Article

General Ideas about Octonions, Quaternions and Twistors

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Abstract

An updated view about $M^8 - H$ duality is discussed. $M^8 - H$ duality allows to deduce $M^4 \times CP_2$ via number theoretical compactification. One important correction is that octonionic spinor structure makes sense only for M^8 whereas for $M^4 \times CP_2$ complexified quaternions characterized the spinor structure.

Octonions, quaternions, quaternionic space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized.

There is a beautiful pattern present suggesting that $H = M^4 \times CP_2$ is completely unique on number theoretical grounds. Consider only the following facts. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. Octonionic projective space OP_2 appears as octonionic twistor space (there are no higher-dimensional octonionic projective spaces). Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing OP_2 to 12-D $G_2/U(1) \times U(1)$ having same dimension as the the twistor space $CP_3 \times SU(3)/U(1) \times U(1)$ of H assignable to complexified quaternionic representation of gamma matrices.

A further fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically six-sphere. Also the analogy of quaternionicity of preferred extremals in TGD with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program. Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

1 Introduction

Octonions, quaternions, quaternionic space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized. This summary involves several corrections to the picture which has been developing for a decade or so.

A brief updated view about $M^8 - H$ duality and twistorialization is in order. There is a beautiful pattern present suggesting that $M^8 - H$ duality makes sense and that $H = M^4 \times CP_2$ is completely unique on number theoretical grounds.

1. $M^8 - H$ duality allows to deduce $M^4 \times CP_2$ via number theoretical compactification. For the option with minimal number of conjectures the associativity/co-associativity of the space-time surfaces in M^8 guarantees that the space-time surfaces in M^8 define space-time surfaces in H. The

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tangent/normal spaces of quaternionic/hyper-quaternionic surfaces in M^8 contain also an integrable distribution of hyper-complex tangent planes $M^2(x)$.

An important correction is that associativity/co-associativity does not make sense at the level of H since the spinor structure of H is already complex quaternionic and reducible to the ordinary one by using matrix representations for quaternions. The associativity condition should however have some counterpart at level of H. One could require that the induced gamma matrices at each point could span a *real*-quaternionic sub-space of complexified quaternions for quaternionicity and a purely imaginery quaternionic sub-space for co-quaternionicity. One might hope that it is consistent with - or even better, implies - preferred extremal property. I have not however found a viable definition of quaternionic "reality". On the other hand, it is possible assigne the tangent space M^8 of H with octonion structure and define associativity as in case of M^8.

The delicacies coming from the signature of imbedding space metric are discussed and the conjecture that real-octonion-analyticity could define quaternionic surfaces in M^8 is considered as also the variant of this hypothesis for H.

2. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. M^8 allows twistor space for octonionic spinor structure obtained by direct generalization of the standard construction for M^4. $M^4 \times CP_2$ spinors can be regarded as tensor products of quaternionic spinors associated with M^4 and CP_2: this trivial observation forces to challenge the earlier rough vision, which however seems to stand up the challenge.

3. Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing 12-D $T(M^8) = G_2/U(1) \times U(1)$ to the 12-D twistor space $T(H) = CP_3 \times SU^3/U(1) \times U(1)$. The interpretation of the twistor space in the case of M^8 is as the space of choices of quantization axes for the 2-D Cartan algebra of G_2 acting as octonionic automorphisms. For CP_2 one has space for the chocies of quantization axes for the 2-D $SU(3)$ Cartan algebra.

4. It is also possible that the dualities extend to a sequence $M^8 \rightarrow H \rightarrow H...$ by mapping the associative/co-associative tangent space to CP_2 and M^4 point to M^4 point at each step. One has good reasons to expect that this iteration generates fractal as the limiting space-time surface.

5. A fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically 7-sphere. Second analogous structure is the 7-D Lie algebra like structure defined by octonionic analogs of sigma matrices.

