Article

FRW Cosmology for Decay Law with Time-varying G, Λ & q

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Abstract

In this paper, we study the Einstein's field equations with time-dependant gravitational and cosmological constants in homogeneous and isotropic flat FRW metric containing the matter of the form perfect fluid which conserves the matter tensor. Solutions of the Einstein field equations are obtained by considering the time varying deceleration parameter q. For this we study the FRW mode with G and Λ by using decay laws $G\rho \propto \frac{\Lambda}{8\pi}$ and $\Lambda \propto H^2$, and consider the γ - fluid equation of state. Numerical studies of the evolution of the scale-factor have tended to focus on the radiation like case $(\gamma = \frac{4}{3})$, the dust-like case $(\gamma = 1)$ and stiff matter $(\gamma = 2)$. Some physical and kinematical parameters are studied.

Keywords: Cosmology, deceleration parameter, Gravitational Constant, perfect fluid.

Introduction

Einstein's theory of gravitation describes how space and time are affected by the gravitational field of matter. The theory predicts that gravitational field change the geometry of space and time causing it to become a curved. Einstein's field equations relate the distribution of matter and energy with the curvature of space-time. He proposed the static universe solution by introducing a cosmological constant term Λ. The presence of cosmological constant Λ in a given cosmology prolongs the age of the universe.

One of the motivations for introducing Λ term is to reconcile the age of parameter and density parameter of the universe with current observational data. In general relativity the cosmological constant Λ may be regarded as the measure of energy density of the vacuum and can lead to reduce the big-bang singularity, which is a characteristic of Friedmann-Robertson-Walker (FRW) models. Albert Einstein added this cosmological constant term to his field equations [1],

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which describe general relativity, to balance out the contraction of gravity. This term allowed the field equations to describe the Universe as being in a stationary state.

Thirteen years later, observations from Edwin Hubble revealed that the Universe was not static, but was expanding [2]. After this discovery, Einstein realized that his original theory was correct and there was no need for this additional term, and therefore abandoned the concept of the cosmological constant. Similarly the Newtonian constant of gravitation G plays the role of a coupling constant between the geometry and matter in the Einstein field equations. Dirac [3] proposed a theory with variable G as a function of time on the basis of large number of hypothesis. Many other extensions of Einstein theory with variable G have been proposed in order to achieve a possible unification of gravitation and elementary particles physics or incorporate Mach's principle in general relativity [4-9]. The FRW cosmological model variable G and Λ have been studied by various authors [10-18].

Some of the recent study on the cosmological constant and its consequence on cosmology with a time-dependent cosmological constant are investigated by Carrol *et.al*. [19]. Berman and Som [20] and Abdel-Rahman [21] have discussed the relation $\Lambda = Bt^{-2}$ which plays an central role in modern cosmology. Chen and Wu [22] studied the relation of the form $\Lambda \propto R^{-2}$ where R is the scale factor in FRW metric. Berman [23, 24] discussed the time varying cosmological term Λ with G as constant or variable using the law of variation $\Lambda \propto t^{-2}$ rather than the law recently suggested by Chen and Wu [22]. The authors [25-35] have constructed homogeneous isotropic cosmological models with variable cosmological constant and gravitational constant satisfying the present day observational data.

However, all vacuum decaying cosmological models do not predict acceleration. Some of the authors [36-38] have proposed a cosmological model with a cosmological constant of form $\Lambda = B\frac{\ddot{R}}{R}$, where B is a constant. Following the same decay law, recently [39, 40 has investigated cosmic acceleration with positive cosmological constant and suggested for introducing Λ term to reconcile the age parameter and the density parameter of the universe with recent observational data. Khadekar *et.al*. in [41] obtained a cosmological model with cosmological term of the form $\Lambda \propto \frac{\ddot{R}}{R}$. Singh [42] discussed a model with variables Λ and G while the conservation law for the energy-momentum tensor is still valid with the assumptions that $\Lambda \propto R^{-2}$and $G\rho \propto \frac{\Lambda}{8\pi}$.

