Article

Three Semi-Analytical Methods for Ninth-Order Korteweg-de Vries Equation

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Abstract

In this work, three powerful semi analytical methods, called Adomian decomposition method (ADM),Homotopy analysis method (HAM) and homotopy-perturbation method (HPM) are presented to obtain the series solutions of the ninth-order Korteweg-de Vries(nKdV) equation for certain initial condition. The results obtained by these methods are then compared. The results show that these methods are very efficient and convenient and can be applied to a large class of problems. The numerical solutions are compared with the known analytical solutions.The comparison among these methods shows that although the numerical results of these methods are the same, because of the auxiliary parameter h and auxiliary $H(x,t)$ function, HAM provides a better convergence than the other numerical methods.

Key Words: ninth-order, Korteweg-de Vries equation, Adomian decomposition method, homotopy analysis method, homotopy perturbation method, semi-analytical methods, series solutions.

1. Introduction

Nonlinear partial differential equations can describe many physical problems in different field of science. These linear and nonlinear models play important roles in applied science, so finding their analytical solutions has fundamental significance in various field of science and engineering.But,it may not always be possible to obtain analytical solutions of nonlinear partial differential equations. In this case, we use some semi analytical methods giving series solution.A problem encountered in semi-analytical methods which are given series solution is the concept of convergence of the series. Many semi-analytical methods have been developed to overcome this problem.Among these,Adomian's decomposition method [1-8], Homotopy analysis method [9-12] and Homotopy perturbation method [13-16] have been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. These methods have proved to be powerful, effective, and can easily handle a wide class of linear and nonlinear problems.

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The aim of this work is to employ ADM, HAM and HPM to obtain the series solution for Ninth-order Korteweg-de Vries equation [17], which occurs in various areas of physics.In this paper, we consider the following nonlinear dispersive, integrable equation,

$$u_t + 45u_x u_{6x} + 45uu_{7x} + 210u_{3x}u_{4x} + 210u_{2x}u_{5x} + 1575u_x u_{2x}^2 + 3150uu_{2x}u_{3x} + 1260uu_x u_{4x} + 630u^2 u_{5x} + 9450u^2 u_x u_{2x} + 3150u^3 u_{3x} + 4725u^4 u_x + u_{9x} = 0, \quad (1)$$

where u_{kx} is $\frac{\partial^k u}{\partial x^k}$ $k = 1,2,3,...$ and $u(x,t)$ is the unknown function depending on the temporal variable *t* and the spatial variable *x*. This equation contains both linear dispersive term u_{9x} and the nonlinear terms $45u_x u_{6x}$, $45uu_{7x}$, $210u_{3x}u_{4x}$, $210u_{2x}u_{5x}$, $1575u_x u_{2x}^2$, $3150uu_{2x}u_{3x}$, $1260uu_x u_{4x}$, $630u^2 u_{5x}$, $9450u^2 u_x u_{2x}$, $3150u^3 u_{3x}$, $4725u^4 u_x$. Eq. (1) be called ninth-order Korteweg-de Vries equation in literatures. Many physicists and mathematicians have paid their attentions to the ninth-order Korteweg-de Vries equation in recently years due to its appearance in scientific applications [17-19].

2. Analysis of Semi Analytical Methods

2.1. Adomian decomposition method (ADM)

Adomian decomposition method depends on decomposing the nonlinear differential equation

$$Fu(t) = g(t) \quad (2)$$

into the two components

$$Lu + Ru + Nu = g, \quad (3)$$

where L and N are the linear and the nonlinear parts of F respectively. The remainder of the linear operator is R. The operator L is assumed to be an invertible operator. Solving for Lu leads to

$$Lu = g - Ru - Nu, \quad (4)$$

Applying the inverse operator L^{-1} on both sides of Eq. (4) yields

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \quad (5)$$

Now assuming that the solution u can be represented as infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (6)$$

Furthermore, suppose that the nonlinear term Nu can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$Nu = \sum_{n=0}^{\infty} A_n \ , \tag{7}$$

where the Adomian polynomials A_n of Nu ;

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{8}$$

Then substituting Eqs. (6) and (7) in Eq. (5) gives

$$\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \ .$$

Then equating the terms in the linear system of Eq. (8) gives the recurrent relation

$$u_0 = A + Bt + L^{-1}g \ ,$$

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_{n,} \quad n \geq 0.$$

However, in practice all the terms of series (8) cannot be determined, and the solution is approximated by the truncated series $\sum_{n=0}^{\infty} u_n$.

2.2. Homotopy analysis method (HAM)

To show the basic idea, let us consider the following differential equations $N[V(t)] = 0,$ where N is a nonlinear operator, t denotes independent variable, $V(t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [9] constructs the so-called zero-order deformation equation

$$(1-q)\mathrm{L}\left[\Phi(t;q) - V_0(t)\right] = hqH(t)\mathrm{N}\left[\Phi(t;q)\right] \ , \tag{9}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $V_0(t)$ is an initial guess $V(t)$, $\Phi(t;q)$ is a unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$ it holds

$$\lim_{q\to 0} \Phi(t;q) = \Phi(t;0) = V_0(t) \ ,$$

$$\lim_{q\to 1} \Phi(t;q) = \Phi(t;1) = V(t) \ ,$$

respectively. Thus as q increases from 0 to 1, the solution $\Phi(t;q)$ varies from the initial guess $V_0(t)$ to the solution $V(t)$ .Expanding $\Phi(t;q)$ in Taylor series with respect to q, one has

$$\Phi(t;q) = \Phi(t;0) + \sum_{m=1}^{+\infty} \frac{1}{m!} \frac{\partial^m \Phi(t;q)}{\partial q^m}\bigg|_{q=0} q^m , \tag{10}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (10) converges at q=1, one has

$$\Phi(t;1) = V_0(t) + \sum_{m=1}^{+\infty} V_m(t) ,$$

which must be one of solutions of original nonlinear equation, as proved by Liao [10].According to definition (10), the governing equation can be deduced from the zero-order deformation Eq. (9). Define the vector

$$V_0(t), V_1(t), V_2(t), \ldots, V_{m-1}(t) ,$$

Differentiating Eq. (9) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathrm{L}\left[V_m(t) - \chi_m V_{m-1}(t)\right] = hH(t) R_m(\vec{V}_{m-1}) , \tag{11}$$

where

$$R_m(\vec{V}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\Phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} ,$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 , \\ 1, & m > 1 . \end{cases}$$