The analogy of quaternionicity of M^8 pre-images of preferred extremals and quaternionicity of the tangent space of space-time surfaces in H with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity at the level of M^8 indeed could define basic dynamical principle of TGD.

In the following some general view about these topics distilled during years are summarized. The first section deals with $M^8 - H$ duality and second second with the various manners to define twistors. Third section is devoted to the recent view about number theoretic vision: the key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizes the ordinary notion of adele: this picture relates closely to Langlands probram. Associativity would force sub-algebras of octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP2$. This summary involves several corrections to the picture which has been developing for a decade or so.

2 Number theoretic compactification and $M^8 - H$ duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally $M^8 - H$ duality was introduced as a number theoretic explanation for $H = M^4 \times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4 \subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M_c^8 using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of $M^8 - H$ duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of $M^8 - H$ duality.

The strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M_c^8 composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M_c^8 identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M_c^8 must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere K̈hler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3) \times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4 \times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering $M^8 - H$ duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M_c^8 = O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j. If complexified quaternions are used for H, Minkowskian signature requires the introduction of two commuting imaginary units j and i meaning double complexification.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under

multiplication. One can however however define the notion of associativity for the sub-space of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q = q_0 + q^k I_k$. Hyper-quaternions are expressible as $q = q_0 + jq^k I_k$ and form a subspace of complexified quaternions $Q_c = Q \oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O \oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H = q_0 + jq^k I_k + jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4 \subset M^8$ implies unique decomposition $M^8 = M^4 \times E^4$ needed to define $M^8 - H$ duality uniquely. This applies also to M^8_c. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCWs associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M^8_c) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

   The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *imbedding space level* and *space-time level*. One must have imbedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \to H \to H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of modified gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the modified gamma matrices need not span the entire tangent space. The space spanned by the modified gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the modified gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

If the space spanned by modified gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For modified gamma matrices the notion of co-associativity can produce problems since modified gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by modified gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the modified gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the modified gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

2.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M_c^8. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M_c^8) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \to S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \to M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M^8_c: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M^8_c$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a manner to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [6]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [7] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of modified Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent plane makes it possible to define $M^8 - H$ duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

2.2 Hyper-octonionic Pauli "matrices" and the definition of associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [11]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define

associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \to H \to H....$ This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and modified gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

2.3 Are Kähler and spinor structures necessary in M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the $M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

2.3.1 Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

2.3.2 Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naive replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

2.3.3 Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1+1+3+\overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1 + kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2 + y^2 + z^2 + t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1 + ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

2.3.4 What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of imbedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

2.4 How could one solve associativity/co-associativity conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \to H \to H$... iteration generating new solutions from existing ones.

2.4.1 Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O \oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1+io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M^8_c but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M^8_c (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

2.4.2 Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a, b, c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a manner that one of the tangent vectors corresponds to real unit (in the imbedding map imbedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition

looks very simple - perhaps too simple!- since it involves only first derivatives of the imbedding space vectors.

One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of imbedding space coordinates (rather than involving imbedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$\begin{aligned} e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\ A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\ e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\ \Gamma_k &= e^A_k \gamma_A \ . \end{aligned} \tag{2.1}$$

The very naive idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{2.2}$$

One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see `http://en.wikipedia.org/wiki/Hyperdeterminant`). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD} x^B y^C z^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [2] expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product e_1e_2 (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x)e_1e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$e_0 = a \times 1 + b^i e_i$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

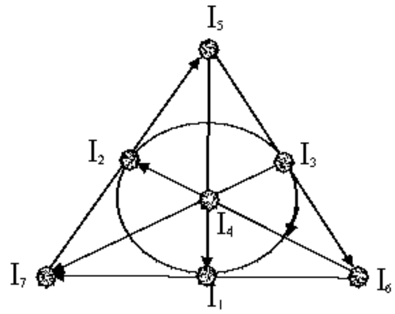

Figure 1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

2.5 Quaternionicity at the level of imbedding space quantum numbers

From the multiplication table of octonions as illustrated by Fano triangle [2] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2:s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

2.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

2.6.1 Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of M^8-H duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the modified gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k\gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the modified gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection modified gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2\times S^2 \subset M^4\times CP_2$. It seems that associativity is satisfied by all known extremals. Hence modified gamma matrices are flexible enough to realize associativity in H.

