Article

Plane Gravitational Waves in Generalized Peres Space-Time

S. W. Bhaware[*], S. R. Bhoyar [#], & A. G. Deshmukh[&]

[*]Department of Mathematics, Shri R.L.T.College of Science Akola- 444001, India
[#]Department of Mathematics, Phulsing Naik College, Pusad-445204
[&]Government Science College, Gadchiroli (M.S.), India.

Abstract

In this paper the generalized Peres space-time is considered for the (t/z)-type plane gravitational waves and the Mathematical solutions of generalized Maxwell's equations are obtained. The solutions resemble with those of Takeno (1961).

Keywords: general relativity, plane gravitational waves, Maxwell's electromagnetic theory, Peres spacetime.

1. Introduction

Takeno (1961)[1] exposed mathematically the plane gravitational waves and obtained numerous results thereof in general relativity. He considered (z-t) and (t/z)-type waves and obtained the plane wave solutions of field equations in G.R. Furthermore he considered the Peres space-time

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x,y,Z)(dz-dt)^2 \ , \qquad (Z = z - t) \qquad (1.1)$$

and obtained a necessary and sufficient condition that P be Minkowskian. Lal and Ali (1970)[3] considered the modified Peres space-time

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - (C-E)dz^2 - 2Edzdt + (C+E)dt^2 \qquad (1.2)$$

where A,B,C,D are the functions of

$$Z = Z(z-t) \text{ and } E = E(x,y,Z) \qquad (1.3)$$

and obtained the solutions of the field equations

$$G_{ij} = -8\pi E_{ij} \qquad (1.4)$$

[*] Correspondence Author: Suresh W. Bhaware, Department of Mathematics, Shri R.L.T.College of Science Akola, 444001, India.
Email: swbhaware@gmail.com

where G_{ij} is the Ricci tensor of space-time (1.2) and E_{ij} is the electromagnetic energy tensor defined by

$$E_{ij} = (1/4) g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl} \tag{1.5}$$

where F_{ij} is antisymmetric electromagnetic field tensor satisfying generalized Maxwell equations

$$\text{(a)} \quad F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \qquad \text{(b)} \; F^{ij}{}_{;j} = 0 \tag{1.6}$$

S.R.Bhoyar and A.G.Deshmukh (2012)[4] cosidered the generalized metric

$$ds^2 = -Adx^2 - Bdy^2 - Z^2(1-E)dz^2 - 2ZEdzdt + (1+E)dt^2 \tag{1.7}$$

where $A = A(Z),$ $B = B(Z),$ and $E = E(x, y, Z),$ and using $Z = t/z$ obtained the condition that the field equation (1.6) possess the solution. In the Present investigation we use the space-time (1.7) with the modification $A = A(x, Z),$ $B = B(y, Z),$ and $E = E(x, y, z, Z),$ and use the $Z = t/z$ -type wave and find the solution of the field equations (1.4).

2. The Curvature Tensor and the Ricci Tensor

According to Takeno's (1961) definition of plane gravitational waves, a plane wave g_{ij} is a non-flat solution of the field equations (1.4) with the property that, in some suitably chosen co-ordinate system,

$$g_{ij} = g_{ij}(Z), \quad Z = Z(x^i), \quad (x^i = x, y, z, t), \tag{2.1}$$

satisfying

$$g^{ij} Z_{,i} Z_{,j} = 0 \;, \tag{2.2}$$

and

$$Z = Z(z,t) \quad , (\; Z_{,3} \neq 0 \quad Z_{,4} \neq 0 \;). \tag{2.3}$$

Following the above definition of plane wave solutions, the fundamental tensors g_{ij} and g^{ij} for the space-time (1.7) are as follows:

$$(g_{ij}) = \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & Z^2(E-1) & -ZE \\ 0 & 0 & -ZE & 1+E \end{bmatrix},$$

$$(g_{ij}) = \begin{bmatrix} -\frac{B}{m} & 0 & 0 & 0 \\ 0 & -\frac{A}{m} & 0 & 0 \\ 0 & 0 & -\frac{1+E}{Z^2} & -\frac{E}{Z} \\ 0 & 0 & -\frac{E}{Z} & (1-E) \end{bmatrix}$$

and $g = -mZ^2$ where $m = AB$. (2.4)