It should be emphasized that $V_m(t)$ for $m \ge 1$ is governed by the linear Eq. (11) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

2.3. The homotopy-perturbation method (HPM)

To illustrate the basic ideas of this method, we consider the following equation [14]

$$A(u) - f(r) = 0 , \quad r \in \Omega , \tag{12}$$

with the boundary condition of

$$B(u, \partial u/\partial n) = 0 , \quad r \in \Gamma ,$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

A can be divided into two parts which are *L* and *N*, where *L* is linear and *N* is nonlinear. Eq. (12) can therefore be rewritten as follows:

$$L(u)+N(u)-f(r)=0\,.$$

Homotopy perturbation structure is shown as follows:

$$H(V,p)=(1-p)\big[L(V)-L(u_0)\big]+p\big[A(V)-f(r)\big]=0, \tag{13}$$

or equivalently

$$H(V,p)=L(V)-L(u_0)+pL(u_0)+p\big[N(V)-f(r)\big]=0, \tag{14}$$

where,

$$V(r,p):\Omega\times[0,1]\to R\,. \tag{15}$$

In Eq. (14), $p\in[0,1]$, is an embedding parameter and u_0, is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (14) can be written as a power series in *p*, as following:

$$V=V_0+pV_1+p^2V_2+\ldots, \tag{16}$$

and the best approximation for solution is

$$u=\lim_{p\to1}V=V_0+V_1+V_2+\ldots. \tag{17}$$

The series (17) is convergent for most cases, and also the rate of convergent depends on *A(v)* [13].

3. Numerical applications

In this section, we apply ADM,HAM and HPM to solve the Ninth-order Korteweg-de Vriesequation. In our work, we use the Mathematica Package to calculate the numerical solutions obtained by these methods.

Consider the Ninth-order Korteweg-de Vries equation

$$\begin{aligned} &u_t+45u_xu_{6x}+45uu_{7x}+210u_{3x}u_{4x}+210u_{2x}u_{5x}+1575u_xu_{2x}^2+3150uu_{2x}u_{3x}\\ &+1260uu_xu_{4x}+630u^2u_{5x}+9450u^2u_xu_{2x}+3150u^3u_{3x}+4725u^4u_x+u_{9x}=0 \end{aligned} \tag{18}$$

subject to the initial condition[17]of

$$u_0 = u(x,0) = \frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2) . \tag{19}$$

a) Implementation of ADM

To illustrate the basic concepts of the Adomian's decomposition method for solving the Ninth-order Korteweg-de Vries equation, first we rewrite it in the following operator form:

$$\begin{aligned} &L_t u + 45\, L_x u L_{6x} u + 45 u L_{7x} u + 210 L_{3x} u L_{4x} u + 210 L_{2x} u L_{5x} u + 1575\, L_x u \left(L_{2x} u\right)^2 \\ &+3150 u L_{2x} u L_{3x} u + 1260 u\, L_x u L_{4x} u + 630 u^2 L_{5x} u + 9450 u^2\, L_x u L_{2x} u + 3150 u^3 L_{3x} u \\ &+4725 u^4\, L_x u + L_{9x} u = 0, \end{aligned} \tag{20}$$

where the notations

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{2x} = \frac{\partial^2}{\partial x^2}, L_{3x} = \frac{\partial^3}{\partial x^3}, L_{4x} = \frac{\partial^4}{\partial x^4}, \dots$$

symbolize the linear differential operators. Assume L_t is invertible; hence the inverse operator L_t^{-1} is given by $L_t^{-1} = \int_0^t (.)dt$ .Operating with the inverse operator on both sides of Eq. (20), we obtain

$$\begin{aligned} u(x,t) = u(x,0) &- 45\, L_t^{-1}\left(L_x u L_{6x} u\right) - 45 L_t^{-1}\left(u L_{7x} u\right) - 210 L_t^{-1}\left(L_{3x} u L_{4x} u\right) - 210 L_t^{-1}\left(L_{2x} u L_{5x} u\right) \\ &-1575 L_t^{-1}\left(L_x u \left(L_{2x} u\right)^2\right) - 3150 L_t^{-1}\left(u L_{2x} u L_{3x} u\right) - 1260 L_t^{-1}\left(u\, L_x u L_{4x} u\right) - 630 L_t^{-1}\left(u^2 L_{5x} u\right) \\ &-9450 L_t^{-1}\left(u^2\, L_x u L_{2x} u\right) - 3150 L_t^{-1}\left(u^3 L_{3x} u\right) - 4725 L_t^{-1}\left(u^4\, L_x u\right) - L_t^{-1}\left(L_{9x} u\right) \end{aligned} \tag{21}$$

Adomian method defines the solution u(x,t) by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) .$$

The nonlinear operators are decomposed as

$$\sum_{n=0}^{\infty} A_n, \sum_{n=0}^{\infty} B_n, \sum_{n=0}^{\infty} C_n, \sum_{n=0}^{\infty} D_n, \sum_{n=0}^{\infty} E_n, \sum_{n=0}^{\infty} F_n, \sum_{n=0}^{\infty} G_n, \sum_{n=0}^{\infty} H_n, \sum_{n=0}^{\infty} J_n, \sum_{n=0}^{\infty} K_n \text{ ve } \sum_{n=0}^{\infty} M_n .$$

These operators, an appropriate Adomian's polynomial which can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [20]. For nonlinearity operator, these polynomials can be calculated using the basic formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}, \quad n \ge 0. \tag{22}$$

This formula is easy to set a computer code to get many polynomials as we need in the calculation of the numerical solution.Substituting the initial condition into (21) identifying the zeroth component u_0 by terms arising from initial condition. Then, to determine the components of $u_n(x,t)$, Adomian decomposition method uses the recursive relation

$$u_0(x,t) = u(x,0) ,$$

$$\begin{aligned} u_{n+1}(x,t) = &-45\, L_t^{-1}(A_n) - 45 L_t^{-1}(B_n) - 210\, L_t^{-1}(C_n) - 210\, L_t^{-1}(D_n) - \\ &- 1575\, L_t^{-1}(E_n) - 3150\, L_t^{-1}(F_n) - 1260\, L_t^{-1}(G_n) - 630\, L_t^{-1}(H_n) - 9450\, L_t^{-1}(J_n) - \\ &- 3150\, L_t^{-1}(K_n) - 4725\, L_t^{-1}(M_n) - L_t^{-1}(L_{9x} u), n \geq 0. \end{aligned} \quad (23)$$

With this relation, the components of $u_n(x,t)$ are easily obtained. This leads to the solution in a series form. The solution in a closed form follows immediately if an exact solution exists.