3. Modified gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4\times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner M^8-H duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the modified gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand M^8-H correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does M^8-H correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

2.6.2 Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \rightarrow k$ duality.

2.6.3 Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

2.6.4 $M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [9] .

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

2.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \to H \to H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

3 Octo-twistors and twistor space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor.

I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall $M^8 - H$ duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [10]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic imbedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

3.1 Two manners to twistorialize imbedding space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1) \times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably, M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-define operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4 \times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

   Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are "real" and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and imbedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

3.2 Octotwistorialization of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H.

3.3 Octonionicity, $SO(1,7)$, G_2, and non-associative Malcev group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1,7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \rightarrow G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (`http://math.ucr.edu/home/baez/octonions/node14.html`). The basic observation is that that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14+14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a, b] = ab - ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}[\gamma_k, \gamma_l] \quad . \tag{3.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy

Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

This algebra has decomposition $g = h + t$, $[h, t] \subset t$, $[t, t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1,7)$.

3.4 Octonionic spinors in M^8 and real complexified-quaternionic spinors in H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.
2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

   The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

   One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc) - (ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

3.5 What the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices could mean?

The basic implication of octonionization is the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4 \times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4 \times CP_2$ but interepret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

3.5.1 Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ \ , \ \ \gamma^i = \gamma^i \otimes \sigma_2 \ \ , \ \ i = 1,..,7 \ \ . \tag{3.2}$$

   7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1, ..., 6 \ \ , \ \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ \ . \tag{3.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \otimes \sigma_1 \ \ , \ \ \gamma_i = e_i \otimes \sigma_2 \ \ . \tag{3.4}$$

   where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = j e_i \times \sigma_3 \ \ , \ \ \Sigma_{ij} = j f_{ij} \ {}^k e_k \otimes 1 \ \ . \tag{3.5}$$

   Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [1] one finds $e_4 e_5 = e_1$ and $e_6 e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [5]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

3.5.2 Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the modified Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://www.tgdtheory.fi/public_html/articles/genesis.pdf`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of

hyper-complex numbers) such that the second complexified modified gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

Does this mean that the difference between quarks and leptons becomes visible only at the imbedding space level where ground states of super-conformal representations correspond to to imbedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

3.5.3 Octo-spinors and their relation to ordinary imbedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \\ \Psi_{q,i} &= e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \end{aligned} \tag{3.6}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3)\subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1\pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2\pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4\pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6\pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{3.7}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon=\pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon=1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon=-1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic modified Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2\subset M^4$.

4 Abelian class field theory and TGD

The context leading to the discovery of adeles (`http://en.wikipedia.org/wiki/Adele_ring`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \ mod \ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (`http://en.wikipedia.org/wiki/Class_field_theory`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

4.1 Adeles and ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://en.wikipedia.org/wiki/Class_field_theory`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence,[A1, 4, 3]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where imbedding space is replaced with Cartesian product of real imbedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see http://en.wikipedia.org/wiki/Adele_ring) is defined as $A_Z = R\times\hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q\otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R\times\prod'_p Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (http://en.wikipedia.org/wiki/Idele_class_groupIdele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

4.2 Questions about adeles, ideles and quantum TGD

The intriguing general result of class field theory (http://en.wikipedia.org/wiki/Class_field_theory) is that the the maximal Abelian extension for rationals is homomorphic with the multiplicative group of ideles. This correspondence plays a key role in Langlands correspondence.

