Article

Fractal Spacetime & the Dynamic Generation of Mass Scales in Field Theory

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Abstract

As of today, the mechanism underlying the generation of mass scales in field theory remains elusive. Here we show how the concept of fractal spacetime having minimal deviations from four-dimensionality (the so-called *minimal fractal manifold* defined through $\varepsilon = 4 - D$, with $\varepsilon \ll 1$) can naturally account for the onset of these scales. A counterintuitive outcome of this analysis is the deep link between the minimal fractal manifold and the holographic principle.

Key words: fractal spacetime, dimensional flow, dimensional reduction, electroweak scale, QCD scale, holographic principle, holographic bound.

1. Introduction and motivation

One of the many unsettled questions raised by field theory revolves around the vast hierarchy of scales in Nature [2-3, 32]. A large numerical disparity exists between the Planck scale (M_{Pl}), the electroweak scale (M_{EW}), the hadronization scale of Quantum Chromodynamics (Λ_{QCD}) and the cosmological constant scale ($\Lambda_{cc}^{1/4}$, with Λ_{cc} expressed as energy density in 3+1 dimensions). The goal of this work is to suggest that the answer to this question may lie in the fractal geometry of spacetime near or above M_{EW}.

It has been long known that perturbative quantum field theory (QFT) cannot provide a complete description of Nature since its formalism entails divergences at both ends of the energy spectrum [1-3]. For instance, many textbooks emphasize that the singular behavior of momentum integrals in the ultraviolet (UV) sector arises from the poorly understood spacetime structure at short distances [2-3]. Lattice field models handle infinities through discretization of the spacetime continuum on a grid of spacing "Δ". This procedure naturally bounds the maximal momentum allowed to propagate through the lattice, namely,

$$p \leq p_{\max} \sim (2\Delta)^{-1} \tag{1}$$

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The downside of lattice models is that they generally fail to be either gauge or Poincaré invariant [1-4]. Restoring formal consistency is further enabled via the Renormalization Group program (RG) [2-3, 15]. RG regulates the *n*-th order momentum integrals of the generic form

$$I_n(p) = \int dp\, f(p^{2n}) \tag{2}$$

by either inserting an arbitrary momentum cutoff $0 < \Lambda \sim \Delta^{-1} < \infty$ *or* by continuously "deforming" the four-dimensional spacetime via the dimensional parameter $\varepsilon = 4 - D$, ε << 1. The resulting theory is free from divergences and operates with a finite number of redefined physical parameters. Restoring the continuum spacetime is done at the end by taking the limit $\Lambda \to \infty$ or $\varepsilon \to 0$.

Regularization techniques employed in RG are not independent from each other. The connection between dimensional and UV cutoff regularizations ($\Lambda = \Lambda_{UV}$) is given by [13, 15, 18]

$$\log \frac{\Lambda_{UV}^2}{\mu^2} = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi + \frac{5}{6} \tag{3}$$

Here, γ_E stands for the Euler constant and μ for the observation (or "sliding") scale. It is more convenient to present (3) is a slightly different form, that is,

$$\varepsilon \sim \frac{1}{\log(\frac{\Lambda_{UV}^2}{\mu^2})} \tag{4}$$

If the numerical disparity between μ and Λ_{UV} is large enough, one can reasonably approximate ε as in

$$\varepsilon \sim (\frac{\mu}{\Lambda_{UV}})^2 \tag{5a}$$

Following [18, 31, 37], the far infrared (IR) scale of field theory set by the cosmological constant ($\Lambda_{cc}^{1/4}$), the electroweak scale (M_{EW}) and the far UV scale fixed by the Planck mass (M_{Pl}) satisfy the constraint

$$\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \sim \frac{M_{EW}}{M_{Pl}} = O(\varepsilon) \tag{5b}$$

It is apparent from (4) or (5a) that the four-dimensional spacetime continuum is recovered in either one of these limits:

a) $\Lambda_{UV} \to \infty$ and $0 < \mu << \Lambda_{UV}$,

b) $\Lambda_{UV} < \infty$ and $\mu \to 0$

However, both limits are disfavored by our current understanding of the far UV and the far IR boundaries of field theory (see e.g. [3]). Theory and experimental data alike tell us that the notions of infinite *or* zero energy are, strictly speaking, meaningless. This is to say that either infinite energies (point-like objects) or zero energy (infinite distance scales) are *unphysical idealizations*. Indeed, there is always a finite cutoff at both ends of either energy or energy density scale (far UV = Planck scale, far IR = finite radius of the observable Universe or the non-vanishing energy density of the vacuum set by cosmological constant). These observations are also consistent with the estimated infinitesimal (yet non-vanishing) photon mass, as highlighted in [23-24].