Considering the given initial condition, we can assume

$u_0(x,t) = \frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2)$ as an initial approximation. We next use the recursive relations (22)-(23) to obtain the rest of components of $u_n(x,t)$.

$$u_0 = \frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2),$$

$$u_1 = \frac{3479691175\ 8272\, b^{\frac{11}{2}} i k^{11} t \sec h[\sqrt{b}kx]^2 \tanh[\sqrt{b}kx]}{9796423},$$

$$u_2 = \frac{1513531334\ 8912102132\ 0053248\, b^{10} k^{20} t^2 (-2 + \cosh[2\sqrt{b}kx]) \sec h[\sqrt{b}kx]^4}{9596990359\ 4929},$$

$$\begin{aligned} u_3 = &(945957084\ 3070063832\ 503328\, b^{\frac{29}{2}} k^{29} t^3 \sec h[\sqrt{b}kx]^{13} (576118951\ 475102 \sinh[\sqrt{b}kx] \\ &+ 2405656077\ 78742 \sinh[3\sqrt{b}kx] - 9656086124\ 7595 \sinh[5\sqrt{b}kx] \\ &- 5025398894\ 2927 \sinh[7\sqrt{b}kx] - 4308691386\ 269 \sinh[9\sqrt{b}kx] \\ &+ 5437017462\ 23 \sinh[11\sqrt{b}kx]))/282048\ 5312655435\ 416901 \quad , \\ &\vdots \end{aligned}$$

and the rest of the components of iteration formula (23) are obtained. The approximate solution which involves four terms is given by

$$\begin{aligned} u(x,t) = &\frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2) + \frac{3479691175\ 8272\, b^{\frac{11}{2}} i k^{11} t \sec h[\sqrt{b}kx]^2 \tanh[\sqrt{b}kx]}{9796423} \\ &+ \frac{1513531334\ 8912102132\ 0053248\, b^{10} k^{20} t^2 (-2 + \cosh[2\sqrt{b}kx]) \sec h[\sqrt{b}kx]^4}{9596990359\ 4929} \end{aligned}$$

$$+(9459570843070063832503328\, b^{\frac{29}{2}} k^{29} t^3 \operatorname{sech}[\sqrt{b}kx]^{13}(576118951475102 \sinh[\sqrt{b}kx]$$
$$+240565607778742 \sinh[3\sqrt{b}kx] - 96560861247595 \sinh[5\sqrt{b}]$$
$$-50253988942927 \sinh[7\sqrt{b}kx] - 4308691386269 \sinh[9\sqrt{b}kx]$$
$$+543701746223 \sinh[11\sqrt{b}kx]))/2820485312655435416901 + \cdots. \quad (24)$$

b) Implementation of HPM

Considering the given initial condition, we can assume

$u_0 = u(x,0) = \dfrac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2)$ as an initial approximation. In order to solve Eq. (18), using HPM, we construct the following homotopy for these equations

$$(1-p)\left[\dot{Y} - \dot{u}_0\right] + p[\dot{Y} + 45Y'Y^{(6)} + 45YY^{(7)} + 210Y'''Y^{(4)} + 210Y''Y^{(5)} + 1575Y'(Y'')^2 + $$
$$+3150YY''Y''' + 1260YY'Y^{(4)} + 630Y^2Y^{(5)} + 9450Y^2Y'Y'' + 3150Y^3Y''' + 4725Y^4Y' + Y^{(9)}] = 0, \quad (25)$$

where

$$\dot{Y} = \frac{\partial Y}{\partial t},\ Y' = \frac{\partial Y}{\partial x},\ Y'' = \frac{\partial^2 Y}{\partial x^2},\ Y''' = \frac{\partial^3 Y}{\partial x^3},\ Y^{(4)} = \frac{\partial^4 Y}{\partial x^4},\ \dots ,$$

and

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + \dots ,$$
$$\dot{Y} = \dot{Y}_0 + p\dot{Y}_1 + p^2 \dot{Y}_2 + p^3 \dot{Y}_3 + \dots ,$$
$$Y' = Y_0' + pY_1' + p^2 Y_2' + p^3 Y_3' + \dots ,$$
$$Y'' = Y_0'' + pY_1'' + p^2 Y_2'' + p^3 Y_3'' + \dots ,$$
$$Y''' = Y_0''' + pY_1''' + p^2 Y_2''' + p^3 Y_3''' + \dots ,$$
$$\vdots$$

Substituting and equating the coefficients of like powers of p, we obtain series of inhomogeneous linear differential equations to solve. i.e.

$$p^0 : \dot{Y}_0 - \dot{u}_0 = 0 \Rightarrow Y_0 = u_0 = \frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2),$$

$$p^1 : \dot{Y}_1 + \dot{u}_0 + 45Y_0'Y_0^{(6)} + 45Y_0Y_0^{(7)} + 210Y_0'''Y_0^{(4)} + 210Y_0''Y_0^{(5)} + 1575Y_0'(Y_0'')^2 + 3150Y_0Y_0''Y_0'''$$
$$+1260Y_0Y_0'Y_0^{(4)} + 630Y_0^2Y_0^{(5)} + 9450Y_0^2Y_0'Y_0'' + 3150Y_0^3Y_0''' + 4725Y_0^4Y_0' + Y_0^{(9)} = 0,$$

$$p^2 : \dot{Y}_2 + 45(Y_0'Y_1^{(6)} + Y_1'Y_0^{(6)}) + 45(Y_0Y_1^{(7)} + Y_1Y_0^{(7)}) + 210(Y_0'''Y_1^{(4)} + Y_1'''Y_0^{(4)})$$
$$+210(Y_0''Y_1^{(5)} + Y_1''Y_0^{(5)}) + 1575(Y_1'(Y_0'')^2 + 2Y_1''Y_0''Y_0') + 3150(Y_0Y_0''Y_1''' + Y_0Y_1''Y_0''' + Y_1Y_0''Y_0''')$$
$$+1260(Y_0Y_0'Y_1^{(4)} + Y_0Y_1'Y_0^{(4)} + Y_1Y_0'Y_0^{(4)}) + 630(Y_0^2Y_1^{(5)} + 2Y_1Y_0Y_0^{(5)}) + 9450(Y_0^2Y_1'Y_0''$$

$$+2Y_1Y_0Y_0^{'}Y_0^{''}+Y_0^2Y_0^{'}Y_1^{''})+3150(Y_0^3Y_1^{'''}+3Y_0^2Y_1Y_0^{'''})+4725(Y_0^4Y_1^{'}+4Y_0^3Y_1Y_0^{'})+Y_1^{(9)}=0\,,$$

