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A Note on Inextensible Flows of Space-like Curves in Lightlike Cone

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Abstract. In this paper, we study inextensible flows of space-like curves in 3-dimensional light-like cone of E_1^4.We give necessary and sufficient conditions for inextensible flows of space-like curves in 3-dimensional lightlike cone of E_1^4.

Keywords: Inextensible flows, Minkowski space-time.

1. Introduction

Nowadays, the study of the motion of inextensible curves has arisen in a number of diverse engineering applications. The flow of a curve is said to be inextensible if the arc length is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of a physical applications. Many authors have studied flow problems in the different area such as [3,4,6,14,15,17].

Firstly, Kwon and Park studied inextensible flows of curves and developable surfaces, which its arclength is preserved, in Euclidean 3-space [10]. Inextensible flows of curves are studied in many different spaces. Gürbüz have examined inextensible flows of spacelike, timelike and null curves in [5]. After this work Ögrenmiş et al. have studied inextensible curves in Galilean space [16] and Yıldız et al. have studied inextensible flows of curves according to Darboux frame in Euclidean 3-space [18] . Moreover Latifi et al. (2008) studied inextensible flows of curves in Minkowski 3-space [11].

In [1], [2], [7] and [8] the authors focused on timelike and space like curves in E_1^3 and E_1^4 .In the recent work [19], [20] Ö.G. Yıldız et al. gave necessary and sufficient conditions for inextensible flows of non-null curves in E^n and E_1^n .

In the present paper following [8], [10],[20], we define inextensible flows of spacelike curves in $Q^3 \subset E_1^4$ .We give neccessary and sufficent conditions for inextensible flows of spacelike curves in $Q^3 \subset E_1^4$. In fact, we prove the following results

Theorem 1.1 Let $\{\mathbf{x}, \alpha, \beta, \mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x in $Q^3 \subset E_1^4$ and $\frac{\partial x}{\partial t} = \lambda_1 \mathbf{x} + \lambda_2 \alpha + \lambda_3 \beta + \lambda_4 \mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3 \subset E_1^4$.The differentiations of $\{\mathbf{x}, \alpha, \beta, \mathbf{y}\}$ with respect to t is

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\psi_1}{\tau}\beta, \tag{1.1}$$

$$\frac{\partial \alpha}{\partial t} = \psi_1 \mathbf{x} + \psi_2 \beta, \tag{1.2}$$

$$\frac{\partial \beta}{\partial t} = \psi_3 \mathbf{x} - \psi_2 \alpha - \frac{\psi_1}{\tau}\mathbf{y}, \tag{1.3}$$

$$\frac{\partial \mathbf{y}}{\partial t} = -\psi_1 \alpha - \psi_3 \beta. \tag{1.4}$$

where

$$\psi_1 = < \frac{\partial \alpha}{\partial t}, \mathbf{y} > = \frac{1}{\tau}\left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau\right), \quad \psi_2 = < \frac{\partial \alpha}{\partial t}, \beta >, \quad \psi_3 = < \frac{\partial \beta}{\partial t}, \mathbf{y} > .$$

Theorem 1.2 Let $\{\mathbf{x}, \alpha, \beta, \mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x and $\frac{\partial x}{\partial t} = \lambda_1 \mathbf{x} + \lambda_2 \alpha + \lambda_3 \beta + \lambda_4 \mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3 \subset E_1^4$.If $\lambda_1 = \cos\tan t$ and λ_2 and λ_3 linear dependend, then spacelike curves x in $Q^3 \subset E_1^4$ is a helix.

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2. Curves in the Lightlike Cone

In the following we use thesimilar notations and concepts in [9,12,13] unless otherwise stated.

The Minkowski 4-space E_1^4 is the Euclidean 4-space E^4 provided with the standart flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$is a rectangular coordinate system in E_1^4 .

Let c be a fixed point in E_1^4 and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_1^3(c,r) = \{x \in E_1^4 \ : \ < x-c, x-c > \ = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_1^2(c,r) = \{x \in E_2^4 \ : \ < x-c, x-c > \ = -r^2\};$$

the pseudo-Riemannian null cone (quadric cone) is defined by

$$Q^3(c) = \{x \in E_1^4 \ : \ < x-c, x-c > \ = 0\}.$$

When $c = 0$ and $q = 1$, we simply denote $Q_1^n(0)$ by Q^n and call it the lightlike cone (or simply the light cone).

