Article

Kantowski-Sachs Dark Energy Cosmological Model in General Scalar Tensor Theory of Gravitation

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Abstract

Spatially homogeneous Kantowski - Sachs dark energy cosmological models in Nordtvedt (1970) general scalar tensor theory of gravitation with the help of a special case proposed by Schwinger (1970) are obtained. These anisotropic as well as isotropic exact models are free from singularities. Some important features of the models, thus obtained, have been discussed. These exact models represent not only the early stages of evolution but also the present universe.

Keywords: Dark Energy, Kantowski-Sachs metric, General Scalar -Tensor Theory of Gravitation.

1. Introduction

Nordtvedt (1970) proposed a general class of scalar-tensor gravitational theories in which the parameter ω of the BD theory is allowed to be an arbitrary function of the scalar field $[\omega \to \omega(\phi)]$.

The field equations of general scalar-tensor theory proposed by Nordtvedt (1970) are

$$R_{ij} - \frac{1}{2} g_{ij} R = -8\pi\phi^{-1} T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\,\phi_{,j} - \frac{1}{2} g_{ij}\phi_{,k}\,\phi^{,k}\right) - \phi^{-1}(\phi_{i;j} - g_{ij}\phi^{,k}_{;k}) \tag{1.1}$$

$$\phi^{,k}_{;k} = \frac{8\pi T}{3+2\omega} - \frac{1}{(3+2\omega)}\frac{d\omega}{d\phi}\phi_{,i}\,\phi^{,i} \tag{1.2}$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, T_{ij} is the stress energy tensor of the matter, comma and semicolon denote partial and covariant differentiation respectively.

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Also, we have the energy conservation equation

$$T^{ij}_{;j} = 0. \tag{1.3}$$

Based on Supernovae (SNe) type Ia observations, cosmologists have accepted the idea of dark energy, which is a fluid with negative pressure making up around 70% of the present universe. Current studies to extract the properties of a dark energy component of the universe from observational data focus on the determination of its equation of state $w(t)$, which is the ratio of the dark energy's pressure to its energy density $w(t) = p/\rho$ and is not necessarily a constant. Recently, the parameter $w(t)$ has been calculated with some reasoning and the simplest dark energy candidate is the vacuum energy $(w = -1)$, which is mathematically equivalent to the cosmological constant (Λ). The other conventional alternatives, which can be described by minimally coupled scalar fields, are quintessence $(w > -1)$, phantom energy $(w < -1)$ and quintom (that can across from phantom region to quintessence region as evolved) and have time dependent EoS parameter. Ray et al. (2010), Yadav and Yadav (2010), Kumar (2010) and Pradhan et al. (2011) are some of the authors who have investigated dark energy models in general relativity with variable EoS parameter in different contexts. Rao and Sreedevi Kumari (2012) have discussed Bianchi type-I perfect fluid cosmological model and Rao et al. (2012a) have obtained Kaluza-Klein radiating model in this theory. Rao et al. (2012b,c) have respectively studied Bianchi type- I dark energy model in Saez – Ballester (1986) theory of gravitation and LRS Bianchi type-I dark energy cosmological model in Brans-Dicke theory of gravitation. Rao et al. (2013a, b) have respectively discussed Bianchi type-II, VIII & IX dark energy cosmological models in Saez - Ballester theory of gravitation and perfect fluid dark energy cosmological models in Saez - Ballester and general theory of gravitation. Rao and Neelima (2013a, b) have obtained LRS Bianchi type-I dark energy cosmological models in general scalar tensor theory of gravitation using two different expressions for $\omega(\phi)$. Rao and Neelima (2013c, d) have also obtained Bianchi types-II, VIII & IX and Kantowski-Sachs string with bulk viscous cosmological models respectively in this theory. Recently Rao and Neelima (2013e) have discussed dark energy cosmological model coupled with perfect fluid in Saez-Ballester theory of gravitation.

In this paper, we study spatially homogeneous Kantowski - Sachs dark energy cosmological models in Nordtvedt (1970) general scalar tensor theory with the help of a special case proposed by Schwinger (1970), i.e. $3 + 2\omega(\phi) = \dfrac{1}{\lambda\phi}$, where λ is a constant.

2. Metric and Energy Momentum Tensor

We consider a spatially homogeneous Kantowski - Sachs metric of the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.1}$$

where $A \,\&\, B$ are the functions of time t only.