The components of Christoffel symbols are as given below

$$\Gamma_{11}^{k} = \left[\frac{BA_x}{2m}, 0, \frac{\bar{A}}{2t}, \frac{\bar{A}}{2z}\right] \quad ,$$

$$\Gamma_{12}^{k} = [0,0,0,0] \quad ,$$

$$\Gamma_{13}^{k} = \left[\frac{-\bar{A}BZ}{2mz}, 0, \frac{-E_x}{2}, \frac{-ZE_x}{2}\right] \quad ,$$

$$\Gamma_{14}^{k} = \left[\frac{\bar{A}B}{2mz}, 0, \frac{E_x}{2Z}, \frac{E_x}{2}\right] \quad ,$$

$$\Gamma_{22}^{k} = \left[0, \frac{AB_y}{2m}, \frac{\bar{B}}{2t}, \frac{\bar{B}}{2z}\right] \quad ,$$

$$\Gamma_{23}^{k} = \left[0, \frac{-A\bar{B}Z}{2mz}, \frac{-E_y}{2}, \frac{-ZE_y}{2}\right] \quad ,$$

$$\Gamma_{24}^{k} = \left[0, \frac{A\bar{B}}{2mz}, \frac{E_y}{2Z}, \frac{E_y}{2}\right] \quad ,$$

$$\Gamma_{33}^{k} = \left[\frac{BZ^2E_x}{2m}, \frac{AZ^2E_y}{2m}, \frac{1}{z}(K-E-1), \frac{Z}{z}(K-2E+1)\right],$$

$$\Gamma^k_{34} = \left[\frac{-BZE_x}{2m}, \frac{-AZE_y}{2m}, \frac{1}{t}(1-K), \frac{1}{z}(E-K) \right],$$

$$\Gamma^k_{44} = \left[\frac{BE_x}{2m}, \frac{AE_y}{2m}, \frac{1}{Zt}(K+E), \frac{K}{t} \right], \quad (2.5)$$

where $$K = \left(E^2 + \frac{\overline{E}Z}{2} \right) \text{,and } m = AB \quad (2.6)$$

The components of covariant curvature tensor R_{hijk} and those of Ricci tensor R_{ij} are obtained as:

$$R_{1313} = -ZR_{1314} = Z^2 R_{1414} = \frac{Z^2\alpha}{2z^2},$$

$$R_{1323} = -ZR_{1324} = -ZR_{1423} = Z^2 R_{1424} = \frac{-Z^2 E_{xy}}{2},$$

$$R_{2323} = -ZR_{2324} = Z^2 R_{2424} = \frac{Z^2\beta}{2z^2},$$

$$R_{1334} = -ZR_{1434} = \frac{ZE_x}{z},$$

$$R_{2334} = -ZR_{2434} = \frac{ZE_y}{z}, \quad (2.7)$$

$$R_{13} = -ZR_{14} = \frac{E_x}{z}$$

$$R_{23} = -ZR_{24} = \frac{E_y}{z}$$

$$R_{33} = -ZR_{34} = Z^2 R_{44} = \frac{Z^2}{2z^2}\left(\frac{\alpha}{A} + \frac{\beta}{B}\right) \quad (2.8)$$

where, $\alpha = \left[\overline{\overline{A}} - z^2 E_{xx} - \frac{\overline{A}^2}{2A} + \frac{z^2 A_x E_x}{2A} + \frac{\overline{A}E}{Z} \right]$, and

$$\beta = \left[\overline{\overline{B}} - z^2 E_{yy} - \frac{\overline{B}^2}{2B} + \frac{z^2 B_y E_y}{2B} + \frac{\overline{B}E}{Z} \right]. \tag{2.9}$$

3. Electromagnetic Field Tensor

We take the components of electromagnetic potential

$$k_i = (-F, -G, -H, \phi) \tag{3.1}$$

where F, G, H, ϕ are functions of *(x,y,Z),* *Z= t/z* where ϕ is a scalar potential and (F, G, H) is a vector potential of electromagnetic field F_{ij}. The electromagnetic field F_{ij} is described by the components of electric force (X_1, Y_1, Z_1) and the components of magnetic force (μ, η, λ).

The usual relations connecting field with potential are

$$X_1 = -\frac{\partial \phi}{\partial x} - \frac{\partial F}{\partial t}, \quad Y_1 = -\frac{\partial \phi}{\partial y} - \frac{\partial G}{\partial t}, \quad Z_1 = -\frac{\partial \phi}{\partial z} - \frac{\partial H}{\partial t},$$

and

$$\mu = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad \eta = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad \lambda = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \tag{3.2}$$

which in tensor notations can be put as $F_{ij} = \frac{\partial k_i}{\partial x^j} - \frac{\partial k_j}{\partial x^i} = k_{i,j} - k_{j,i}$ (3.3)

From equations (3.2) and (3.3) we have

$$X_1 = F_{14} = -\frac{\partial F}{\partial t} - \frac{\partial \phi}{\partial x}, \quad Y_1 = F_{24} = -\frac{\partial G}{\partial t} - \frac{\partial \phi}{\partial y}, \quad Z_1 = F_{34} = -\frac{\partial H}{\partial t} - \frac{\partial \phi}{\partial z},$$

$$\mu = F_{23} = -\frac{\partial G}{\partial z} + \frac{\partial H}{\partial y}, \quad \eta = -F_{13} = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad \lambda = F_{12} = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \tag{3.4}$$

