Article

Matrix Approach to Petrov Classification

B. E. Carvajal-Gámez[1], J. López-Bonilla[*2] & R. López-Vázquez[2]

[1] SEPI-ESCOM, Instituto Politécnico Nacional (IPN), Av. Bátiz S/N, 07738, México DF
[2] ESIME-Zacatenco, IPN, Edif. 5, Col. Lindavista CP 07738, México DF

Abstract

We exhibit the matrix method of Petrov to obtain the algebraic type of the Weyl tensor.

Key words: Conformal tensor, Petrov's classification, Newman-Penrose's quantities.

1. Introduction

The Petrov classification (PC) is an invariant characterization of the solutions of Einstein field equations. Hence the PC is important in general relativity, which leads to search efficient methods to realize the PC. In Sec. 2 we exhibit the matrix approach introduced by Petrov [1-5] for the classification of any tensor of 4th order with the algebraic symmetries of the Weyl tensor, and in Sec. 3 we indicate the corresponding null tetrad version.

2. Petrov's Matrix Method

The PC is local because it is realized in a given event of the space-time, that is, the Petrov type can change over our Riemannian 4-space. If we calculate the traces of the curvature tensor we obtain the conformal tensor C_{abcd} with the symmetries:

$$C_{abcd} = -C_{bacd} = -C_{abdc}\,, \qquad C_{abcd} + C_{acdb} + C_{adbc} = 0, \tag{1}$$

$$C^{a}{}_{bca} = 0, \tag{2}$$

where we use the summation convention on repeated indices introduced by Dedekind (1868) [6, 7] and Einstein; the properties (1) imply the symmetry:

$$C_{abcd} = C_{cdab}\,. \tag{3}$$

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* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, México DF
E-mail: jlopezb@ipn.mx

At a given event of the space-time we construct an arbitrary orthonormal real tetrad [8] ${e_{(c)}}^r$, $c = 1, \dots, 4$:

$$ {e_{(a)}}^r e_{(b)r} = \eta_{(a)(b)} , \qquad \eta_{4x4} = \left(\eta_{(a)(b)}\right) = Diag(1,1,1,-1), \tag{4} $$

with the inverse matrix:

$$ \eta^{-1} = \left(\eta^{(a)(b)}\right) = \eta , \tag{5} $$

which permits to define the following dual orthonormal tetrad:

$$ e^{(a)r} = \eta^{(a)(b)} {e_{(b)}}^r \qquad \therefore \qquad e^{(a)r} {e^{(c)}}_r = \eta^{(a)(c)} . \tag{6} $$

Now we project the Wey tensor over the tetrads (4) and (6):

$$ {\tilde{C}^{ab}}_{jk} = {C^{pq}}_{ht} {e^{(a)}}_p {e^{(b)}}_q {e_{(j)}}^h {e_{(k)}}^t , \tag{7} $$

hence we construct the real matrix $Q_{6x6} = ({Q^A}_B)$ via the association:

$$ {Q^A}_B = {\tilde{C}^{ab}}_{jk} , \qquad \begin{matrix} ab: & 23 & 31 & 12 & 14 & 24 & 34 \\ A: & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \tag{8} $$

for example, ${Q^1}_3 = {\tilde{C}^{23}}_{12}$, ${Q^5}_2 = -{\tilde{C}^{42}}_{31}$, etc.

The real matrix Q leads to the complex matrix $P_{3x3} = M + i\,N$, $i = \sqrt{-1}$, such that:

$$ M = \begin{pmatrix} {Q^1}_1 & {Q^1}_2 & {Q^1}_3 \\ {Q^1}_2 & {Q^2}_2 & {Q^2}_3 \\ {Q^1}_3 & {Q^2}_3 & {Q^3}_3 \end{pmatrix} , \qquad N = \begin{pmatrix} {Q^1}_4 & {Q^2}_4 & {Q^3}_4 \\ {Q^2}_4 & {Q^2}_5 & {Q^3}_5 \\ {Q^3}_4 & {Q^3}_5 & {Q^3}_6 \end{pmatrix} . \tag{9} $$

The properties (1), (2) and (3) imply that the matrices (9) are symmetric with null trace, therefore:

$$ P = P^T, \qquad tr\,P = 0 . \tag{10} $$

Petrov [1, 2] studied the eigenvalue problem of P and thus he proved the existence of only six algebraic types (I, II, III, D, N and O) for the conformal tensor, which can be characterized in terms of P via the following flux diagram (I is the 3x3 unit matrix):

$$\begin{array}{ccccccccc} P=0 & \xrightarrow{\text{no}} & P^2=0 & \xrightarrow{\text{no}} & tr\,P^2 = tr\,P^3 = 0 & \xrightarrow{\text{no}} & (tr\,P^2)^3 = 6(tr\,P^3)^2 & \xrightarrow{\text{no}} & \text{I} \\ \downarrow \text{yes} & & \downarrow \text{yes} & & \downarrow \text{yes} & & \downarrow \text{yes} & & \\ 0 & & \text{N} & & \text{III} & & & & \end{array} \qquad (11)$$

$$\text{D} \xleftarrow{\text{yes}} \begin{array}{c} P^2 + \lambda\,P - 2\lambda^2 I = 0, \\ \lambda^2 = \frac{1}{6} tr\,P^2,\ \lambda^3 = -\frac{1}{6} tr\,P^3 \end{array} \xrightarrow{\text{no}} \text{II}$$