Reinforcing this viewpoint, some authors argue that the idea of smooth spacetime stands in manifest conflict with the basic premises of quantum theory [11]. To confine an event within a region of extension Δ requires a momentum transfer on the order of Δ^{-1} which, in turn, generates a local gravitational field. If the density of momentum transfer is comparable in magnitude with the right hand side of Einstein's equation, the local curvature of space-time (~ R_0^{-2}) induced by this transfer is given by (in natural units, $\hbar = c = 1$)

$$R_0^{-2} \sim G_N \Delta^{-4} \tag{6}$$

However, collapse of the event within a short region of extent $\Delta = O(R_0)$ amounts to trapping outgoing light signals and preventing direct observation.

All these considerations invariably point out to the following challenge: on the one hand, a continuum model of spacetime near or below M_{EW} serves as an effective paradigm that is likely to fail at large probing energies. Yet on the other, any discrete model of spacetime typically violates Poincaré or gauge symmetries. It seems only natural, in this context, to take a fresh look at (4) and (5a) and appreciate the message it conveys: if either Λ_{UV} stays finite *or* ε << 1 is arbitrarily small but non-vanishing, the spacetime dimensionality becomes a non-integer arbitrarily close to four. Stated differently, in the neighborhood of M_{EW}, conventional spacetime turns into a *minimal fractal manifold* (MFM) [13, 15-18].

On closer examination, this finding is hinted by a number of alternative theoretical arguments:

a) It is well known that the principle of *general covariance* lies at the core of classical relativistic field theory. An implicit assumption of general covariance is that any coordinate transformation and its inverse are *smooth* functions that can be differentiated arbitrarily many times. However, as it is also known, there is a plethora of non-differentiable curves and surfaces in Nature, as repeatedly discovered since the introduction of fractal geometry in 1983 [29, 31]. The unavoidable conclusion is that relativistic field theory assigns a preferential status to differentiable transformations and the smooth geometry of spacetime, which is at odds with the very spirit of general covariance.

b) On the mathematical front, significant effort was recently invested in the development of *q*-deformed Lie algebras, non-commutative field theory, quantum groups, fractional field theory and its relationship to the MFM [1, 5-6, 12, 27-28]. It is instructive to note that all these contributions appear to be directly or indirectly related to fractal geometry [13, 31]. Moreover, the condition ε << 1, defined within the framework of MFM, is the only sensible setting where fractal geometry asymptotically approaches all consistency requirements mandated by QFT and the Standard Model [16, 30].

c) Demanding that phenomena associated with gravitational collapse follow the postulates of quantum theory implies that the world is no longer four-dimensional near M_{Pl}. This statement has lately received considerable attention and forms the basis for *dimensional reduction* and for the *holographic principle* of Quantum Gravity theories [10, 25-28, 32-35]. If we accept that the four-dimensional continuum is an emergent property of the electroweak scale and below ($\mu < M_{EW}$), the holographic principle implies that spacetime dimensionality evolves with the energy scale between M_{EW}, where ε << 1, and M_{Pl}, where space is expected to become two-dimensional ($\varepsilon = O(1)$) (27-28, 34-35).

Our paper is organized as follows: next section introduces the concept of holographic bound and derives the relationship involving the IR and UV cutoffs of field theory. Building on these premises, section 3 presents a comparison between mass scales estimated using our approach and their currently known values.

2. The holographic bound

Consider an effective QFT confined to a spacetime region with characteristic length scale L and assume that the theory makes valid predictions up to an UV cutoff scale $\Lambda_{UV} \gg L^{-1}$. It can be shown that the entropy associated with this effective QFT takes the form [10]

$$S \sim \Lambda_{UV}^3 L^3 \tag{7}$$

To understand the significance of (7), consider an ensemble of fermions living on a periodic space lattice with characteristic size L and period Λ_{UV}^{-1}. One finds that (7) simply follows from counting the number of occupied states for this system, which turns out to be $N = 2^{(L\Lambda_{UV})^3}$ [10]. The holographic principle stipulates that (7) must not exceed the corresponding black hole entropy S_{BH}, that is,

$$L^3\Lambda_{UV}^3 \leq S_{BH} = \frac{A_{BH}}{4l_{Pl}^2} = \pi R^2 M_{Pl}^2 \tag{8}$$

in which A_{BH} is the area of the spherical event horizon of radius R. Introducing a new reference length scale Δ defined as

$$\Delta = \frac{L^3}{R^2} \tag{9}$$

leads to the condition

$$\Delta \leq \pi\Lambda_{UV}^{-3} M_{Pl}^2 \tag{10}$$

On the other hand, since the maximum energy density in a QFT bounded by the UV cutoff is Λ_{UV}^4, the holography bound (8) leads to [7-8]

$$\Lambda_{UV}^4 \sim \frac{(\pi^{-1}\Delta)\, M_{Pl}^2}{(\pi^{-1}\Delta)^3} = \pi^2 \frac{M_{Pl}^2}{\Delta^2} \Rightarrow \Lambda_{UV}^2 \sim \pi \frac{M_{Pl}}{\Delta} \tag{11}$$