Does this mean that it is not absolutely necessary to introduce p-adic numbers? This is actually not so. The Galois group of the maximal abelian extension is rather complex objects (absolute Galois group, AGG, defines as the Galois group of algebraic numbers is even more complex!). The ring $\hat{Z}$ of adeles defining the group of ideles as its invertible elements homeomorphic to the Galois group of maximal Abelian extension is profinite group (http://en.wikipedia.org/wiki/Profinite_group). This means that it is totally disconnected space as also p-adic integers and numbers are. What is intriguing that p-dic integers are however a continuous structure in the sense that differential calculus is possible. A concrete example is provided by 2-adic units consisting of bit sequences which can have literally infinite non-vanishing bits. This space is formally discrete but one can construct differential calculus since the situation is not democratic. The higher the pinary digit in the expansion is, the less significant it is, and p-adic norm approaching to zero expresses the reduction of the insignificance.

1. Could TGD based physics reduce to a representation theory for the Galois groups of quaternions and octonions?

Number theoretical vision about TGD raises questions about whether adeles and ideles could be helpful in the formulation of TGD. I have already earlier considered the idea that quantum TGD could reduce to a representation theory of appropriate Galois groups. I proceed to make questions.

1. Could real physics and various p-adic physics on one hand, and number theoretic physics based on maximal Abelian extension of rational octonions and quaternions on one hand, define equivalent formulations of physics?

2. Besides various p-adic physics all classical number fields (reals, complex numbers, quaternions, and octonions) are central in the number theoretical vision about TGD. The technical problem is that p-adic quaternions and octonions exist only as a ring unless one poses some additional conditions. Is it possible to pose such conditions so that one could define what might be called quaternionic and octonionic adeles and ideles?

   It will be found that this is the case: p-adic quaternions/octonions would be products of rational quaternions/octonions with a p-adic unit. This definition applies also to algebraic extensions of rationals and makes it possible to define the notion of derivative for corresponding adeles. Furthermore, the rational quaternions define non-commutative automorphisms of quaternions and rational octonions at least formally define a non-associative analog of group of octonionic automorphisms [12].

3. I have already earlier considered the idea about Galois group as the ultimate symmetry group of physics. The representations of Galois group of maximal Abelian extension (or even that for algebraic numbers) would define the quantum states. The representation space could be group algebra of the Galois group and in Abelian case equivalently the group algebra of ideles or adeles. One would have wave functions in the space of ideles.

   The Galois group of maximal Abelian extension would be the Cartan subgroup of the absolute Galois group of algebraic numbers associated with given extension of rationals and it would be natural to classify the quantum states by the corresponding quantum numbers (number theoretic observables).

   If octonionic and quaternionic (associative) adeles make sense, the associativity condition would reduce the analogs of wave functions to those at 4-dimensional associative sub-manifolds of octonionic adeles identifiable as space-time surfaces so that also space-time physics in various number fields would result as representations of Galois group in the maximal Abelian Galois group of rational octonions/quaternions. TGD would reduce to classical number theory! One can hope that WCW spinor fields assignable to the associative and co-associative space-time surfaces provide the adelic representations for super-conformal algebras replacing symmetries for point like objects.

   This of course involves huge challenges: one should find an adelic formulation for WCWin terms octonionic and quaternionic adeles, similar formulation for WCW spinor fields in terms of adelic induced spinor fields or their octonionic variants is needed. Also zero energy ontology, causal diamonds, light-like 3-surfaces at which the signature of the induced metric changes, space-like 3-surfaces and partonic 2-surfaces at the boundaries of CDs, M^8-H duality, possible representation of space-time surfaces in terms of of O_c-real analytic functions (O_c denotes for complexified octonions), etc. should be generalized to adelic framework.

4. Absolute Galois group is the Galois group of the maximal algebraic extension and as such a poorly defined concept. One can however consider the hierarchy of all finite-dimensional algebraic extensions (including non-Abelian ones) and maximal Abelian extensions associated with these and obtain in this manner a hierarchy of physics defined as representations of these Galois groups homomorphic with the corresponding idele groups.