Beside the Bianchi type metrics, the Kantowski – Sachs (1966) models are also describing spatially homogeneous universes. For a review of Kantowski - Sachs of metrics one can refer to MacCallum (1971).These metrics represent homogeneous but anisotropically expanding (or contacting) cosmologies and provide models where the effects of anisotropic can be estimated and compared with all well known Friedmann-Roberston-Walker class of cosmologies. Wang Xing-Xiang (2005) has obtained Kantowski - Sachs string cosmological model with bulk viscosity in general relativity. Kandalkar et al. (2009) have discussed Kantowski-Sachs viscous fluid cosmological model with a varying Λ. Kandalkar et al. (2011) have obtained string cosmology in Kantowski- Sachs space-time with bulk viscosity and magnetic field.
The Energy Momentum tensor components of the fluid can be written in anisotropic diagonal form as

$$T_j^i = diag[T_0^0, T_1^1, T_2^2, T_3^3] \tag{2.2}$$

We can parameterize the components of the Energy Momentum tensor as follows:

$$\begin{aligned} T_j^i &= diag[\rho, -p_x, -p_y, -p_z] \\ &= diag[1, -w_x, -w_y, -w_z]\rho \\ &= diag[1, -w, -(w+\gamma), -(w+\delta)]\rho \end{aligned} \tag{2.3}$$

where ρ is the energy density of the fluid. p_x, p_y and p_z are the pressures, w_x, w_y and w_z are the directional equation of state (EoS) parameters of the fluid along x, y and z axes respectively and $w(t) = \frac{p}{\rho}$ is the deviation free EoS parameter of the fluid.

Here we have parameterized the deviation from isotropy by setting $w_x = w$. Also $\gamma \,\&\, \delta$ are the skewness parameters, which is a deviation from w along y and z axes. The parameters $w, \gamma \,\&\, \delta$ are not necessarily constants and can be functions of the cosmic time *t*.

Here u^i and x^i satisfy the equations

$$g_{ij}u^i u^j = 1, g_{ij}x^i x^j = -1, \quad \text{and} \quad u^i x_i = 0 \tag{2.4}$$

3. Solution of the field equations

Using commoving coordinates, the field equations (1.1) to (1.3) for the metric (2.1) with the help of equations (2.2) to (2.4) can be written as

$$2\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{1}{B^2}+\frac{\omega\dot{\phi}^2}{2\phi^2}+\frac{\ddot{\phi}}{\phi}+2\frac{\dot{\phi}\dot{B}}{\phi B}=-\frac{8\pi}{\phi}w\rho \tag{3.1}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{\omega\dot{\phi}^2}{2\phi^2}+\frac{\ddot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)=-\frac{8\pi}{\phi}(w+\gamma)\rho \tag{3.2}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{\omega\dot{\phi}^2}{2\phi^2}+\frac{\ddot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)=-\frac{8\pi}{\phi}(w+\delta)\rho \tag{3.3}$$

$$2\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}+\frac{1}{B^2}-\frac{\omega\dot{\phi}^2}{2\phi^2}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)=\frac{8\pi\rho}{\phi} \tag{3.4}$$

$$\ddot{\phi}+\dot{\phi}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)=\frac{8\pi}{(3+2\omega)}(1-\gamma-\delta-3w)-\frac{1}{(3+2\omega)}\frac{d\omega}{d\phi}\dot{\phi}^2 \tag{3.5}$$

$$\dot{\rho}+\rho\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)+w\rho\frac{\dot{A}}{A}+2(w+\gamma)\rho\frac{\dot{B}}{B}=0. \tag{3.6}$$

Here the over head dot denotes differentiation with respect to *t*.

From equations (3.2) & (3.3), we get

$$\gamma=\delta \tag{3.7}$$

Using (3.7) and by using the transformation $dt = AB^2 dT$, the above field equations (3.1) to (3.6) can be written as

$$2\frac{B''}{B}-3\frac{B'^2}{B^2}-2\frac{A'B'}{AB}+A^2B^2+\frac{\omega}{2}\frac{\phi'^2}{\phi^2}+\frac{\phi''}{\phi}-\frac{\phi'}{\phi}\frac{A'}{A}=-\frac{8\pi}{\phi}w\rho\left(A^2B^4\right) \tag{3.8}$$

$$\frac{A''}{A}-\frac{A'^2}{A^2}+\frac{B''}{B}-2\frac{B'^2}{B^2}-2\frac{A'B'}{AB}+\frac{\omega}{2}\frac{\phi'^2}{\phi^2}+\frac{\phi''}{\phi}-\frac{\phi'}{\phi}\frac{B'}{B}=\frac{-8\pi}{\phi}(w+\gamma)\rho\left(A^2B^4\right) \tag{3.9}$$

$$2\frac{A'B'}{AB}+\frac{B'^{2}}{B^{2}}+A^{2}B^{2}-\frac{\omega}{2}\frac{\phi'^{2}}{\phi^{2}}+\frac{\phi'}{\phi}(\frac{A'}{A}+2\frac{B'}{B})=\frac{8\pi}{\phi}\rho\left(A^{2}B^{4}\right) \tag{3.10}$$

$$(3+2\omega)\phi''=8\pi(1-2\gamma-3w)\rho(A^{2}B^{4})-\phi'^{2}\frac{d\omega}{d\phi} \tag{3.11}$$

$$\rho'+\rho\left(\frac{A'}{A}+2\frac{B'}{B}\right)+w\rho\frac{A'}{A}+2(w+\gamma)\rho\frac{B'}{B}=0. \tag{3.12}$$

