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Bulk Viscous Cosmological Models Coupled with a Scalar Field in Einstein Theory of Gravitation

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Abstract

We have considered and investigated spatially homogeneous Bianchi types II, VIII and IX bulk viscous cosmological models coupled with zero-mass scalar field in Einstein's theory of gravitation. Some important features of the models, thus obtained, have been discussed. All the models are anisotropic, expanding, non-rotating and accelerating.

Keywords: Bianchi type metrics, Bulk viscosity, Zero-mass scalar field, Einstein's theory of gravitation.

1. Introduction

In order to study the evolution of the universe, many authors constructed cosmological models containing a viscous fluid. The presence of viscosity in the fluid introduces many interesting features in the dynamics of homogeneous cosmological models. The possibility of bulk viscosity leading to inflationary like solutions in general relativistic FRW models have been earlier discussed by several authors like Barrow, (1986), Padmanabhan and Chitre, (1987), Pavon et al. (1991), Lima et al. (1993), Martens (1995). Later Roy and Tiwari (1983), Mohanty and Pattanaik (1991), Mohanty and Pradhan (1992), Singh and Shri Ram (1996) and Singh (2005) have investigated cosmological models with bulk viscosity in general relativity. Also Bali and Dave (2002), Bali and Pradhan (2007), Tripathy et al. (2009, 2010) and Rao et al. (2011) have studied various Bianchi type string cosmological models in the presence of bulk viscosity in several theories of gravitation. Rao and Sireesha (2012) have studied Bianchi type - II, VIII and IX string cosmological models with bulk viscosity in Brans-Dicke theory of gravitation. Rao et al. (2012) have discussed Bianchi type - II, VIII and IX string cosmological models with bulk viscosity in a theory of gravitation.

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Rao and Neelima (2013) have studied Kantowski-Sachs string cosmological model with bulk viscosity in general scalar tensor theory of gravitation. Recently Rao et al. (2013, 2014) have investigated Bianchi type-I string cosmological model with bulk viscosity in bimetric theory of gravitation and modified theory of gravity proposed by Harko et al. (2011) respectively.

Bianchi type space-times play a vital role in understanding and description of the early stages of evolution of the universe. In particular, the study of Bianchi types II, VIII and IX universes is important because familiar solutions like FRW universes with positive curvature, the de-Sitter universe, the Taub-NUT solutions, and so forth correspond to these space-times. Rao et al. (2008a, b, c) have studied Bianchi types II, VIII and IX cosmological models in different theories of gravitation. Rao and Vijaya Santhi (2012) have obtained Bianchi types II, VIII and IX magnetized cosmological models in Brans-Dicke theory of gravitation. Rao and Sireesha (2012 a, b) have studied Bianchi types II, VIII and IX string cosmological models with bulk viscosity in some scalar tensor theories of gravitation. Recently Rao and Neelima (2013) have discussed string cosmological models with bulk viscosity in Nordtvedt general scalar tensor theory of gravitation.

The study of interacting fields, one of the fields being a zero-mass scalar field, is basically an attempt to look into the yet unsolved problem of the unification of the gravitational and quantum theories. Zero-mass scalar field has acquired particular importance since Weinberg (1978) and Wilczek (1978) proposed the existence of a low-mass (< 1 MeV) scalar boson, the so-called axion. Such particles will explain the absence of charge conjugation and party (CP) non conservation in strong interactions in particle physics as pointed out by Pecci and Quinn (1979). Any light particle has a potential for playing a major role in stellar energy loss; so there may exist a cosmic back ground of these particles. In the centre of the stars where the gravitational field is strong, a scalar field may have some effects on scalar configurations. Such as effect becomes important only when the general relativistic effect itself becomes important. It is, therefore, conceivable that stellar configuration will be appreciably affected by its own scalar field in the case of a neutron star and more particularly in the case of pulsars. Bramhachary (1960), Rao et al. (1972), Koijam Manihar Singh (1989), Reddy and Rao (1983), Krori et al. (1984), Reddy and Innaiah (1986), Singh and Deo (1986), Venkateswarlu and Reddy (1989) are some of the authors who have earlier investigated various aspects in the presence of zero-mass scalar fields. Rao and Sanyasi Raju (1992) have discussed the Bianchi type VIII and IX models in zero-mass scalar fields. Ibotombi Singh et al. (2009) have studied bulk viscous cosmological models with variable deceleration parameter in Lyra geometry. Rao et al. (2013) have studied Bianchi type-I cosmological viscous fluid universes coupled with zero-mass scalar field in Einstein's theory of gravitation. Recently Venkateswarlu and Satish (2014) and Venkateswarlu and Sreenivas (2014) have discussed Kantowski-Sachs bulk viscous string cosmological models in the presence of zero-mass scalar field and anisotropic Bianchi type-I & II bulk viscous string cosmological models coupled with zero-mass scalar field respectively.