$\vdots$

From the above equations, we can obtain

$$Y_0=\frac{2}{273}bk^2(692-273\tanh[\sqrt{b}kx]^2)\,,$$

$$Y_1=\frac{3479691175\ 8272\,b^{\frac{11}{2}}ik^{11}t\,\mathrm{sec}h[\sqrt{b}kx]^2\tanh[\sqrt{b}kx]}{9796423}\,,$$

$$Y_2=\frac{1513531334\ 8912102132\ 0053248\,b^{10}k^{20}t^2(-2+\cosh[2\sqrt{b}kx])\,\mathrm{sec}h[\sqrt{b}kx]^4}{9596990359\ 4929}\,,$$

$$\begin{aligned}Y_3=&(945957084\ 3070063832\ 503328\,b^{\frac{29}{2}}k^{29}t^3\,\mathrm{sec}h[\sqrt{b}kx]^{13}(576118951\ 475102\sinh[\sqrt{b}kx]\\&+2405656077\ 78742\sinh[3\sqrt{b}kx]-9656086124\ 7595\sinh[5\sqrt{b}kx]\\&-5025398894\ 2927\sinh[7\sqrt{b}kx]-4308691386\ 269\sinh[9\sqrt{b}kx]\\&+5437017462\ 23\sinh[11\sqrt{b}kx]))/282048\,5312655435\ 416901\end{aligned}$$

$\vdots$

and

$$u(x,t)=\lim_{p\to1}\sum_{k=0}^{\infty}p^kY_k\,,$$

$$u(x,t)=Y_0+Y_1+Y_2+Y_3+\ldots$$

$$\begin{aligned}u(x,t)=&\frac{2}{273}bk^2(692-273\tanh[\sqrt{b}kx]^2)+\frac{3479691175\ 8272\,b^{\frac{11}{2}}ik^{11}t\,\mathrm{sec}h[\sqrt{b}kx]^2\tanh[\sqrt{b}kx]}{9796423}\\&+\frac{1513531334\ 8912102132\ 0053248\,b^{10}k^{20}t^2(-2+\cosh[2\sqrt{b}kx])\,\mathrm{sec}h[\sqrt{b}kx]^4}{9596990359\ 4929}\\&+(945957084\ 3070063832\ 503328\,b^{\frac{29}{2}}k^{29}t^3\,\mathrm{sec}h[\sqrt{b}kx]^{13}(576118951\ 475102\sinh[\sqrt{b}kx]\\&+2405656077\ 78742\sinh[3\sqrt{b}kx]-9656086124\ 7595\sinh[5\sqrt{b}]\\&-5025398894\ 2927\sinh[7\sqrt{b}kx]-4308691386\ 269\sinh[9\sqrt{b}kx]\\&+5437017462\ 23\sinh[11\sqrt{b}kx]))/282048\,5312655435\ 416901+\ldots.(26)\end{aligned}$$

which is the same as that obtained by ADM.

So we can see clearly that the obtained solution is of high accuracy and by continuation of homotopy perturbation method we can obtain more terms of Taylor expansion, rapidly.

b) Implementation of HAM

Considering the given initial condition, we can assume $u_0 = u(x,0) = \frac{2}{273} bk^2 (692 - 273 \tanh[\sqrt{b}kx]^2)$ as an initial approximation. In order to solve Eq. (18), using HAM

$$\mathrm{L}\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] = hH(x,t) R_n[\vec{u}_{n-1}(x,t)], \quad (27)$$

where

$$H(x,t) = 1, \quad \chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases} \quad \text{and}$$

$$R_n[\vec{u}_{n-1}(x,t)] = \frac{\partial u_{n-1}(x,t)}{\partial t} + 45\sum_{k=0}^{n-1} \frac{\partial u_k(x,t)}{\partial x} \frac{\partial^6 u_{n-1-k}(x,t)}{\partial x^6} + 45\sum_{k=0}^{n-1} u_k(x,t) \frac{\partial^7 u_{n-1-k}(x,t)}{\partial x^7}$$
$$+ 210\sum_{k=0}^{n-1} \frac{\partial^3 u_k(x,t)}{\partial x^3} \frac{\partial^4 u_{n-1-k}(x,t)}{\partial x^4} + 210\sum_{k=0}^{n-1} \frac{\partial^2 u_k(x,t)}{\partial x^2} \frac{\partial^5 u_{n-1-k}(x,t)}{\partial x^5}$$
$$+1575\sum_{k=0}^{n-1}\left(\frac{\partial u_k(x,t)}{\partial x} \sum_{i=0}^{n-1-k} \frac{\partial^2 u_i(x,t)}{\partial x^2} \frac{\partial^2 u_{n-1-k-i}(x,t)}{\partial x^2}\right) + 3150\sum_{k=0}^{n-1}\left(u_k(x,t) \sum_{i=0}^{n-1-k} \frac{\partial^2 u_i(x,t)}{\partial x^2} \frac{\partial^3 u_{n-1-k-i}(x,t)}{\partial x^3}\right)$$
$$+1260\sum_{k=0}^{n-1}\left(u_k(x,t) \sum_{i=0}^{n-1-k} \frac{\partial u_i(x,t)}{\partial x} \frac{\partial^4 u_{n-1-k-i}(x,t)}{\partial x^4}\right) + 630\sum_{k=0}^{n-1}\left(\frac{\partial^5 u_k(x,t)}{\partial x^5} \sum_{i=0}^{n-1-k} u_i(x,t) u_{n-1-k-i}(x,t)\right) (27)$$
$$+9450\sum_{k=0}^{n-1}\left(\frac{\partial u_k(x,t)}{\partial x} \sum_{i=0}^{n-1-k}\left(\frac{\partial^2 u_i(x,t)}{\partial x^2} \sum_{j=0}^{n-1-k-i} u_j(x,t) u_{n-1-k-i-j}(x,t)\right)\right)$$
$$+3150\sum_{k=0}^{n-1}\left(\frac{\partial^3 u_k(x,t)}{\partial x^3} \sum_{i=0}^{n-1-k}\left(u_i(x,t) \sum_{j=0}^{n-1-k-i} u_j(x,t) u_{n-1-k-i-j}(x,t)\right)\right)$$
$$+4725\sum_{k=0}^{n-1}\left(\frac{\partial u_k(x,t)}{\partial x} \sum_{i=0}^{n-1-k}\left(u_i(x,t) \sum_{j=0}^{n-1-k-i}\left(u_j(x,t) \sum_{t=0}^{n-1-k-j} u_t(x,t) u_{n-1-k-i-j-t}(x,t)\right)\right)\right) + \frac{\partial^9 u_{n-1}(x,t)}{\partial x^9}.$$