From here after the over head dash denotes differentiation with respect to *T*.

The field equations (3.8) to (3.11) are four independent equations with seven unknowns $A, B, \omega, \phi, w, \gamma$ & ρ.

From equations (3.8) to (3.11), we have:

$$(3+2\omega)\phi''+\frac{d\omega}{d\phi}\phi'^{2}=2\phi\left[\frac{A''}{A}-\frac{A'^{2}}{A^{2}}+2\frac{B''}{B}-3\frac{B'^{2}}{B^{2}}-2\frac{A'B'}{AB}+A^{2}B^{2}\right]+\omega\frac{\phi'^{2}}{\phi}+3\phi'' \tag{3.13}$$

Here we obtain dark energy cosmological model in Nordtvedt's general scalar-tensor theory with the help of a special case proposed by Schwinger (1970) in the form

$$3+2\omega(\phi)=\frac{1}{\lambda\phi}, \qquad \lambda = \text{constant} \tag{3.14}$$

From (3.13) & (3.14), we get

$$\frac{1}{\lambda}\left[\frac{\phi''}{\phi}-\frac{\phi'^{2}}{\phi^{2}}\right]+\frac{3}{2}\frac{\phi'^{2}}{\phi}-3\phi''=2\phi\left[\frac{A''}{A}-\frac{A'^{2}}{A^{2}}+2\frac{B''}{B}-3\frac{B'^{2}}{B^{2}}-2\frac{A'B'}{AB}+A^{2}B^{2}\right] \tag{3.15}$$

In order to solve the above equation completely, we assume that the expansion scalar is proportional to shear scalar. This condition leads to

$$A=B^{n}, n\neq 0 \tag{3.16}$$

From (3.15) & (3.16), we get

$$\phi=e^{(k_1T+k_2)} \tag{3.17}$$

where k_1 & k_2 are arbitrary constants and

$$\frac{B''}{B}-\frac{3(n+1)}{n+2}\frac{B'^{2}}{B^{2}}=\frac{-1}{(n+2)}B^{2n+2}-\frac{3}{4(n+2)}k_1^{2} \tag{3.18}$$

From (3.18), we get

$$B=\left[\left(\frac{k_4}{k_3}\right)Sech(n+1)k_4T\right]^{\frac{1}{n+1}},n\neq-1 \tag{3.19}$$

where $k_3{}^2=\frac{1}{(n^2+n+1)}$ & $k_4{}^2=\frac{3k_1{}^2}{4(2n+1)}$

From (3.16) & (3.19), we get

$$A=\left[\left(\frac{k_4}{k_3}\right)Sech(n+1)k_4T\right]^{\frac{n}{n+1}} \tag{3.20}$$

From (3.10), we get the string energy density

$$\rho=\frac{1}{8\pi k_5}\left[Cosh(n+1)k_4T\right]^{\frac{2(n+2)}{n+1}}\left\{e^{(k_1T+k_2)}\begin{bmatrix}(2n+1)k_4{}^2Tanh^2(n+1)k_4T\\+\frac{k_4{}^2}{k_3{}^2}Sech^2(n+1)k_4T\\-(n+2)k_1k_4Tanh(n+1)k_4T+\frac{3}{4}k_1{}^2\end{bmatrix}-\frac{k_1{}^2}{4\lambda}\right\} \tag{3.21}$$

where $k_5=\left(\frac{k_4}{k_3}\right)^{\frac{2(n+2)}{n+1}}$

From (3.8), we get the EoS parameter

$$w=-\frac{\begin{bmatrix}(2n+1)k_4{}^2Tanh^2(n+1)k_4T-(n^2+n+1)k_4{}^2Sech^2(n+1)k_4T\\-nk_1k_4Tanh(n+1)k_4T-\frac{k_1{}^2}{4}\end{bmatrix}-\frac{k_1{}^2}{4\lambda}e^{-(k_1T+k_2)}}{\begin{bmatrix}(2n+1)k_4{}^2Tanh^2(n+1)k_4T+\frac{k_4{}^2}{k_3{}^2}Sech^2(n+1)k_4T\\-(n+2)k_1k_4Tanh(n+1)k_4T+\frac{3k_1{}^2}{4}\end{bmatrix}-\frac{k_1{}^2}{4\lambda}e^{-(k_1T+k_2)}} \tag{3.22}$$