In this paper, we will investigate spatially homogeneous Bianchi types II, VIII and IX bulk viscous cosmological models coupled with zero-mass scalar field in Einstein's theory of gravitation.

2. Metric and Energy Momentum Tensor

We consider spatially homogeneous Bianchi type - II, VIII & IX metrics in the form

$$ds^2 = dt^2 - R^2[d\theta^2 + f^2(\theta)d\varphi^2] - S^2[d\psi + h(\theta)d\varphi]^2 \tag{2.1}$$

where θ, φ *and* ψ are the Eulerian angles. Also R *and* S are functions of t only.
It represents:

Bianchi type-II if $f(\theta) = 1$ and $h(\theta) = \theta$

Bianchi type-VIII if $f(\theta) = Cosh\theta$ and $h(\theta) = Sinh\theta$

Bianchi type- IX if $f(\theta) = Sin\theta$ and $h(\theta) = Cos\theta$

The Einstein field equations corresponding to interacting zero-mass scalar field are given by

$$G_{ij} = -8\pi T_{ij} - (\phi_i\phi_j - \tfrac{1}{2} g_{ij}\phi_m\phi^m) \tag{2.2}$$

and

$$\phi^k_{;k} = 0 \tag{2.3}$$

where $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ is an Einstein tensor, R is the scalar curvature and T_{ij} is the stress energy tensor of the matter.

The energy momentum tensor is given by

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p} g_{ij} \tag{2.4}$$

and

$$\bar{p} = p - 3\xi H \tag{2.5}$$

where $p = \in_0 \rho \ (0 \le \in_0 \le 1)$. Here $\bar{p}$ is the total pressure which includes the proper pressure p, ρ is the rest energy density of the system, $\xi(t)$ is the coefficient of bulk viscosity, $3\xi H$ is usually known as bulk viscous pressure, H is the Hubble parameter and u_i is a four velocity vector.

The four velocity vector u^i and the space-like vector x^i, which represents the anisotropic directions of the string, will satisfy the equations

$$g_{ij}u^i u^j = 1, \quad g_{ij}x^i x^j = -1 \text{ and } u^i x_i = 0 \tag{2.6}$$

In a commoving coordinate system, we get

$$T_1^1 = -\bar{p} = T_2^2 = T_3^3 \quad \text{and} \quad T_4^4 = \rho \tag{2.7}$$

where ρ and $\bar{p}$ are functions of time t only.

3. Solutions of Field equations

Now with the help of (2.3) to (2.7), the field equations (2.2) for the metric (2.1) can be written as (using geometrized units with $c = 1$, $G = 1$)

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} = -8\pi\bar{p} - \frac{\dot{\phi}^2}{2} \tag{3.1}$$

$$2\frac{\ddot{R}}{R} + \frac{(\dot{R}^2 + \delta)}{R^2} - \frac{3S^2}{4R^4} = -8\pi\bar{p} - \frac{\dot{\phi}^2}{2} \tag{3.2}$$

$$2\frac{\dot{R}\dot{S}}{RS} + \frac{(\dot{R}^2 + \delta)}{R^2} - \frac{S^2}{4R^4} = 8\pi\rho + \frac{\dot{\phi}^2}{2} \tag{3.3}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = 0 \tag{3.4}$$

$$\dot{\rho} + (\rho + \bar{p})\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = 0 \tag{3.5}$$

Here the over head dot denotes differentiation with respect to t.