The solution of the *n*th-order deformation equation (26) for $n \ge 1$ reads

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + h\int_0^t R_n[\vec{u}_{n-1}(x,t)]dt,$$

where the constant of integration c_1 is determined by the initial condition(19).Using symbolic computation systems such as Maple or Mathematica, we recursively obtain

$$u_1(x,t) = \int_0^t hR_1[u_0(x,t)]dt$$

$$= \int_0^t h[u_{0_t} + 45u_{0_x}u_{0_{6x}} + 45u_0u_{0_{7x}} + 210u_{0_{3x}}u_{0_{4x}} + 210u_{0_{2x}}u_{0_{5x}} + 1575u_{0_{2X}}^2 u_{0_x} + 3150u_0u_{0_{2x}}u_{0_{3x}}$$

$$+1260u_0u_{0_x}u_{0_{4x}} + 630u_0^2u_{0_{5x}} + 9450u_0^2u_{0_x}u_{0_{2x}} + 3150u_0^3u_{0_{3x}} + 4725u_0^4u_{0_x} + u_{0_{9x}}]dt$$

$$= -\frac{3479691175\ 8272 b^{\frac{11}{2}} hk^{11} t \sec h[\sqrt{b}kx]^2 \tanh[\sqrt{b}kx]}{9796423},$$

$$u_2(x,t) = u_1(x,t) + \int_0^t hR_2[u_1(x,t)]dt$$

$$= \frac{1}{9596990359\ 4929} 1739845587\ 9136 b^{\frac{11}{2}} hk^{11} t \sec h[\sqrt{b}kx]^4 (-1739845587\ 9136 b^{\frac{9}{2}} hk^9 t$$

$$+8699227939\ 568 b^{\frac{9}{2}} hk^9 t \cosh[2\sqrt{b}kx] - 9796423(1+h)\sinh[2\sqrt{b}kx]),$$

$$u_3(x,t) = u_2(x,t) + \int_0^t hR_3[u_2(x,t)]$$

$$= -\frac{h}{2820485312\ 6554354169\ 01}(945957084\ 3070063832\ 503328 b^{\frac{29}{2}} k^{29} t^3 \sec h[\sqrt{b}kx]^{13}$$

$$(576118951\ 475102 \sinh[\sqrt{b}kx] + 2405656077\ 78742 \sinh[3\sqrt{b}kx]$$

$$-9656086124\ 7595 \sinh[5\sqrt{b}kx] - 5025398894\ 2927 \sinh[7\sqrt{b}kx]$$

$$-4308691386\ 269 \sinh[9\sqrt{b}kx] + 5437017462\ 23 \sinh[11\sqrt{b}kx])).$$

The approximate solution which involves four terms is given by

$$u(x,t) = \frac{2}{273} bk^2 (692 - 273\tanh[\sqrt{b}kx]^2) - \frac{3479691175\ 8272 b^{\frac{11}{2}} hk^{11} t \sec h[\sqrt{b}kx]^2 \tanh[\sqrt{b}kx]}{9796423}$$

$$+\frac{1}{9596990359\ 4929} 1739845587\ 9136 b^{\frac{11}{2}} hk^{11} t \sec h[\sqrt{b}kx]^4 (-1739845587\ 9136 b^{\frac{9}{2}} hk^9 t$$

$$+8699227939\ 568 b^{\frac{9}{2}} hk^9 t \cosh[2\sqrt{b}kx] - 9796423(1+h)\sinh[2\sqrt{b}kx])$$

$$-\frac{h}{2820485312\ 6554354169\ 01}(945957084\ 3070063832\ 503328 b^{\frac{29}{2}} k^{29} t^3 \sec h[\sqrt{b}kx]^{13}$$

$$(576118951\ 475102 \sinh[\sqrt{b}kx] + 2405656077\ 78742 \sinh[3\sqrt{b}kx]$$

$$-9656086124\ 7595 \sinh[5\sqrt{b}kx] - 5025398894\ 2927 \sinh[7\sqrt{b}kx]$$

$$-4308691386\ 269 \sinh[9\sqrt{b}kx] + 5437017462\ 23 \sinh[11\sqrt{b}kx])) + \ldots \quad (28)$$

When $h=-1$, the 4th-order approximation

$$u(x,t)=\frac{2}{273}bk^2(692-273\tanh[\sqrt{b}kx]^2)-\frac{3479691175\ 8272\,b^{\frac{11}{2}}hk^{11}t\,\mathrm{sec}h[\sqrt{b}kx]^2\tanh[\sqrt{b}kx]}{9796423}$$

$$+\frac{1}{9596990359\ 4929}1739845587\ 9136\,b^{\frac{11}{2}}hk^{11}t\,\mathrm{sec}h[\sqrt{b}kx]^4(-1739845587\ 9136\,b^{\frac{9}{2}}hk^9t$$

$$+8699227939\ 568\,b^{\frac{9}{2}}hk^9t\cosh[2\sqrt{b}kx]-9796423(1+h)\sinh[2\sqrt{b}kx])$$

$$-\frac{h}{2820485312\ 6554354169\ 01}(945957084\ 3070063832\ 503328\,b^{\frac{29}{2}}k^{29}t^3\,\mathrm{sec}h[\sqrt{b}kx]^{13}$$

$$(576118951\ 475102\sinh[\sqrt{b}kx]+2405656077\ 78742\sinh[3\sqrt{b}kx]$$

$$-9656086124\ 7595\sinh[5\sqrt{b}kx]-5025398894\ 2927\sinh[7\sqrt{b}kx]$$

$$-4308691386\ 269\sinh[9\sqrt{b}kx]+5437017462\ 23\sinh[11\sqrt{b}kx]))+\ldots$$

is exactly the same as the HPM solution (26).Therefore, the HPM solution is indeed aspecial case of the HAM solution when $h=-1$.