Since $<\ ,>$ is an indefinite metric, it can be spacelike if $< v, v > \ > 0$ or $v = 0,$ timelike if $< v, v > \ < 0$ and null (lightlike) if $< v, v > \ = 0$ and $v \neq 0$.

Similarly. an arbitatry curve $x = x(s)$ can be locally spacelike, timelike or null (lightlike), if all of its velocity $x^{'}(s)$ are respectively spacelike, timelike or null (lightlike).

It is well known that to each unit speed spacelike curve $x = x(s) : I \to Q^3 \subset E_1^4$, one can associate a pseudo orthonormal frame $\{\mathbf{x}, \alpha, \mathbf{y}, \beta\}$. In this situation, the Frenet Frame of unit speed spacelike curve $x = x(s) : I \to Q^3 \subset E_1^4$ are given as

$$\begin{aligned} \mathbf{x}^{'} &= \alpha \\ \alpha^{'} &= \kappa\mathbf{x} - \mathbf{y} \\ \beta^{'} &= \tau\mathbf{x} \\ \mathbf{y}^{'} &= -\kappa\alpha - \tau\beta. \end{aligned} \tag{2.1}$$

The functions κ and τ are defined as

$$\mathbf{k} = -\frac{1}{2} < \mathbf{x}^{''},\ \mathbf{x}^{''} > \tag{2.2}$$

$$(\tau)^2 = \ < \mathbf{x}^{''},\ \mathbf{x}^{''} > -4\,(\mathbf{k}))^2 . \tag{2.3}$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy

$$\mathbf{y} = -\ \mathbf{x}^{''} - \frac{1}{2} < \mathbf{x}^{''},\ \mathbf{x}^{''} > \mathbf{x} \tag{2.4}$$

we have

$$< \mathbf{y},\ \mathbf{y} >=< \mathbf{x},\ \mathbf{x} >=< \mathbf{y},\ \mathbf{x}^{'} >=\ 0,\ \ < \mathbf{x},\ \mathbf{y} >=\ 1. \tag{2.5}$$

Put $\alpha = x^{'}$ and choose β such that

$$\det\ (\mathbf{x},\ \beta,\ \mathbf{y}\ ) = 1.$$

From [13], for any asymtotic orthonormal frame $\{\mathbf{x},\alpha,\mathbf{y},\beta\}$ of the curve $x = x(s) : I \to Q^3 \subset E_1^4$ with

$$\begin{aligned} &< \mathbf{x},\ \alpha >=< \mathbf{x},\ \beta >=< \mathbf{y},\alpha >=< \mathbf{y},\ \beta >=< \alpha,\ \beta >=\ 0\ , \\ &< \alpha,\alpha >=< \beta,\ \beta >= 1. \end{aligned} \tag{2.6}$$

The frame field $\{\mathbf{x},\alpha,\mathbf{y},\beta\}$ is called the cone frenet frame of the curve $x(s)$.

Definition.2.1. The functions k and τ in (2.1) are called the (first) cone curvature and cone torsion (or second cone curvature) of the spacelike curve x in $Q^3 \in E_1^4$.

Definition.2.2. A spacelike curve x such that the functions $\frac{k}{\tau} =\ const.$ is called a general helix.

3. Inextensible Flows of Spacelike Curve in 3-Dimensional Lightlike Cone

Unless otherwise stated we assume that

$$x : [0,l] \times [0,w) \to E_1^4$$

is a one parameter family of smooth spacelike curves in $Q^3 \subset E_1^4$, where l is the arclength of the initial curve. Suppose that u is the curve parametrization variable , $0 \leq u \leq l$. If the speed spacelike curves in $Q^3 \subset E_1^4$ γ is given by $v = \left\|\frac{\partial x}{\partial u}\right\|$, then the arclength of x is given as a function of u by

$$s(u) = \int_0^u \left\|\frac{\partial x}{\partial u}\right\| du = \int_0^u v\ du. \tag{3.1}$$

where

$$\left\|\frac{\partial x}{\partial u}\right\| = \sqrt{\left|< \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} >\right|}.$$

The operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}. \tag{3.2}$$

where $v = \left\|\frac{\partial x}{\partial u}\right\|$.