Equations (3.1) to (3.5) represent a system of four equations with five unknowns $R, S, \rho, \bar{p}$ & ϕ, which are functions of t. In order to get a deterministic solution we take the physical condition, that is, the expansion scalar θ is proportional to the shear scalar σ .This condition leads to

$$S = R^n \tag{3.6}$$

where n is an arbitrary constant.

From the equations (3.1) and (3.2), we have

$$\frac{\ddot{S}}{S} - \frac{\ddot{R}}{R} + \frac{\dot{R}\dot{S}}{RS} - \frac{\left(\dot{R}^2 + \delta\right)}{R^2} + \frac{S^2}{4R^4} = 0 \tag{3.7}$$

From the equations (3.6) and (3.7), we get

$$(n-1)\frac{\ddot{R}}{R} + \frac{n^2\dot{R}^2}{R^2} - \frac{\left(\dot{R}^2 + \delta\right)}{R^2} + R^{2n-4} = 0 \tag{3.8}$$

Bianchi type-II ($\delta = 0$) cosmological model:

If $\delta = 0$, the equation (3.8) becomes

$$\frac{\ddot{R}}{R} + (n+1)\frac{\dot{R}^2}{R^2} + \frac{R^{2n-4}}{(n-1)} = 0 \tag{3.9}$$

From equation (3.9) with a suitable substitution, we get

$$\frac{d}{dR}\left[f^2(R)\right] + 2(n+1)\frac{f^2(R)}{R} = \frac{2}{n-1}R^{2n-3} \tag{3.10}$$

which is a linear differential equation in f^2, where $\dot{R} = f(R)$ *and* $\ddot{R} = f\,\dot{f}$.

The general solution of equation (3.10) is

$$f^2 R^{2n+2} = \frac{1}{2n(1-n)}R^{4n} \tag{3.11}$$

The equation (3.11) is equal to

$$\frac{dR}{R^{n-1}} = \frac{1}{\sqrt{2n(1-n)}}dt .$$

Hence

$$R = \left(K_3\, t + K_4\right)^{\frac{1}{2-n}} \tag{3.12}$$

and

$$S = R^n = \left(K_3\, t + K_4\right)^{\frac{n}{2-n}} \tag{3.13}$$

where $K_3 = \dfrac{2-n}{\sqrt{2n(1-n)}}$, $K_4 = K_2(2-n)$ and $n \neq 0,1,2$.

From (3.4), (3.12) and (3.13), we get

$$\dot{\phi} = K_5 \left(K_3\, t + K_4\right)^{-\left(\frac{2+n}{2-n}\right)} \tag{3.14}$$

where K_5 is an integration constant and without loss of generality, let us take $K_5 = 1$.
From (3.14), we have

$$\phi = \frac{(n-2)}{2n}\left(K_3\, t + K_4\right)^{\frac{2n}{n-2}} \tag{3.15}$$

From equations (3.3), (3.12) - (3.15), we get
the energy density

$$8\pi\rho = K_6 \left(K_3\, t + K_4\right)^{-2} - \tfrac{1}{2}\left(K_3\, t + K_4\right)^{-\left(\frac{4+2n}{2-n}\right)} \tag{3.16}$$

where $K_6 = \dfrac{(n+1)(n+2)}{4n(1-n)}$.

From equations (3.1), (3.2), (3.12) - (3.14), we get the total pressure

$$8\pi\, \bar{p} = K_7 \left(K_3\, t + K_4\right)^{-2} - \tfrac{1}{2}\left(K_3\, t + K_4\right)^{-\left(\frac{4+2n}{2-n}\right)} \tag{3.17}$$

where $K_7 = \dfrac{4K_3^{\,2}\left(1-2n^2\right) - (2-n)^2}{4(2-n)^2}$.