4. Comparison among ADM, HAM and HPM

In this chapter, it is shown three dimensional figure of $u(x,t)$ obtaining by using ADM, HAM and HPM. Moreover, it is given error data between numerical solution and analytical solution of Ninth-order Korteweg-de Vries equation on tables and graphs. Later, by drawn graph of auxiliary h parameter, it is determined convergence range for approximate solution of Ninth-order Korteweg-de Vries equation by using HAM. By given values to auxiliary h parameter in points within the specified convergence range, it is shown best approach in which values of the auxiliary h parameter. For numerical comparisons purposes, we consider Ninth-order Korteweg-de Vries equation. The formula of numerical results for ADM, HAM and HPM (when $H(x,t)=1$) are given as follow

$$\lim_{n\to\infty}\psi_n=u(x,t)\ \text{where}\ \psi_n(x,t)=\sum_{k=0}^{n}u_k(x,t),\ \ n\geq 0 \tag{29}$$

The numerical results obtained with ADM, HAM and HPM for Ninth-order Korteweg-de Vries equation are shown in Figure 1, 3 Table 1-3. As seen in Table 1-3, series of solutions $u(x,t)$ of Ninth-order Korteweg-de Vries equation are very close to the analytical solution by considering only the six-term. As can be seen in Table 1-3, the numerical values calculated with ADM, HAM and HPM are the same. However, this situation occurs when $h=-1$. That is, if it chooses $h=-1$

for considered the problem; the numerical results of ADM and HPM converge to the numerical results of HAM. As seen in Figure 2, it is obtained convergence range of series solution by drawn figure of h parameter in the numerical solutions obtained with HAM for the best convergence. According to Figure 2, the convergence range of series solutions $u(x,t)$ is $-1.4 \le h \le -0.6$ approximately. It is seen in Table 4 that the best approach to the analytical solution is at $h=-1$. The choice of parameter h in this way, it can be seen as an advantage of HAM. Thus, the auxiliary parameter h plays an important role within the frame of the HAM. In light of these findings, we can say that ADM and HPM are in fact a special case of HAM, when $h=-1$. This situation is also shown in [21-24].

a) b)

c) **d)**

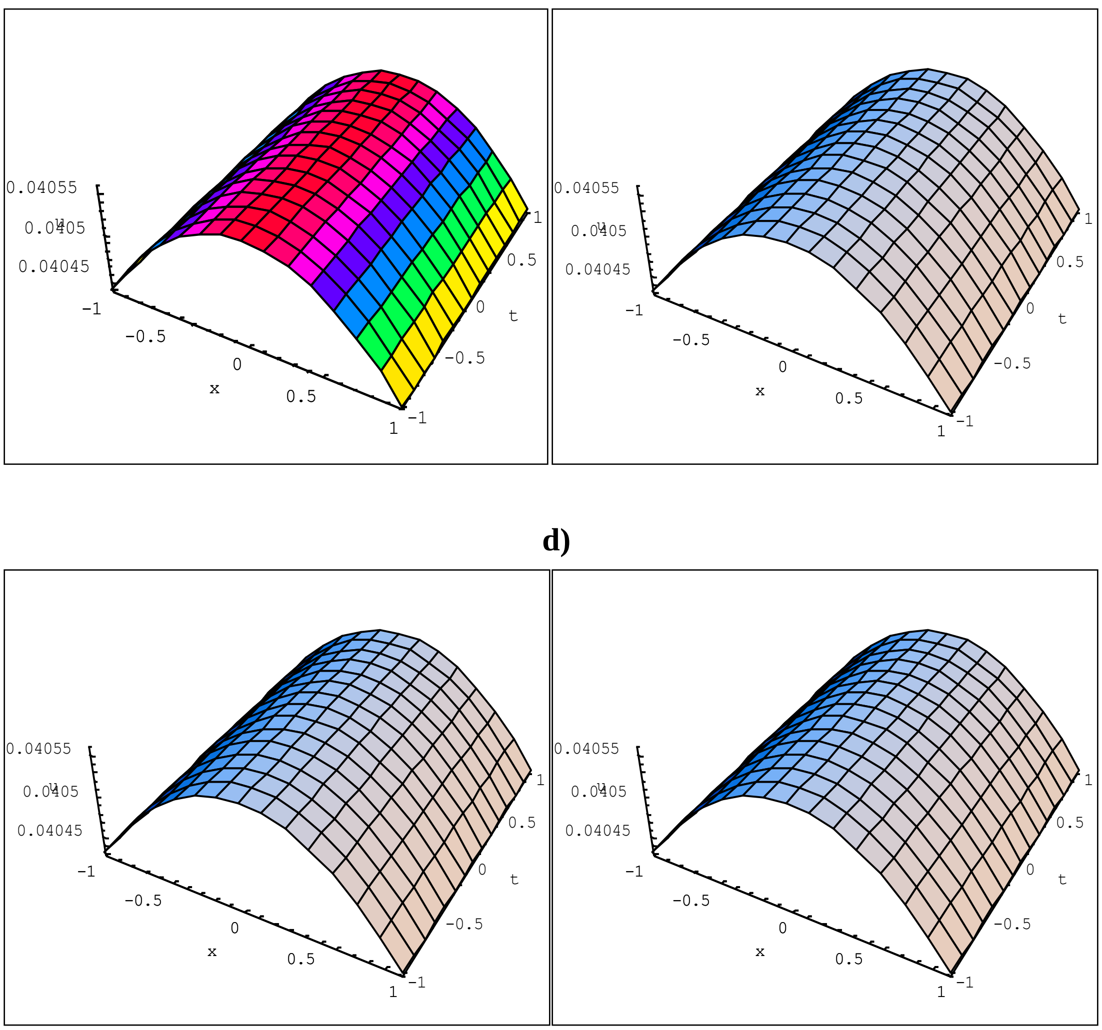


FIG. 1. Comparison of 3-dimensional the exact and approximate solutions of nKdV equation.**a)**Exact solution;**b)**Approximate solutions by means of ADM; **c)**Approximate solutions by means of HAM $(h=-1)$;**d)** Approximate solutions by means of HPM. $(b=0.2, k=0.2)$