From (3.8) & (3.9), we get the skewness parameter

$$\gamma = \frac{n(2n+1)k_4^{\,2}Sech^2(n+1)k_4T + (n-1)k_1k_4Tanh(n+1)k_4T}{\begin{bmatrix} (2n+1)k_4^{\,2}Tanh^2(n+1)k_4T + \dfrac{k_4^{\,2}}{k_3^{\,2}}Sech^2(n+1)k_4T \\ -(n+2)k_1k_4Tanh(n+1)k_4T + \dfrac{3k_1^{\,2}}{4} \end{bmatrix} - \dfrac{k_1^{\,2}}{4\lambda}e^{-(k_1T+k_2)}} \tag{3.23}$$

The metric (2.1) can now be written as

$$\begin{aligned} ds^2 = \frac{1}{k_5}\left[Cosh(n+1)k_4T\right]^{\frac{2(n+2)}{n+1}}dT^2 - \left[\left(\frac{k_4}{k_3}\right)Sech(n+1)k_4T\right]^{\frac{2n}{n+1}}dr^2 \\ - \left[\left(\frac{k_4}{k_3}\right)Sech(n+1)k_4T\right]^{\frac{2}{n+1}}\left(d\theta^2 + Sin^2\theta\, d\phi^2\right) \end{aligned} \tag{3.24}$$

The metric (3.24) together with (3.17) & (3.21) to (3.23) constitutes a Kantowski - Sachs anisotropic dark energy cosmological model in Nordtvedt's general scalar-tensor theory of gravitation with a special case proposed by Schwinger.

Isotropic cosmological model:

For $n = 1$, the metric (3.24) together with (3.17) & (3.21) to (3.23) constitutes a Kantowski - Sachs isotropic dark energy cosmological model in Nordtvedt's general scalar-tensor theory of gravitation with a special case proposed by Schwinger.

4. Some important features of the model

The volume element of the model (3.24) is given by

$$V = (-g)^{1/2} = \sqrt{k_5}\left[Sech(n+1)k_4T\right]^{\frac{(n+2)}{n+1}} Sin\theta \tag{4.1}$$

The expression for the expansion scalar θ is given by

$$\theta = u^i{}_{;i} = -k_4(n+2)Tanh(n+1)k_4T \tag{4.2}$$

and the shear σ is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{7}{18}k_4{}^2(n+2)^2 Tanh^2(n+1)k_4T \tag{4.3}$$

The deceleration parameter q is given by

$$q = -3\theta^{-2}(\theta_{,i}u^i + \frac{1}{3}\theta^2)$$

$$q = -\left(\frac{-12(n+1)}{(n+2)e^{2(n+1)k_4T}\left(1-e^{-2(n+1)k_4T}\right)^2}+1\right) \tag{4.4}$$

The Hubble parameter H is given by

$$H = \frac{-k_4(n+2)Tanh(n+1)k_4T}{3} \tag{4.5}$$

The tensor of rotation $w_{ij} = u_{i,j} - u_{j,i}$ is identically zero and hence this universe is non-rotational.

5. Conclusions

In this paper we have presented Kantowski-Sachs dark energy cosmological models in Nordtvedt (1970) general scalar tensor theory with the help of a special case proposed by Schwinger (1970). The models presented here are free from singularities and the spatial volume vanishes as $T \to \infty$. We observe that, as T approaches to infinity, the expansion scalar θ leads to a constant value. This shows that the universe expands homogeneously. We also observe that the shear scalar σ and the Hubble parameter H approaches to constant value as $T \to \infty$. We can also observe that the energy density ρ diverges as $T \to \infty$. The EoS parameter w, skewness parameter γ approach constant value as $T \to \infty$. For this model, the deceleration parameter $q = -1$ for large values of T and hence the universe expands exponentially. The models obtained here remain anisotropic throughout the evolution of the universe for $n \neq 1$. Study of anisotropies connects and unravels fundamental issues in various fields of astrophysics and cosmology. Anisotropic features can reveal key information on the structure and the nature of the components of the universe, and provide hints on the origin of high energy emission. Hence anisotropic space-times are important. But for $n = 1$, the model reduces to isotropic universe.

These anisotropic as well as isotropic exact models are free from singularities. Also the models presented here represent not only the early stages of evolution but also the present universe.

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