The proper pressure is given by

$$8\pi\, p = 8\pi \in_0 \rho = K_6 \in_0 \left(K_3\, t + K_4\right)^{-2} - \tfrac{1}{2} \in_0 \left(K_3\, t + K_4\right)^{-\left(\frac{4+2n}{2-n}\right)} \tag{3.18}$$

The coefficient of bulk viscosity is given by

$$\xi = \frac{(2-n)}{(n+2)K_3}\left[\left(\frac{(1-\in_0)}{2}\left(K_3\, t + K_4\right)^{\frac{-(3n+2)}{2-n}} + \left(K_3\, t + K_4\right)^{-1}\left(K_6 \in_0 - K_7\right)\right] \tag{3.19}$$

The metric (2.1), in this case can be written as

$$ds^2 = dt^2 - (K_3 t + K_4)^{\frac{2}{2-n}}\left(d\theta^2 + d\varphi^2\right) - (K_3 t + K_4)^{\frac{2n}{2-n}}\left(d\psi + \theta\, d\varphi\right)^2 \tag{3.20}$$

Thus the metric (3.20) together with (3.15) to (3.19) constitutes a Bianchi type - II bulk viscous cosmological model coupled with a zero-mass scalar field in Einstein's theory of gravitation.

Bianchi type-VIII ($\delta = -1$) cosmological model:

If $\delta = -1$, the equation (3.8) becomes

$$\frac{\ddot{R}}{R} + (n+1)\frac{\dot{R}^2}{R} = \left(\frac{1}{1-n}\right)\left(\frac{1}{R} + R^{2n-3}\right) = 0 \tag{3.21}$$

We can solve the above equation only in case of $n = 2$.

So, from equation (3.21) with a suitable substitution, we get

$$\frac{d}{dR}\left(f^2(R)\right) + \frac{6}{R} f^2(R) = -\frac{2}{R} - 2R \tag{3.22}$$

which is a linear differential equation in f^2, where $\dot{R} = f(R)$ *and* $\ddot{R} = f\,\dot{f}$.

The general solution of (3.22) is

$$f^2 = -\frac{1}{3} - \frac{R^2}{4} \tag{3.23}$$

From equation (3.23), we get

$$R^2 = \frac{\omega^2}{\gamma^2}\sin^2(\gamma t) \tag{3.24}$$

where $\omega^2 = -\frac{1}{3}, \gamma^2 = \frac{1}{4}$.

From (3.24) and (3.6), we have

$$S^2 = \frac{\omega^4}{\gamma^4}\sin^4(\gamma t) \tag{3.25}$$

Using (3.24) and (3.25) in (3.4), we get

$$\dot{\phi} = \frac{K_1}{SR^2} = K_1\left(\frac{\gamma}{\omega}\right)^4 \operatorname{cosec}^4(\gamma t)$$

and

$$\phi = K_8 \cot(\gamma t)\left(1+\frac{\cot^2(\gamma t)}{3}\right), \tag{3.26}$$

where $K_8 = -\frac{K_1\gamma^3}{\omega^4}$.

From equations (3.4), (3.23) - (3.25), we get the energy density

$$8\pi\rho = -\frac{K_1{}^2}{2}\frac{\gamma^8}{\omega^8}\cos ec^8(\gamma t) + 5\gamma^2\cot^2(\gamma t) - \frac{\gamma^2}{\omega^2}\cos ec^2(\gamma t) - \frac{1}{4} \tag{3.27}$$

From equations (3.1), (3.2), (3.23) - (3.25), we getthe total pressure

$$8\pi\bar{p} = -\frac{K_1{}^2}{2}\frac{\gamma^8}{\omega^8}\cos ec^8(\gamma t) - \gamma^2\cot^2(\gamma t) + \frac{\gamma^2}{\omega^2}\cos ec^2(\gamma t) + \frac{3}{4} \tag{3.28}$$