In this case; the arclength is as follows $ds = vdu$.

Definition3.1. Let x be a spacelike curves in $Q^3 \subset E_1^4$ and $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of x in Minkowski space-time. Any flow of the spacelike curve can be expressed as follows

$$\frac{\partial x}{\partial t} = \lambda_1\mathbf{x} + \lambda_2\alpha + \lambda_3\beta + \lambda_4\mathbf{y}. \tag{3.3}$$

where, λ_i is the i^{th} scalar speed of the spacelike curve x.

Let the arclength variation be

$$s(u,t) = \int_0^u v\ du.$$

In E_1^4 , the requirement that thespacelike curves not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}\ du=0.$$

where $u\in[0,l]$.

Definition 3.2. Let x be a spacelike curves in $Q^3\subset E_1^4$. A spacelike curve evolution $x(u,t)$ and its flow $\frac{\partial x}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial x}{\partial u}\right\|=0. \tag{3.4}$$

Before deriving the necessary and sufficient condition for inelastic spacelike curve flow, we need the following lemma.

Lemma 3.3. Let $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of a spacelike curve x and

$$\frac{\partial x}{\partial t}=\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}.$$

be a smooth flow of a spacelike curves x in $Q^3\subset E_1^4$.If $x(s)$ be unit speed spacelike curve in $Q^3\subset E_1^4$,then we have the following equality

$$\frac{\partial v}{\partial t}=\left(\frac{\partial\lambda_2}{\partial u}-\lambda_4kv+\lambda_1v\right). \tag{3.5}$$

Proof. Suppose that $\frac{\partial \mathbf{x}}{\partial t}$ be a smooth flow of the spacelike curve in $Q^3\subset E_1^4$.Using definition of x, we have

$$v^2=<\frac{\partial x}{\partial u},\frac{\partial x}{\partial u}>. \tag{3.6}$$

Then, by differentiating (3.6), we get

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}<\frac{\partial x}{\partial u},\frac{\partial x}{\partial u}>.$$

On the other hand, as $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute, we have

$$v\frac{\partial v}{\partial t}=<\frac{\partial x}{\partial u},\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial t}\right)>.$$

From (3.3), we obtain

$$v\frac{\partial v}{\partial t}=<\frac{\partial x}{\partial u},\frac{\partial}{\partial u}\left(\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}\right)>.$$

By using (2.1), we have

$$\frac{\partial v}{\partial t}=<\alpha,\left(\frac{\partial\lambda_1}{\partial u}+\lambda_2kv+\lambda_3\tau v\right)\mathbf{x}+\left(\frac{\partial\lambda_2}{\partial u}-\lambda_4kv+\lambda_1v\right)\alpha$$

$$+\left(\frac{\partial\lambda_3}{\partial u}-\lambda_4\tau v\right)\beta\quad+\left(\frac{\partial\lambda_4}{\partial u}-\lambda_2v\right)\mathbf{y}>. \tag{3.7}$$

This clearly forces

$$\frac{\partial v}{\partial t}=\left(\frac{\partial\lambda_2}{\partial u}-\lambda_4kv+\lambda_1v\right).$$

Lemma 3.4. Let $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x and $\frac{\partial \mathbf{x}}{\partial t}=\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3\subset E_1^4$.If x is a unit speed spacelike curve in $Q^3\subset E_1^4$, then we have the following equality

$$\frac{\partial \lambda_2}{\partial u}=(\lambda_4 k-\lambda_1)\,v \tag{3.8}$$

Proof. Let us assume that the spacelike curve flow is inextensible. From (3.4), we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}\,du=\int_0^u \left(\frac{\partial \lambda_2}{\partial u}-\lambda_4 kv+\lambda_1 v\right)\,du=0. \tag{3.9}$$

This clearly forces

$$\frac{\partial \lambda_2}{\partial u}-\lambda_4 kv+\lambda_1 v=0.$$

We now restrict ourselves to arc length parametrized curves.That is, $v=1$ and the local coordinate u corresponds to the curve arc length s. Then, we have the following lemma.