TABLE I. Absolute error for the (24) approximate solution using the ADM. $(b = 0.2, k = 0.2)$

$\lvert u(x,t) - \Phi_4(x,t) \rvert$					
T | ***x***	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0.1	2.08137×10^{-17}	1.45717×10^{-16}	4.92661×10^{-16}	1.16573×10^{-15}	2.27596×10^{-15}
0.2	3.46945×10^{-17}	2.91434×10^{-16}	9.71445×10^{-16}	2.32453×10^{-15}	4.53804×10^{-15}
0.3	5.55112×10^{-17}	4.30211×10^{-16}	1.46411×10^{-16}	3.48332×10^{-15}	6.79318×10^{-15}
0.4	6.93889×10^{-17}	5.82867×10^{-16}	1.94983×10^{-16}	4.6213×10^{-15}	9.0275×10^{-15}
0.5	9.71445×10^{-17}	7.07767×10^{-16}	2.42861×10^{-16}	5.75234×10^{-15}	1.12341×10^{-15}

TABLE II. Absolute error for the (26) approximate solution using the HPM. $(b = 0.2, k = 0.2)$

$\lvert u(x,t) - \Phi_4(x,t) \rvert$					
T | ***x***	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0.1	2.08137×10^{-17}	1.45717×10^{-16}	4.92661×10^{-16}	1.16573×10^{-15}	2.27596×10^{-15}
0.2	3.46945×10^{-17}	2.91434×10^{-16}	9.71445×10^{-16}	2.32453×10^{-15}	4.53804×10^{-15}
0.3	5.55112×10^{-17}	4.30211×10^{-16}	1.46411×10^{-16}	3.48332×10^{-15}	6.79318×10^{-15}
0.4	6.93889×10^{-17}	5.82867×10^{-16}	1.94983×10^{-16}	4.6213×10^{-15}	9.0275×10^{-15}
0.5	9.71445×10^{-17}	7.07767×10^{-16}	2.42861×10^{-16}	5.75234×10^{-15}	1.12341×10^{-15}

TABLE III. Absolute error for the (28) approximate solution using the HAM. $(b = 0.2, k = 0.2, h = -1)$

$\lvert u(x,t) - \Phi_4(x,t) \rvert$					
T | ***x***	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0.1	2.08137×10^{-17}	1.45717×10^{-16}	4.92661×10^{-16}	1.16573×10^{-15}	2.27596×10^{-15}
0.2	3.46945×10^{-17}	2.91434×10^{-16}	9.71445×10^{-16}	2.32453×10^{-15}	4.53804×10^{-15}
0.3	5.55112×10^{-17}	4.30211×10^{-16}	1.46411×10^{-16}	3.48332×10^{-15}	6.79318×10^{-15}
0.4	6.93889×10^{-17}	5.82867×10^{-16}	1.94983×10^{-16}	4.6213×10^{-15}	9.0275×10^{-15}
0.5	9.71445×10^{-17}	7.07767×10^{-16}	2.42861×10^{-16}	5.75234×10^{-15}	1.12341×10^{-15}

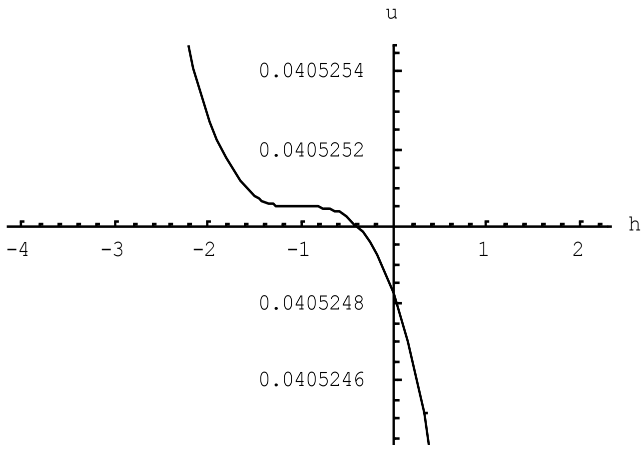
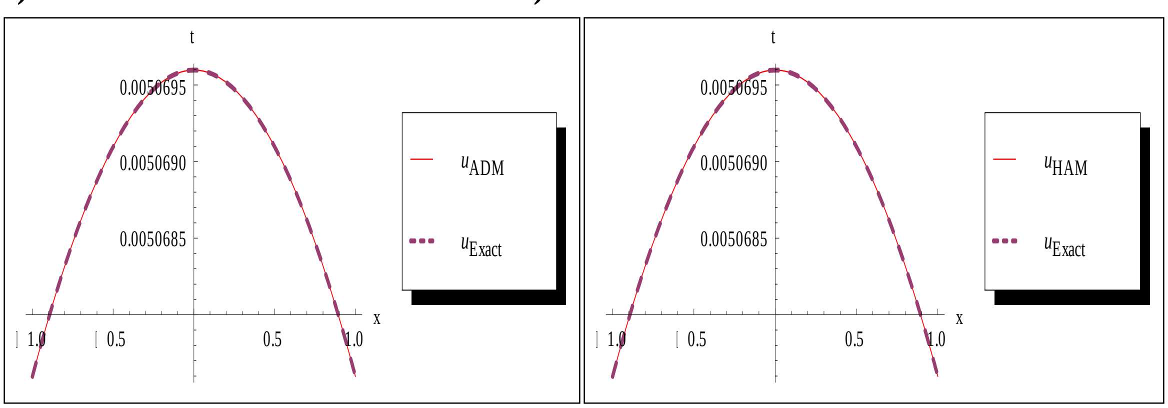


FIG. 2. Curve of arbitrary parameters h for $u(x,t)$

TABLE IV. Absolute error for the (28) approximate solution using the HAM for the value of h in the convergence region of solution series. $(x=0.5, b=0.2, k=0.2)$

T | ***h***	**-1.4**	**-1.2**	**-1**	**-0.8**	**-0.6**
0.1	3.23693×10^{-9}	4.30082×10^{-10}	2.27596×10^{-15}	3.28447×10^{-10}	2.8304×10^{-9}
0.2	6.21498×10^{-9}	8.0253×10^{-10}	4.53804×10^{-15}	7.00807×10^{-10}	5.80811×10^{-9}
0.3	9.18964×10^{-9}	1.17468×10^{-9}	6.79318×10^{-15}	1.07282×10^{-9}	8.78222×10^{-9}
0.4	1.2159×10^{-8}	1.54629×10^{-9}	9.0275×10^{-15}	1.44423×10^{-9}	1.17508×10^{-8}
0.5	1.51212×10^{-8}	1.91713×10^{-9}	1.12341×10^{-15}	1.81482×10^{-9}	1.4712×10^{-8}

a) **b)**



c)

