Review Article

From Sachs-Wolfe Acoustic Theorem to Fractal Laplace-Beltrami Operator

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ABSTRACT

According to Czaja et. al. [2], if one considers the acoustic field propagating in the radiation-dominated (p=ε/3) universe of arbitrary space curvature (K=0, ±1), then the field equations are reduced to the d'Alembert equation in an auxiliary static Robertson-Walker spacetime. This is related to the so-called Sachs-Wolfe acoustic theorem, which can be found useful in the observation and analysis of Cosmic Microwave Background anisotropies. In this paper, I will discuss what Laplace-Beltrami operator for curved space is and how this operator may be extended further to become fractal Laplace-Beltrami Operator. I will also discuss possible implications for dark energy observation.

Key Words: Sach-Wolfe, acoustic theorem, Laplace-Beltrami, fractal operator, dark energy.

A. Introduction

The Sachs–Wolfe theorem contains two separate results formulated for two different equations of state: the first for pressureless matter (p=0) and the second for an ultrarelativistic gas (p=e/3) [1]. According to Czaja, *et. al.* [2], the second theorem can be called as the *acoustic theorem*, to distinguish it with the other.

The Sachs–Wolfe acoustic theorem refers to the spatially flat (K=0), hot (p=ε/3) Friedmann–Robertson–Walker universe and the scalar perturbation propagating in it. The theorem states that with the appropriate choice of the perturbation variable, one can express the propagation equation in the form of d'Alembert's equation in Minkowski spacetime. Scalar perturbations in the flat, early universe propagate like electromagnetic or gravitational waves ([1], p. 79).

On the other hand, the wave equation for the scalar field of the dust (p=0) cosmological model can be transformed into the d'Alembert equation in the static Robertson–Walker spacetime, regardless of the universe's space curvature (see [1]). Therefore, we can suppose that the flatness assumption in the Sachs–Wolfe theorem is not needed and that the theorem is true in the general case. The proof of this fact, formulated as a symbolic computation, is presented in the first section of this paper.

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B. Review of the Sachs–Wolfe acoustic theorem

In accordance with Czaja, Golda, and Woszczyna [2], I begin with Robertson–Walker metrics in spherical coordinates $x^{s=\{h,c,J,f\}}$:

$$g_{(RW)} = a^2(\eta)\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)}{K} & 0 \\ 0 & 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)\sin^2(\vartheta)}{K} \end{bmatrix} \tag{1}$$

with the scale factor a(h) appropriate for the equation of state p=e/3,

$$a(\eta) = \frac{\sin(\sqrt{K}\chi)}{\sqrt{K}}. \tag{2}$$

Let us define a new perturbation variable Y with the help of the second-order differential transformation of the density contrast d,

$$\Psi(x^\sigma) = \frac{1}{\cos(\sqrt{K}\chi)}\frac{\partial}{\partial\eta}\left(\frac{K}{\tan^2(\sqrt{K}\chi)}\frac{\partial}{\partial\eta}\left(\frac{\tan^2(\sqrt{K}\chi)}{K}\cos(\sqrt{K}\chi)\delta(x^\sigma)\right)\right). \tag{3}$$

The function $Y(x^{s)}$ is the solution of the d'Alembert equation

$$\frac{\partial^2}{\partial\eta^2}\Psi(x^\sigma) - \frac{1}{3}\varepsilon\,\Delta\,\Psi(x^\sigma) = 0, \tag{4}$$

with the Beltrami–Laplace operator Δ acting in this space,

$${}^{(3)}g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\sin^2(\sqrt{K}\chi)}{K} & 0 \\ 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)\sin^2(\vartheta)}{K} \end{bmatrix}. \tag{5}$$

The Beltrami–Laplace operator Δ is defined as follows:

$$\Delta = {}^{(3)}g_{mn}\nabla^m\nabla^n. \tag{6}$$

And it can be considered as an extension of Laplace operator for curved space. I will discuss this operator in the following section.

Sachs-Wolfe Acoustic Theorem

Scalar perturbations in the hot ($p=\varepsilon/3$) Friedmann–Robertson–Walker universe of arbitrary space curvature ($K=0,\pm 1$) expressed in terms of the perturbation variable *(3)* obey the wave equation *(4)* in the static Robertson–Walker spacetime $g=diag(-1, {}^{(3)}g)$.

Proof of this theorem has been performed with Mathematica by Czaja, Golda, and Woszczyna [2].

It should be noted that this acoustic theorem may be proved useful in the study and simulation of CMBR anisotropies [3][4][5].

C. What is Laplace-Beltrami Operator?

In differential geometry, the Laplace operator can be generalized to operate on functions defined on surfaces in Euclidean space and, more generally, on Riemannian and pseudo-Riemannian manifolds. This more general operator goes by the name Laplace-Beltrami operator, after Pierre-Simon Laplace and Eugenio Beltrami. Like the Laplacian, the Laplace-Beltrami operator is defined as the divergence of the gradient, and is a linear operator taking functions into functions. The operator can be extended to operate on tensors as the divergence of the covariant derivative. Alternatively, the operator can be generalized to operate on differential forms using the divergence and exterior derivative. The resulting operator is called the Laplace-de Rham operator (named after Georges de Rham).

