mu\nu\alpha\beta}} \quad \leftrightarrow \quad \varepsilon_{AC}\,\varepsilon_{EG}\,\psi_{\dot{B}\dot{D}\dot{F}\dot{H}}\,, \qquad (66)$$

with the notation $\psi_{\dot{B}\dot{D}\dot{F}\dot{H}} = \overline{\psi_{BDFH}}$. The relation (57) gives the Weyl tensor:

$$C_{\mu\nu\alpha\beta} = \frac{1}{2}\left(S_{\mu\nu\alpha\beta} + \overline{S_{\mu\nu\alpha\beta}}\right), \qquad (67)$$

and with (66) we obtain its spinor representation:

$$C_{ACEG\dot{B}\dot{D}\dot{F}\dot{H}} = \psi_{ACEG}\,\varepsilon_{\dot{B}\dot{D}}\,\varepsilon_{\dot{F}\dot{H}} + \varepsilon_{AC}\,\varepsilon_{EG}\,\psi_{\dot{B}\dot{D}\dot{F}\dot{H}}\,, \qquad (68)$$

such that:

$$\psi_{ACEG} = \frac{1}{4}C_{ACEG\dot{B}}{}^{\dot{B}}{}_{\dot{F}}{}^{\dot{F}} = \frac{1}{8}S_{ACEG\dot{B}}{}^{\dot{B}}{}_{\dot{F}}{}^{\dot{F}}\,. \qquad (69)$$

We note that, for empty 4-spaces, the existence of $K_{\mu\nu\alpha}$ gives a simple proof of the relation [39, 40]:

$$\sqrt{-g}\,S_{\mu\nu\alpha\beta}S^{\mu\nu\alpha\beta} = (\sqrt{-g}A^{\mu})_{;\mu} \qquad (70)$$

where A^μ is certain tensor, which implies that the Lanczos invariants [41] $C_{\mu\nu\alpha\beta}\,C^{\mu\nu\alpha\beta}$ and ${}^*C_{\mu\nu\alpha\beta}\,C^{\mu\nu\alpha\beta}$ are exact divergences [42-46].

Our analysis shows that the null tetrad of Newman-Penrose is an excellent platform for the spinorial study of Maxwell, Lanczos and Weyl tensors, because it makes evident the symmetries of the corresponding spinors.

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