The proper pressure is given by

$$8\pi p = 8\pi \in_0 \rho = -\frac{K_1{}^2}{2}\in_0\frac{\gamma^8}{\omega^8}\cos ec^8(\gamma t) + 5\gamma^2\in_0\cot^2(\gamma t) - \frac{\gamma^2}{\omega^2}\in_0\cos ec^2(\gamma t) - \frac{1}{4}\in_0 \tag{3.29}$$

The coefficient of bulk viscosity is given by

$$\xi = \frac{K_1{}^2}{8}(1-\in_0)\frac{\gamma^7}{\omega^8}\cos ec^7(\gamma t)\sec(\gamma t) + \frac{\gamma}{4}(1+5\in_0)\cot(\gamma t) - \frac{\gamma}{4\omega^2}(1+\in_0)\cos ec(\gamma t)\sec(\gamma t) - \frac{(3+\in_0)}{16\gamma\cot(\gamma t)} \tag{3.30}$$

The metric (2.1), in this case can be written as

$$ds^2 = dt^2 + \frac{4}{3}\sin^2\left(\frac{1}{2}t\right)\left(d\theta^2 + \cosh^2\theta\, d\varphi^2\right) - \frac{16}{9}\sin^4\left(\frac{1}{2}t\right)\left(d\psi + \sinh\theta\, d\varphi\right)^2 \tag{3.31}$$

Thus the metric (3.31) together with (3.26) to (3.30) constitutes a homogeneous and anisotropic Bianchi type-VIII bulk viscous cosmological model coupled with a zero-mass scalar field in Einstein's theory of gravitation.

Bianchi type-IX ($\delta = 1$) cosmological model:

If $\delta = 1$**,** equation (3.8) becomes

$$(n-1)\frac{\ddot{R}}{R} + \frac{n^2\dot{R}^2}{R^2} - \frac{(\dot{R}^2+1)}{R^2} + R^{2n-4} = 0, \tag{3.32}$$

We can solve the above equation only in case of $n = 2$.

So, from equation (3.32) with a suitable substitution, we get

$$\frac{d}{dR}\left(f^2(R)\right)+\frac{6}{R}f^2(R)=2\left(\frac{1}{R}-R\right) \tag{3.33}$$

which is a linear differential equation in f^2, where $\dot{R}=f(R)$ *and* $\ddot{R}=f\dot{f}$.

The general solution of (3.33) is

$$f^2=\frac{1}{3}-\frac{R^2}{4} \tag{3.34}$$

From equation (3.34), we get

$$R^2=\frac{\omega^2}{\gamma^2}\sin^2(\gamma t) \text{ , where } \omega^2=\frac{1}{3} \text{ and } \gamma^2=\frac{1}{4} \tag{3.35}$$

From equations (3.35) and (3.6), we have

$$S^2=\frac{\omega^4}{\gamma^4}\sin^4(\gamma t) \tag{3.36}$$

Using (3.35) and (3.36) in (3.4), we get

$$\dot{\phi}=\frac{K_1}{SR^2}=K_1\left(\frac{\gamma}{\omega}\right)^4\operatorname{cosec}^4(\gamma t).$$

Which on integration, we get

$$\phi=K_8\cot(\gamma t)\left(1+\frac{\cot^2(\gamma t)}{3}\right) \text{ , where } K_8=-\frac{K_1\gamma^3}{\omega^4} \tag{3.37}$$

From equations (3.4), (3.35) - (3.37), we get
the energy density

$$8\pi\rho=-\frac{K_1^{\ 2}}{2}\frac{\gamma^8}{\omega^8}\operatorname{cosec}^8(\gamma t)+5\gamma^2\cot^2(\gamma t)+3\gamma^2\operatorname{cosec}^2(\gamma t)-\frac{1}{4} \tag{3.38}$$

From equations (3.1), (3.2), (3.35) - (3.37), we get the total pressure

$$8\pi\bar{p}=-\frac{K_1^{\ 2}}{2}\frac{\gamma^8}{\omega^8}\operatorname{cosec}^8(\gamma t)-\gamma^2\cot^2(\gamma t)-\frac{\gamma^2}{\omega^2}\operatorname{cosec}^2(\gamma t)+\frac{3}{4} \tag{3.39}$$