5. In this approach the symmetries of the theory would have automatically adelic representations and one might hope about connection with Langlands program [8],[A1, 4, 3].

2. Adelic variant of space-time dynamics and spinorial dynamics?

As an innocent novice I can continue to pose stupid questions. Now about adelic variant of the space-time dynamics based on the generalization of Kähler action discussed already earlier but without mentioning adeles ([13]).

1. Could one think that adeles or ideles could extend reals in the formulation of the theory: note that reals are included as Cartesian factor to adeles. Could one speak about adelic space-time surfaces endowed with adelic coordinates? Could one formulate variational principle in terms of adeles so that exponent of action would be product of actions exponents associated with various factors with Neper number replaced by p for Z_p. The minimal interpretation would be that in adelic picture one collects under the same umbrella real physics and various p-adic physics.

2. Number theoretic vision suggests that 4:th/8:th Cartesian powers of adeles have interpretation as adelic variants of quaternions/ octonions. If so, one can ask whether adelic quaternions and octonions could have some number theoretical meaning. Adelic quaternions and octonions are not number fields without additional assumptions since the moduli squared for a p-adic analog of quaternion and octonion can vanish so that the inverse fails to exist at the light-cone boundary which is 17-dimensional for complexified octonions and 7-dimensional for complexified quaternions. The reason is that norm squared is difference $N(o_1) - N(o_2)$ for $o_1 \oplus io_2$. This allows to define differential calculus for Taylor series and one can consider even rational functions. Hence the restriction is not fatal.

   If one can pose a condition guaranteeing the existence of inverse for octonionic adel, one could define the multiplicative group of ideles for quaternions. For octonions one would obtain non-associative analog of the multiplicative group. If this kind of structures exist then four-dimensional associative/co-associative sub-manifolds in the space of non-associative ideles define associative/co-associative adeles in which ideles act. It is easy to find that octonionic ideles form 1-dimensional objects so that one must accept octonions with arbitrary real or p-adic components.

3. What about equations for space-time surfaces. Do field equations reduce to separate field equations for each factor? Can one pose as an additional condition the constraint that p-adic surfaces provide in some sense cognitive representations of real space-time surfaces: this idea is formulated more precisely in terms of p-adic manifold concept [13]. Or is this correspondence an outcome of evolution?

   Physical intuition would suggest that in most p-adic factors space-time surface corresponds to a point, or at least to a vacuum extremal. One can consider also the possibility that same algebraic equation describes the surface in various factors of the adele. Could this hold true in the intersection of real and p-adic worlds for which rationals appear in the polynomials defining the preferred extremals.

4. To define field equations one must have the notion of derivative. Derivative is an operation involving division and can be tricky since adeles are not number field. The above argument suggests this is not actually a problem. Of course, if one can guarantee that the p-adic variants of octonions and quaternions are number fields, there are good hopes about well-defined derivative. Derivative as limiting value $df/dx = \lim (f(x+dx) - f(x))/dx$ for a function decomposing to Cartesian product of real function $f(x)$ and p-adic valued functions $f_p(x_p)$ would require that $f_p(x)$ is non-constant only for a finite number of primes: this is in accordance with the physical picture that only finite number of p-adic primes are active and define "cognitive representations" of real space-time surface. The second condition is that dx is proportional to product $dx \times \prod dx_p$ of differentials dx and dx_p, which are rational numbers. dx goes to xero as a real number but not p-adically for any of the primes involved. dx_p in turn goes to zero p-adically only for Q_p.

5. The idea about rationals as points common to all number fields is central in number theoretical vision. This vision is realized for adeles in the minimal sense that the action of rationals is well-defined in all Cartesian factors of the adeles. Number theoretical vision allows also to talk about

common rational points of real and various p-adic space-time surfaces in preferred coordinate choices made possible by symmetries of the imbedding space, and one ends up to the vision about life as something residing in the intersection of real and p-adic number fields. It is not clear whether and how adeles could allow to formulate this idea.