Lemma 3.5. Let $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x and $\frac{\partial x}{\partial t}=\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3\subset E_1^4$.The differentiations of $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ with respect to t is

$$\frac{\partial \mathbf{x}}{\partial t}=\left(\frac{\partial \lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\mathbf{x}+\left(\frac{\partial \lambda_3}{\partial s}-\lambda_4\tau\right)\beta \;\; +\left(\frac{\partial \lambda_4}{\partial s}-\lambda_2\right)\mathbf{y}, \tag{3.10}$$

$$\frac{\partial \alpha}{\partial t}=\psi_1\mathbf{x}+\psi_2\beta, \tag{3.11}$$

$$\frac{\partial \beta}{\partial t}=\psi_3\mathbf{x}+\psi_2\alpha-\left(\frac{\partial \lambda_3}{\partial s}-\lambda_4\tau\right)\mathbf{y}, \tag{3.12}$$

$$\frac{\partial \mathbf{y}}{\partial t}=\psi_1\alpha-\psi_3\beta-\left(\frac{\partial \lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\mathbf{y}, \tag{3.13}$$

where

$$\psi_1=<\frac{\partial \alpha}{\partial t},\mathbf{y}>,\quad \psi_2=<\frac{\partial \alpha}{\partial t},\beta>,\quad \psi_3=<\frac{\partial \beta}{\partial t},\mathbf{y}>.$$

Proof. From the assumption, we have

$$\frac{\partial x}{\partial t}=\frac{\partial}{\partial t}\frac{\partial x}{\partial s}=\frac{\partial}{\partial s}\left(\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}\right).$$

Thus, it is seen that

$$\begin{aligned}\frac{\partial x}{\partial t}=&\left(\frac{\partial \lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\mathbf{x}+\left(\frac{\partial \lambda_2}{\partial s}-\lambda_4 k+\lambda_1\right)\alpha\\ &+\left(\frac{\partial \lambda_3}{\partial s}-\lambda_4\tau\right)\beta \;\; +\left(\frac{\partial \lambda_4}{\partial s}-\lambda_2\right)\mathbf{y}.\end{aligned} \tag{3.14}$$

Subsituting (3.8) in (3.14), we get

$$\frac{\partial \mathbf{x}}{\partial t}=\left(\frac{\partial \lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\mathbf{x}+\left(\frac{\partial \lambda_3}{\partial s}-\lambda_4\tau\right)\beta \;\; +\left(\frac{\partial \lambda_4}{\partial s}-\lambda_2\right)\mathbf{y},$$

Since

$$\begin{aligned}
&< \mathbf{x},\ \alpha > = 0 \ \Rightarrow < \mathbf{x}, \frac{\partial \alpha}{\partial t} > \ = 0,\\
&< \mathbf{x},\ \beta > = 0 \ \Rightarrow < \mathbf{x}, \frac{\partial \beta}{\partial t} > \ = -\left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau\right),\\
&< \mathbf{x}, \mathbf{y} > \ = 1 \ \Rightarrow < \mathbf{x}, \frac{\partial \mathbf{y}}{\partial t} > = -\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2 k + \lambda_3 \tau\right),\\
&< \mathbf{y},\ \alpha > \ = 0 \ \Rightarrow < \alpha, \frac{\partial \mathbf{y}}{\partial t} > \ = -\psi_1,\\
&< \alpha,\ \beta > \ = 0 \ \Rightarrow < \alpha, \frac{\partial \beta}{\partial t} > \ = -\psi_2,\\
&< \mathbf{y}, \beta > = 0 \ \Rightarrow < \beta, \frac{\partial \mathbf{y}}{\partial t} > \ = -\psi_3.
\end{aligned}$$

we have

$$< \mathbf{y}, \frac{\partial \mathbf{y}}{\partial t} > \ = \ < \alpha, \frac{\partial \alpha}{\partial t} > \ = < \beta, \frac{\partial \beta}{\partial t} > \ = 0.$$