The proper pressure is given by

$$8\pi\, p = 8\pi \in_0 \rho = -\frac{K_1^{\,2}}{2} \in_0 \frac{\gamma^8}{\omega^8} \operatorname{cosec}^8(\gamma t) + 5\gamma^2 \in_0 \cot^2(\gamma t) + 3\gamma^2 \in_0 \operatorname{cosec}^2(\gamma t) - \frac{1}{4} \in_0 \tag{3.40}$$

The coefficient of bulk viscosity is given by

$$\xi = \frac{K_1^{\,2}}{8}(1-\in_0)\frac{\gamma^7}{\omega^8}\operatorname{cosec}^7(\gamma t)\sec(\gamma t) + \frac{\gamma}{4}(1+5\in_0)\cot(\gamma t) - \frac{\gamma}{4}\left(\frac{1}{\omega^2}+3\in_0\right)\operatorname{cosec}(\gamma t)\sec(\gamma t) - \frac{(3+\in_0)}{16\gamma\cot(\gamma t)} \tag{3.41}$$

The metric (2.1), in this case can be written as

$$ds^2 = dt^2 - \frac{4}{3}\sin^2\left(\frac{1}{2}t\right)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) - \frac{16}{9}\sin^4\left(\frac{1}{2}t\right)\left(d\psi + \cos\theta\, d\varphi\right)^2 \tag{3.42}$$

Thus the metric (3.42) together with (3.37) to (3.41) constitutes a homogeneous and anisotropic Bianchi type-IX bulk viscous cosmological model coupled with a zero-mass scalar field in Einstein's theory of gravitation.

4. Some other important features of the models

Bianchi type-II cosmological model ($\delta = 0$):

The spatial volume for the model (3.20) is

$$V = (-g)^{1/2} = SR^2 = \left(K_3 t + K_4\right)^{\left(\frac{n+2}{2-n}\right)} \tag{4.1}$$

The expression for expansion scalar θ calculated for the flow vector u^i is given by

$$\theta = 3H = K_3\left(\frac{n+2}{2-n}\right)\left(K_3 t + K_4\right)^{-1} \tag{4.2}$$

and the shear σ is given by

$$\sigma^2 = \frac{7}{18}K_3^{\,2}\left(\frac{n+2}{2-n}\right)^2\left(K_3 t + K_4\right)^{-2} \tag{4.3}$$

The deceleration parameter q is given by

$$q=\frac{d}{dt}\left(\frac{1}{H}\right)-1=\frac{4(1-n)}{(n+2)}\ ,n\neq-2\,. \tag{4.4}$$

From equation (4.4) we can see that for $1<n<\infty$ & $-\infty<n<-2$, q is negative which may be attributed to the current accelerated expansion of the universe.

The generalized mean Hubble parameter (H) is

$$H=\frac{1}{3}\left(H_x+H_y+H_z\right)=K_3\left(\frac{n+2}{2-n}\right)\left(K_3\,t+K_4\right)^{-1} \tag{4.5}$$

The average anisotropy parameter is defined by

$$A_m=\frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2=\frac{2(n-1)^2}{(n+2)^2},\ \ n\neq-2, \tag{4.6}$$

where $\Delta H_i=H_i-H\ \ (i=1,2,3)$.

The overall density parameter Ω is given by

$$\Omega=\frac{3}{K_3^{\,2}}\left(\frac{n-1}{n+2}\right)^2\left[K_8-4\pi K_5^{\,2}\left(K_3t+K_4\right)^{\frac{4n}{n-2}}\right]\ . \tag{4.7}$$

Bianchi type-VIII ($\delta=-1$) & IX ($\delta=1$) cosmological models:

The spatial volume for both the models (3.31) & (3.42) is

$$V=[(\tfrac{w}{\gamma})\sin(\gamma\,t)]^4\,f(\theta) \tag{4.8}$$

where $f(\theta)=\cosh\theta$ & $\sin\theta$ for Bianchi type-VIII & IX respectively.