6. For adelic variants of imbedding space spinors Cartesian product of real and p-adc variants of imbedding spaces is mapped to their tensor product. This gives justification for the physical vision that various p-adic physics appear as tensor factors. Does this mean that the generalized induced spinors are infinite tensor products of real and various p-adic spinors and Clifford algebra generated by induced gamma matrices is obtained by tensor product construction? Does the generalization of massless Dirac equation reduce to a sum of d'Alembertians for the factors? Does each of them annihilate the appropriate spinor? If only finite number of Cartesian factors corresponds to a space-time surface which is not vacuum extremal vanishing induced Kähler form, Kähler Dirac equation is non-trivial only in finite number of adelic factors.

3. Objections leading to the identification of octonionic adeles and ideles

The basic idea is that appropriately defined invertible quaternionic/octonionic adeles can be regarded as elements of Galois group assignable to quaternions/octonions. The best manner to proceed is to invent objections against this idea.

1. The first objection is that p-adic quaternions and octonions do not make sense since p-adic variants of quaternions and octonions do not exist in general. The reason is that the p-adic norm squared $\sum x_i^2$ for p-adic variant of quaternion, octonion, or even complex number can vanish so that its inverse does not exist.

2. Second objection is that automorphisms of the ring of quaternions (octonions) in the maximal Abelian extension are products of transformations of the subgroup of $SO(3)$ (G_2) represented by matrices with elements in the extension and in the Galois group of the extension itself. Ideles separate out as 1-dimensional Cartesian factor from this group so that one does not obtain 4-field (8-fold) Cartesian power of this Galois group.

One can define quaternionic/octonionic ideles in terms of rational quaternions/octonions multiplied by p-adic number. For adeles this condition produces non-sensical results.

1. This condition indeed allows to construct the inverse of p-adic quaternion/octonion as a product of inverses for rational quaternion/octonion and p-adic number. The reason is that the solutions to $\sum x_i^2 = 0$ involve always p-adic numbers with an infinite number of pinary digits - at least one and the identification excludes this possibility. The ideles form also a group as required.

2. One can interpret also the quaternionicity/octonionicity in terms of Galois group. The 7-dimensional non-associative counterparts for octonionic automorphisms act as transformations $x \rightarrow gxg^{-1}$. Therefore octonions represent this group like structure and the p-adic octonions would have interpretation as combination of octonionic automorphisms with those of rationals.

3. One cannot assign to ideles 4-D idelic surfaces. The reason is that the non-constant part of all 8-coordinates is proportional to the same p-adic valued function of space-time point so that space-time surface would be a disjoint union of effectively 1-dimensional structures labelled by a subset of rational points of M^8. Induced metric would be 1-dimensional and induced Kähler and spinor curvature would vanish identically.

4. One must allow p-adic octonions to have arbitrary p-adic components. The action of ideles representing Galois group on these surfaces is well-defined. Number field property is lost but this feature

comes in play as poles only when one considers rational functions. Already the Minkowskian signature forces to consider complexified octonions and quaternions leading to the loss of field property. It would not be surprising if p-adic poles would be associated with the light-like orbits of partonic 2-surfaces. Both p-adic and Minkowskian poles might therefore be highly relevant physically and analogous to the poles of ordinary analytic functions. For instance, n-point functions could have poles at the light-like boundaries of causal diamonds and at light-like partonic orbits and explain their special physical role.

The action of ideles in the quaternionic tangent space of space-time surface would be analogous to the action of of adelic linear group $Gl_n(A)$ in n-dimensional space.

5. Adelic variants of octonions would be Cartesian products of ordinary and various p-adic octonions and would define a ring. Quaternionic 4-surfaces would define associative local sub-rings of octonion-adelic ring.

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