Since the IR cutoff is fixed by $\Lambda_{IR} = \Delta^{-1}$, (11) yields the scaling behavior

$$\boxed{\frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi M_{Pl}}} \tag{12}$$

Although conventional wisdom suggests that the Standard Model retains its validity all the way up in the far UV sector of particle physics, there are indications that it may break at a scale that is at least an order of magnitude lower than M_{Pl}, that is, $\Lambda'_{UV} < M_{Pl}$ [see e.g 14]. Relation (12) may be conveniently reformulated at $\Lambda'_{UV} > \Lambda_{UV}$ as in

$$\frac{\Lambda_{UV}}{\pi M_{Pl}} = \frac{\Lambda_{UV}}{\pi\Lambda'_{UV}} \frac{\Lambda'_{UV}}{M_{Pl}} \tag{13}$$

such that

$$\frac{M_{Pl}}{\Lambda'_{UV}}\frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi\Lambda'_{UV}} \tag{14}$$

or

$$\boxed{\frac{\Lambda'_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi\Lambda'_{UV}}} \tag{15}$$

in which $\Lambda'_{IR} > \Lambda_{IR}$ is a new IR scale given by

$$\Lambda'_{IR} = \frac{M_{Pl}\Lambda_{IR}}{\Lambda'_{UV}} \tag{16}$$

A glance at (5b), (12) and (15) reveals deep similarities between the holographic principle and the minimal fractal manifold (MFM). They all represent scaling relations that mix and constrain largely separated mass scales. We next use (12) and (15) to derive numerical estimates and compare them with experimental data.

3. Numerical estimates

Tab. 1 displays currently known values for the representative scales of QFT and classical field theory. The electroweak scale (M_{EW}) is set by the vacuum expectation value of the Higgs boson, the far UV scale is set by either Planck mass (M_{Pl}) or the unification scale (M_{GUT}). The near UV cutoff is assumed to be close to the so-called Cohen-Kaplan threshold ($\Lambda_{CK} \sim 10^2$ TeV), according to [7-8, 19-21].

Scale	**Name**	**Magnitude**
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	Cosmological constant scale	$\leq \sim 10^{-3}$ eV
$\Lambda'_{IR} = \Lambda_{QCD}$	QCD scale	~ 200 MeV
$\Lambda_{UV} = M_{EW}$	EW scale	~ 246 GeV
$\Lambda'_{UV} = \Lambda_{CK}$	UV cutoff	~ 10^2 TeV
M_{GUT}	GUT scale	~ 10^{16} GeV
M_{Pl}	Planck scale	~ 10^{19} GeV

Tab. 1: The spectrum of mass scales in field theory

Tab. 2 shows numerical results. We find that:

a) the cosmological constant scale is consistent with its experimentally determined value and with the scale of neutrino masses [36].

b) the near IR scale is consistent with the QCD scale (Λ_{QCD}). This conclusion may shed light into the long-standing problem of the QCD mass gap as well as on the non-perturbative properties of strongly coupled gauge theory [9, 22, 38].

Mass scale	Estimated	Units	Comments
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.6\times10^{-6}$	eV	from M_{Pl}
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.9\times10^{-3}$	eV	from M_{GUT}
$\Lambda'_{IR} = \Lambda_{QCD}$	~ 193	MeV	from Λ_{CK}

Tab 2: Estimated values of the cosmological constant and QCD scales (assuming the electroweak scale at $M_{EW} \approx 246$ GeV and the Cohen-Kaplan cutoff at $\Lambda_{CK} \approx 10^2$ TeV)

The hierarchy of mass scales derived above can be conveniently summarized in the following diagram:

$$\Lambda_{cc}^{1/4}(\text{far IR Cutoff}) << \Lambda_{QCD}(\text{near IR cutoff}) < M_{EW} < \Lambda_{CK}(\text{near UV cutoff}) << M_{Pl}(\text{far UV cutoff})$$

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