The Laplace-Beltrami operator, like the Laplacian, is the divergence of the gradient:

$$\nabla^2 f = \nabla \cdot \nabla f. \tag{7}$$

An explicit formula in local coordinates is possible.

Combining the definitions of the gradient and divergence, the formula for the Laplace-Beltrami operator applied to a scalar function f is, in local coordinates

$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right). \tag{8}$$

D. Towards Laplace-Beltrami Operator for Fractional Brownian Surface

In their recent paper, Gelbaum and Titus [7] simulate fractal surfaces as random series of eigenfunctions, using the spectral decomposition of the Laplace-Beltrami operator. This approach allows them to generate random fields over smooth manifolds of arbitrary dimension, generalizing previous work with fractional Brownian motion with multidimensional parameter. According to them, the Spectral Theorem and the functional calculus associated with it then yield [7]:

$$-\Delta(f) = \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k \rangle \phi_k, \tag{9}$$

Moreover, since the integrated Sachs-Wolfe effect can be related to dark energy observation, then it seems that we can also expect some kind of zigzagging in dark energy caused by fractional Brownian surface of the boundary of the Universe (See [8]).

There are some questions worth to explore further, for example:

(1) How the random surface is identified?
As a first guess, random surface could be for instance graph of map from surface to 1-D space (think of plane deformed randomly in vertical direction) and that the value of field at given position and as function of time is random variable say given by Brownian motion. I understood than one starts from Laplace-Beltrami operator and adds a random source term to it representing noise represented in terms of the correlation function, which is typically that for Laplacian. This noise could correspond to fluctuations in dark energy. The formula (8) represents this kind of alternative but does not specify what kind of noise the function f represents; that is its correlation function.

(2) What one means with randomness?
The article of Gelbaum et al. suggests that randomness should be described as a random source term in d'Alembert equation for which correlation function is known. One can consider various kinds of noises. This would give rise to a response in the observable involved, say acoustic wave or density fluctuation.

(3) Is randomness induced by the randomness of the 3-D metric due to dark energy fluctuations?
Could a perturbation of RW metric give an inhomogeneity to the d'Alembertian? Einstein's equations govern the evolution of cosmological for density perturbations in slowly varying RW metric. In accordance with the idea what S-W effect is the cosmological term in Einstein's equations.

This issue concerning how to define rigorously what is Laplace-Beltrami Operator for fractional Brownian Surface remains an open question, as I cannot find a good paper or books except [7]. Therefore further research is recommended.

E. Conclusions

I have discussed a number of ideas in this paper related to the Sachs-Wolfe acoustic theorem. I also suggest that perhaps we should generalize Laplace-Beltrami Operator to become fractal Laplace-Beltrami Operator for fractional Brownian Surface. The latter will permit us to verify whether our basic hypothesis of smooth surface in Riemannian geometry is still valid, or whether we could expect to observe further effect in CMBR anisotropies caused by surface imperfection of the boundary of the Universe. In order to do that, first we should define rigorously what Laplace-Beltrami Operator for fractional Brownian Surface is, and how to derive eigenvalues in this case. This issue concerning how to define rigorously what is Laplace-Beltrami Operator for fractional Brownian Surface remains an open question, as I cannot find a good paper or books except [7]. Therefore further research is recommended.

Since integrated Sachs-Wolfe effect can be related to dark energy observation, then we can also expect some kind of zigzagging in dark energy caused by fractional Brownian surface. These issues are left for future investigation.

Note:

The heavens declare the glory of God;
The skies proclaim the work of His hands.
Day after day they pour forth speech;
Night after night they display knowledge.
There is no speech or language where their voice is not heard.
Their voice goes out into all the earth,
Their words to the ends of the world.
(Psalm 19:1-4; NIV)

The verses 1-4 of Psalm 19 as quoted above express clearly how the Universe propagates a set of unheard voices yet proclaiming the greatness of God. In my reading, these remind us to the Prolegomena of John (verse 1:1) where it is said that the Logos or Wisdom of God was there since the beginning and He played a prominent role in the creation of the Universe. Combining these two passages then it seems we can conclude that the Logos or Christ in His pre-existence was the mediator of creation process by the Father in Heaven. The next thing that I would like to offer is that such an interpretation seems to agree with the idea of Sacred Voice as declared in Hinduism: it is said that the Universe was created by utterances of Brahman. Similarly, in Islamic belief, it is stated that the Universe was created as God spoke (Kun Fayakun). Therefore this line of thoughts may form a good basis for religious dialogue among different religions and cultures. Now the question is: can this line of thought of Sacred Utterance in the Beginning be reconciled with modern cosmology? We suggest here that such a dialogue is perhaps possible through studying the Sachs-Wolfe acoustic theorem, which can be found useful in the observation and analysis of Cosmic Microwave Background anisotropies.

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