and

$$\begin{aligned}
\frac{\partial \alpha}{\partial t} &= \psi_1 \mathbf{x} + \psi_2 \beta,\\
\frac{\partial \beta}{\partial t} &= \psi_3 \mathbf{x} + \psi_2 \alpha - \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau\right) \mathbf{y},\\
\frac{\partial \mathbf{y}}{\partial t} &= \psi_1 \alpha - \psi_3 \beta - \left(\frac{\partial \lambda_1}{\partial s} + \lambda_2 k + \lambda_3 \tau\right) \mathbf{y},
\end{aligned}$$

where

$$\psi_1 = < \frac{\partial \alpha}{\partial t},\ \mathbf{y} >, \quad \psi_2 = < \frac{\partial \alpha}{\partial t},\ \beta >, \quad \psi_3 \ = < \frac{\partial \beta}{\partial t},\ \mathbf{y} > .$$

Theorem 3.6. Let $\{\mathbf{x}, \alpha, \beta, \mathbf{y}\}$ be the cone Frenet frame of a spacelike curve x and $\frac{\partial x}{\partial t} = \lambda_1 \mathbf{x} + \lambda_2 \alpha + \lambda_3 \beta + \lambda_4 \mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3 \subset E_1^4$.Then, there exists the following system of partially differential equations.

$$\frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial (\lambda_2 k)}{\partial s} + \frac{\partial (\lambda_3 \tau)}{\partial s} + \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau\right) \tau = \psi_1, \tag{3.15}$$

$$\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2 k + \lambda_3 \tau\right) - \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) k = 0, \tag{3.16}$$

$$\left(\frac{\partial^2 \lambda_3}{\partial s^2} - \frac{\partial (\lambda_4 \tau)}{\partial s}\right) - \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) \tau = \psi_2 \tag{3.17}$$

$$\left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right) = 0. \tag{3.18}$$

Proof From Lemma 3.4, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial x}{\partial t} = &\left(\frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial (\lambda_2 k)}{\partial s} + \frac{\partial (\lambda_3 \tau)}{\partial s}\right) \mathbf{x} + \left(\frac{\partial^2 \lambda_3}{\partial s^2} - \frac{\partial (\lambda_4 \tau)}{\partial s}\right) \beta \ + \left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right) \mathbf{y}\\
&+ \left(\frac{\partial \lambda_1}{\partial s} + \lambda_2 k + \lambda_3 \tau\right) \alpha + \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau\right) \tau\ \mathbf{x} \ + \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) (-k\ \alpha - \tau\ \beta)
\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial x}{\partial t} &= \left(\frac{\partial^2\lambda_1}{\partial s^2}+\frac{\partial(\lambda_2 k)}{\partial s}+\frac{\partial(\lambda_3\tau)}{\partial s}+\left(\frac{\partial\lambda_3}{\partial s}-\lambda_4\tau\right)\tau\right)\mathbf{x}\\ &+\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau-\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)k\right)\alpha\\ &+\left(\frac{\partial^2\lambda_3}{\partial s^2}-\frac{\partial(\lambda_4\tau)}{\partial s}-\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)\tau\right)\beta+\left(\frac{\partial^2\lambda_4}{\partial s^2}-\frac{\partial\lambda_2}{\partial s}\right)\mathbf{y}\end{aligned} \tag{3.19}$$

Note that

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial s}\right)=\frac{\partial\alpha}{\partial t}=\psi_1\mathbf{x}+\psi_2\beta \tag{3.20}$$

Hence from (3.19) and (3.20), we get

$$\frac{\partial^2\lambda_1}{\partial s^2}+\frac{\partial(\lambda_2 k)}{\partial s}+\frac{\partial(\lambda_3\tau)}{\partial s}+\left(\frac{\partial\lambda_3}{\partial s}-\lambda_4\tau\right)\tau=\psi_1,$$

$$\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)-\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)k=0,$$

$$\left(\frac{\partial^2\lambda_3}{\partial s^2}-\frac{\partial(\lambda_4\tau)}{\partial s}\right)-\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)\tau=\psi_2$$

$$\left(\frac{\partial^2\lambda_4}{\partial s^2}-\frac{\partial\lambda_2}{\partial s}\right)=0.$$

This completes the proof.