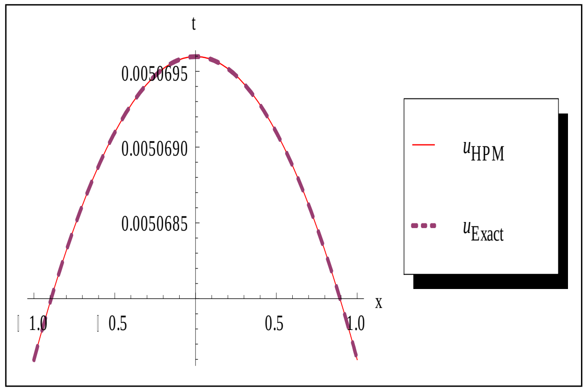


FIG. 3 Comparison of the exact and approximate solutions of nKdV equation by means of ADM, HAM and HPM.**a)** Comparison of the exact and approximate solutions of nKdV equation by means of ADM;**b)**Comparison of the exact and approximate solutions of nKdV equation by means of HAM $(h=-1)$; **c)**Comparison of the exact and approximate solutions of nKdV equation by means of HPM.

5. Conclusion and Remarks

A clear conclusion can be drawn from the numerical results that the ADM, HAM and HPM algorithms provide highly accurate numerical solutions for nonlinear partial differential equations. It is also worth noting that the advantage of the approximation of the series methodologies displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depends on the character and behavior of the solutions just as in a closed form solutions. Because of the auxiliary parameter h and auxiliary $H(x,t)$ function, HAM provides a better convergence than the other numerical methods. When it is taken $h=-1$, ADM and HPM are derived from HAM.

References

[1] Adomian, G., A Review of the Decomposition Method and Some Recent Results for Nonlinear Equations, Mathematical and Computer Modelling, 13, (1990) 17-43.
[2] Wazwaz, A. M., Adomian Decomposition Method for aEliable Treatment of theBratu-Type Equations, Applied Mathematics and Computation, 166, (2005) 652-663.
[3] Wazwaz, A. M., Gorguis, A., Exact Solutions of Heat-Like and Wave-LikeEquations with Variable Coefficients, Applied Mathematics and Computation, 149, (2004) 15-29.
[4] Wazwaz, A. M., A New Algorithm for Calculating Adomian Polynomials forNonlinear Operators, Applied Mathematics and Computation, 111, (2000) 33-51.
[5] Lesnic, D., Blow-up solutions obtained using the decomposition method, ChaosSolitons Fractals, 28, (2006) 776-787.
[6] Kaya, D., 2005. An Implementation of the ADM for Generalized One Dimensional KleinGordon Equation, Applied Mathematics and Computation, 166, 426-433.

[7] Adomian, G., Solving Frontier Problems of Physics: The decomposition Method,Kluwer Academic Publishers, Boston, 1994.

[8] Adomian, G., A Review of the Decomposition Method in Applied Mathematics, Journal of Mathematical Analysis and Applications, 135, (1998) 501-544.

[9] Liao, S. J., The Proposed Homotopy Analysis Technique for the Solution ofNonlinear Problems, PhD Thesis, Shanghai Jiao Tong University, 2003.

[10] Liao, S. J., Beyond Perturbation: İntroduction to the Homotopy Analysis Method,Boca Raton: Chapman Hall CRC/Press, 2003.

[11] Wang, Z., Zou, L. and Zhang, H., Applying Homotopy Analysis Method forSolving Differential-Difference Equation, Physics Letters A, 369, (2007) 77-84.

[12] Abbasbandy, S., The Application of Homotopy Analysis Method to NonlinearEquations Arising in Heat Transfer, Physics Letters A, 360, (2006) 109-113.

[13] He, J. H., Homotopy-Perturbation Technique, Computer Methods in AppliedMechanics and Engineering, 178, (1999) 257-262.

[14] He, J. H., Perturbation Methods: Basic and Beyond, Elsevier, Amsterdam, 2006.

[15] Sadighi, A. and Ganji, D. D., Exact Solutions of Laplace Equation by HomotopyPerturbation and Adomian Decomposition Methods, Physics Letter A, 367, (2007) 83-87.

[16] Olayiwola, M. O., Gbolagade, A. W., Adesanya, A. O. and Akinpelu, F. O.,Application of Homotopi Perturbation Method for the Large Angle Period of NonlinearOscillator, Researcher, 1(1), (2009) 1-5.

[17] Inan, I. E., Uğurlu, Y. and Duran, S., Complex Solutions for Ninth-OrderKorteweg-deVries (nKdv) Equation and 2-dimensional Burgers Equation, Turkish Journal ofScience&Technology, 5(1), (2010) 37–42.

[18] Wazwaz, A. M., Multiple-Soliton Solutions for the Ninth-Order KdVEquation and SixthOrder Boussinesq Equation, Applied Mathematics and Computation, 203, (2008)277–283.

[19] Wazwaz, A. M., Erratum to: ‘‘Multiple-Soliton Solutions for the Ninth-OrderKdVEquation and Sixth-Order Boussinesq Equation”, Applied Mathematics andComputation, 206,(2008) 1005.

[20] Cherrualt, Y. and Adomian, G., Decomposition Methods: A New Proof of Convergence,Mathematical and Computer Modelling, 18, (1993) 103-106.

[21] Liang,S. and Jeffrey, D.J., Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation,Communications in Nonlinear Science and Numerical Simulation 14, (2009) 4057-4064.

[22] Allan, F.M., Derivation of the Adomian decomposition method using the homotopy analysis method, Applied Mathematics and Computation, 190, (2007) 6-14.

[23] Liao,S., Comparison between the homotopy analysis method and homotopy perturbation method, Applied Mathematics and Computation 169, (2005) 1186-1194.

[24] Bataineh, A.S., Noorani, M.S.M. and Hashim, I., Solutions of time-dependent Emden–Fowler type equations by homotopy analysis method, Physics Letters A, 371, (2007) 72- 82.