It is clear that, for all Petrov types, P satisfies its characteristic polynomial (Cayley-Hamilton theorem) [9, 10]:

$$P^3 - \frac{1}{2}(tr\,P^2)\,P - \frac{1}{3}(tr\,P^3)\,I = 0. \qquad (12)$$

We note that the Petrov type is independent of the coordinate system for any real tetrad, thus we say that the PC is invariant.

3. PC via Null Tetrads of Newman-Penrose

Here we deduce the null tetrad transcription of the diagram (11) to obtain a non-matrix algorithm in terms of the Newman-Penrose (NP) projections of Weyl tensor. In fact, the real tetrad $e_{(b)r}$ verifying (4) permits to construct the null tetrad [11]:

$$m^r = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^r - i e_{(2)}{}^r\right),\ \ \overline{m}^r = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^r + i e_{(2)}{}^r\right),\ \ l^r = \frac{1}{\sqrt{2}}\left(e_{(4)}{}^r - e_{(3)}{}^r\right),$$
$$n^r = \frac{1}{\sqrt{2}}\left(e_{(4)}{}^r + e_{(3)}{}^r\right). \qquad (13)$$

The symmetries (1) and (2) imply that C_{abpq} have 10 independent real components which are equivalent to the following 5 complex NP quantities [5, 11-14]:

$$\psi_0 = C_{abpq} n^a m^b n^p m^q, \qquad \psi_1 = C_{abpq} n^a l^b n^p m^q, \qquad \psi_2 = -C_{abpq} l^a \overline{m}^b n^p m^q,$$
$$\psi_3 = C_{abpq} l^a n^b l^p \overline{m}^q, \qquad \psi_4 = C_{abpq} l^a \overline{m}^b l^p \overline{m}^q, \qquad (14)$$

that is, the geometric information into conformal tensor is stored in the $\psi_a,\ a = 0, \dots, 4$. Then from (6), …. , (9), (13) and (14) we obtain the matrix P in terms of the complex scalar quantities of Newman-Penrose:

$$P = \begin{pmatrix} \frac{1}{2}(2\psi_2 - \psi_0 - \psi_4) & \frac{i}{2}(\psi_4 - \psi_0) & \psi_1 - \psi_3 \\ \frac{i}{2}(\psi_4 - \psi_0) & \frac{1}{2}(2\psi_2 + \psi_0 + \psi_4) & i(\psi_1 + \psi_3) \\ \psi_1 - \psi_3 & i(\psi_1 + \psi_3) & -2\psi_2 \end{pmatrix}, \tag{15}$$

in harmony with (10). The expression (15) was deduced by Ludwig [15] employing spinors in a complex 5-space.

If we use (15) into the diagram (11) we obtain an algorithm for the PC via the NP formalism [16]:

$$\begin{array}{ccccccccc} \psi_r = 0,\ r = 0, \dots, 4 & \overset{\text{no}}{\longrightarrow} & G_r = 0,\ r = 0, \dots, 5 & \overset{\text{no}}{\longrightarrow} & I = J = 0 & \overset{\text{no}}{\longrightarrow} & I^3 = 27\, J^2 & \overset{\text{no}}{\longrightarrow} & \text{I} \\ \downarrow \text{ yes} & & \downarrow \text{ yes} & & \downarrow \text{ yes} & & \downarrow \text{ yes} & & \\ \text{O} & & \text{N} & & \text{III} & & & & \end{array} \tag{16}$$

$$\text{D} \overset{\text{yes}}{\longleftarrow} \begin{array}{c} G_2 + 2G_5 + 3\lambda\psi_2 = 0, \\ G_r + \lambda\, \psi_r = 0,\ r = 0,1,3,4 \end{array} \overset{\text{no}}{\longrightarrow} \text{II}$$

where:

$$G_0 = 2(\psi_0\psi_2 - \psi_1^2),\ \ G_1 = \psi_0\psi_3 - \psi_1\psi_2,\ \ G_2 = \psi_2^2 + \psi_0\psi_4 - 2\psi_1\psi_3,$$
$$G_3 = \psi_1\psi_4 - \psi_2\psi_3,\ \ \lambda^3 = -J, \tag{17}$$

$G_4 = 2(\psi_2\psi_4 - \psi_3^2),\ G_5 = 2(\psi_1\psi_3 - \psi_2^2),\ \ I = G_2 - G_5,\ \ J = -\psi_3 G_1 + \frac{1}{2}(\psi_2 G_5 + \psi_4 G_0),\ \ \lambda^2 = \frac{1}{3} I.$

The algorithm (17) is employed by Differential Geometry (Maple software package) [17] to determine the Petrov type of the Weyl tensor.

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