Theorem 3.7. Let $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of a spacelike curve x and $\frac{\partial x}{\partial t}=\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}$ be a smooth flow of a spacelike curves x in $Q^3\subset E_1^4$.Then, there exists the following system of partially differential equations.

$$\frac{\partial k}{\partial t}=-\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)k+\frac{\partial\psi_1}{\partial s}+\psi_2\tau \tag{3.21}$$

$$\frac{\partial\psi_2}{\partial s}=\left(\frac{\partial\lambda_3}{\partial s}-\lambda_4\tau\right)k+\psi_3, \tag{3.22}$$

$$\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)k+\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)=0 \tag{3.23}$$

Proof Noting that $\frac{\partial}{\partial s}\left(\frac{\partial\alpha}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial s}\right)$, we have the equations (3.21), (3.22)and (3.23).

Theorem 3.8. Let $\{\mathbf{x},\alpha,\beta,\mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x in $Q^3\subset E_1^4$. and $\frac{\partial x}{\partial t}=\lambda_1\mathbf{x}+\lambda_2\alpha+\lambda_3\beta+\lambda_4\mathbf{y}$ be a smooth flow of a spacelike curve x in $Q^3\subset E_1^4$.Then, there exists the following system of partially differential equation.

$$\frac{\partial\tau}{\partial t}=-\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\tau+\frac{\partial\psi_3}{\partial s}-\psi_2 k \tag{3.24}$$

Proof. By same way above and considering $\frac{\partial}{\partial s}\left(\frac{\partial\beta}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial\beta}{\partial s}\right)$ we reach.

$$\frac{\partial\tau}{\partial t}=-\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2 k+\lambda_3\tau\right)\tau+\frac{\partial\psi_3}{\partial s}-\psi_2 k$$

Proof of Theorem 1. 1. Since

$$< \mathbf{x},\ \mathbf{x} > = 0\ \Rightarrow < \mathbf{x}, \frac{\partial \mathbf{x}}{\partial t} > \ = 0,$$

and from (3.10), we obtain

$$\frac{\partial \lambda_4}{\partial s} - \lambda_2 = 0. \tag{3.25}$$

Thus, from (3.16) and (3.15), we get

$$\frac{\partial \lambda_1}{\partial s} + \lambda_2 k + \lambda_3 \tau = 0 \tag{3.26}$$

and

$$\psi_1 = \tau \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4 \tau \right). \tag{3.27}$$

Therefore, from the equations (3.25) - (3.27) and (3.10)-(3.13) it is seen that

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial t} &= \frac{\psi_1}{\tau} \beta, \\ \frac{\partial \alpha}{\partial t} &= \psi_1 \mathbf{x} + \psi_2 \beta, \\ \frac{\partial \beta}{\partial t} &= \psi_3 \mathbf{x} - \psi_2 \alpha - \frac{\psi_1}{\tau} \mathbf{y}, \\ \frac{\partial \mathbf{y}}{\partial t} &= -\psi_1 \alpha - \psi_3 \beta. \end{aligned}$$

Corollary 3.9. Let $\{\mathbf{x}, \alpha, \beta, \mathbf{y}\}$ be the cone Frenet frame of a spacelike curves x in $Q^3 \subset E_1^4$. and $\frac{\partial x}{\partial t} = \lambda_1 \mathbf{x} + \lambda_2 \alpha + \lambda_3 \beta + \lambda_4 \mathbf{y}$ be a smooth flow of a spacelike curve x in $Q^3 \subset E_1^4$.Then, there exists the following system of partially differential equation

$$\frac{\partial k}{\partial t} = \frac{\partial \psi_1}{\partial s} + \psi_2 \tau,$$

and

$$\frac{\partial \tau}{\partial t} = \frac{\partial \psi_3}{\partial s} - \psi_2 k$$

Proof of Theorem 1. 2. From hypotesis and (3.26), we get

$$\frac{k}{\tau} = \ const.$$

Therefore, theorem is proved.

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