The expression for expansion scalar θ and the shear σ for the models (3.31) & (3.42) are given by

$$\theta=3H=4\gamma\cot(\gamma\,t) \tag{4.9}$$

$$\sigma^2=\frac{56}{9}\gamma^2\cot^2(\gamma\,t) \tag{4.10}$$

The deceleration parameter q for the models (3.31) & (3.42) is given by

$$q=\frac{d}{dt}\left(\frac{1}{H}\right)-1=\frac{3}{4}\sec^2(\gamma\,t)-1 \tag{4.11}$$

From equation (4.11), we can observe that the deceleration parameter q is always negative and hence they represent accelerating universe.

The components of the Hubble parameter H_x, H_y & H_z for the models (3.31) & (3.42) are given by

$$H_x = H_y = \frac{\dot{R}}{R} = \gamma \cot(\gamma t), H_z = \frac{\dot{S}}{S} = 2\gamma \cot(\gamma t)$$

Therefore the generalized mean Hubble parameter (H) is

$$H = \frac{1}{3}\left(H_x + H_y + H_z\right) = \frac{4}{3}\gamma \cot(\gamma t) \quad (4.12)$$

The average anisotropy parameter for the models (3.31) & (3.42) are given by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 = \frac{1}{8} \quad (4.13)$$

where $\Delta H_i = H_i - H \ (i = 1,2,3)$.

The overall density parameter Ω for the models (3.31) & (3.42) are given by

$$\Omega = \frac{15}{144\pi} + \frac{9}{144\pi}\sec^2(\gamma t) - \frac{243}{32}K_1^{\ 2}\gamma^6 \operatorname{cosec}^6(\gamma t)\sec^2(\gamma t) \quad (4.14)$$

The tensor of rotation

$w_{ij} = u_{i,j} - u_{j,i}$ is identically zero and hence these universes are non-rotational.

5. Discussion and Conclusions

In this paper we have presented spatially homogeneous Bianchi type –II, VIII & IX bulk viscous cosmological models coupled with a zero-mass scalar field in Einstein's theory of gravitation.
The following are the observations and conclusions:

1. For Bianchi type-II cosmological model, at $t = \frac{-k_4}{k_3}$, the spatial volume vanishes and increases continuously with time while all other parameters diverge for $0 < n < 2$. This shows that at the initial epoch the universe starts with zero volume and expands continuously approaching to infinite volume.
2. The model (3.20) has no initial singularity at $t = \frac{-k_4}{k_3}$ for $0 < n < 2$ and has cigar type singularity for $n < 0 \ \ and \ \ n > 2$.

3. From (3.16) & (3.18), we can see that matter pressure and density will vanish with the increase of cosmic time for $-2 < n < 2$.
4. For Bianchi type-II cosmological model, the deceleration parameter q is negative for $1 < (n \neq 2) < \infty$ & $-\infty < n < -2$, which may be attributed to the current accelerated expansion of the universe. Also the deceleration parameter q is always negative for Bianchi type-VIII & IX cosmological models and hence they too represent accelerating universes.
5. For Bianchi type-VIII & IX cosmological models, the spatial volume increase with time and also the models have no initial singularity at $t = 0$.
6. From (4.6) & (4.13), we can observe that $A_m \neq 0$, which indicates that these Bianchi type-II, VIII & IX models are always anisotropic.
7. The expansion scalar θ, shear scalar σ, the Hubble parameter $\overline{H}$ and the overall density parameter Ω vanish as $t \to \infty$.
8. The models obtained and presented here remain anisotropic throughout the evolution of the universe. We know that the present day universe accelerates and is better described by homogeneous isotropic space-times. However, experiments show that there is a certain amount of anisotropy in the universe. Hence anisotropic space-times are also important.
9. All the models are expanding, non-rotating and accelerating.
10. It is observed that bulk viscosity coupled with zero-mass scalar field will play a significant role in getting an accelerated universe.

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