Since the electromagnetic waves are transverse in character hence $F_{12} = F_{34} = 0$. This imposes the condition on the components of k_i, namely

$$\text{(i)} \quad \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 0 \quad \text{and} \quad \overline{H} = Z\overline{\phi}. \tag{3.5}$$

From (3.5)(i)and (3.4) we get

$$\frac{\partial F_{13}}{\partial y}=\frac{\partial F_{23}}{\partial x}, \qquad \frac{\partial F_{14}}{\partial y}=\frac{\partial F_{24}}{\partial x} . \tag{3.6}$$

From (3.5)(ii)and (3.4) we get,

$$zF_{13}+tF_{14}=C_1, \qquad zF_{23}+tF_{24}=C_2 \tag{3.7}$$

and assuming $C_1 = C_2 = 0$ we get

$$-zF_{13}=tF_{14}=\sigma, \quad zF_{23}=tF_{24}=\rho \tag{3.8}$$

where σ and ρ are some functions of (x,y,Z) which reduce the equation (3.6) to

$$\frac{\partial \rho}{\partial x}+\frac{\partial \sigma}{\partial y}=0 \tag{3.9}$$

Hence the components of electromagnetic field tensor F_{ij} are given as

$$(F_{ij})=\begin{bmatrix} 0 & 0 & -\frac{\sigma}{z} & \frac{\sigma}{t} \\ 0 & 0 & \frac{\rho}{z} & \frac{-\rho}{t} \\ \frac{\sigma}{z} & -\frac{\rho}{z} & 0 & 0 \\ -\frac{\sigma}{t} & \frac{\rho}{t} & 0 & 0 \end{bmatrix} \tag{3.10}$$

The corresponding components of F^{ij} are

$$(F^{ij})=\begin{bmatrix} 0 & 0 & -\frac{B\sigma}{mtZ} & \frac{-B\sigma}{mt} \\ 0 & 0 & \frac{A\rho}{mtZ} & \frac{A\rho}{mt} \\ \frac{B\sigma}{mtz} & -\frac{A\rho}{mtZ} & 0 & 0 \\ \frac{B\sigma}{mt} & \frac{-A\rho}{mt} & 0 & 0 \end{bmatrix} \tag{3.11}$$

Equations (3.10), (3.11)and (1.5) give the values of non vanishing components of electromagnetic energy tensor as:

$$E_{33} = -ZE_{34} = Z^2 E_{44} = \frac{1}{z^2}\left(\frac{\sigma^2}{A} + \frac{\rho^2}{B}\right) \tag{3.12}$$

4. Solutions of the Field Equations

Substituting the values of R_{ij} and E_{ij} into equation (1.4) we get

$$\bar{\bar{m}} - \frac{\bar{m}^2}{2m} + \frac{\bar{m}E}{Z} - \bar{A}\bar{B} - z^2(AE_{yy} + BE_{xx}) + \frac{z^2}{2m}(B^2 A_x E_x + A^2 B_y E_y) = \frac{-16\pi}{z^2}(A\rho^2 + B\sigma^2) \tag{4.1}$$

The field equation (1.6) (a) is satisfied identically by the components of F_{ij}

while the equation (1.6) (b) is satisfied only if

$$A\frac{\partial\rho}{\partial y} - B\frac{\partial\sigma}{\partial x} = 0 \tag{4.2}$$

5. Conclusions

Thus, following Takeno's definition the plane wave solutions of the field equations (1.4)and that of the Maxwell's generalized equations (1.6) are composed of g_{ij} given by (2.4) and F_{ij} given by (3.10) under the condition (4.1) and (4.2).

References

[1]. H.Takeno: The mathematical theory of plane gravitational waves in general relativity, *Sci. Rep. Res. Inst. Theor. Phys, Hiroshima Univ.,**1** (1961).*
[2]. A.Peres : Some gravitational waves, *Phys. Rev.Let., 3(1959), 571-572.*
[3]. Lal and Ali : The wave solutions of the field equations of general relativiry in a Takeno generalized space-time, *Tensor,N.S.;V0l.21(1970) pp134-138*
[4]. S. R. Bhoyar and A. G. Deshmukh : The Z=(t/z)-type plane wave-like solutions of the field equations of general relativity in a generalized Peres space-time, *The Afri. Rev. Phy.(2012) pp225-228.*
[5]. J. L. Synge: Relativity, The special theory, *Amsterdam,(1956), Chap XI.*