In this paper, motivated by above mentioned studied, Einstein's field equations with time-dependant gravitational and cosmological constants are considered in homogeneous and isotropic flat FRW metric in presence of perfect fluid matter which conserves the matter tensor. Solutions of the Einstein field equations are obtained by considering the time varying deceleration parameter q. For this we study the FRW mode with G and Λ by using decay laws $G\rho \propto \frac{\Lambda}{8\pi}$ and $\Lambda \propto H^2$, and consider the γ - law equation of state. Numerical studies of the evolution of the scale-factor have tended to focus on the radiation like case $(\gamma = \frac{4}{3})$, the dust-like case $(\gamma = 1)$ and stiff matter ($\gamma = 2$). Some physical and kinematical parameters are studied.

The Field Equations

We considered the homogeneous and isotropic flat FRW metric in the form

$$ds^2 = dt^2 - a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{1}$$

where $a(t)$ is the scale factor of the spatial expansion.

We take the energy momentum tensor of a perfect fluid in the form as

$$T_j^i = (\rho + P)u^i u_j - P g_j^i \tag{2}$$

together with the relation $u^i u_j = 1$.

Here P is the pressure in the fluid and ρ is the energy density of the fluid and u^i is the four velocity vector defined by $u^i = \delta_4^i$, where i = 1, 2, 3, 4. We assume the co-ordinate to be co-moving so that $u^i = (0, 0, 0, 1)$.

The FRW model is often used as a realistic description of the evolution of the universe in cosmology. In this model the matter distribution is spatially homogeneous and isotropic and has the perfect fluid form (2).

We take the γ - fluid equation of state

$$P = (\gamma - 1)\rho, \quad \text{with } 1 \le \gamma \le 2, \text{ where } \gamma \text{ is constant.} \tag{3}$$

The Einstein's field equations with time-dependant gravitational and cosmological constants are given by [43]

$$R_j^i - \frac{1}{2} R g_j^i = -8\pi G(t) T_j^i - \Lambda(t) g_j^i \,. \tag{4}$$

The field equations (4) foe the metric (1) with equation (2) lead to the following set of equations:

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = -8\pi G P + \Lambda \tag{5}$$

and $$3(\frac{\dot{a}}{a})^2 = 8\pi G\rho + \Lambda \,. \tag{6}$$

where the dots denote the differentiation with respect to cosmic time t.

In view of vanishing of divergence of Einstein tensor, we have

$$\left(R_j^i - \frac{1}{2} R g_j^i\right)_{;i} = 0 \quad \text{lead to} \quad \left(8\pi G(t) T_j^i + \Lambda(t) g_j^i\right)_{;i} = 0 \,.$$

This implies that

$$8\pi G\left[\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a}\right] + 8\pi \dot{G}\rho + \dot{\Lambda} = 0. \tag{7}$$

We assume that the law of conservation law ($T_{j;i}^i = 0$) gives

$$\left[\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a}\right] = 0. \tag{8}$$

From equations (7) and (8), we find

$$8\pi \dot{G}\rho + \dot{\Lambda} = 0. \tag{9}$$

This implying that Λ is a constant whenever G is constant.

From equation (5), we write

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{3} a^2. \tag{10}$$

Equations (3), (5) and (11) yield

$$\ddot{a} = \frac{8\pi G}{3}\left(1 - \frac{3}{2}\gamma\right) a + \frac{\Lambda}{3} a \,. \tag{11}$$

Equation (11) shows that a positive density ρ acts to decelerate the expansion of the universe when $\gamma > \frac{2}{3}$ and, on the other hand, when $\gamma < \frac{2}{3}$, then $\ddot{a} > 0$ this means that matter density actually accelerates the expansion of the universe. This phenomenon is commonly known as inflation.

Eliminating ρ from equations (10) and (11), we obtain a differential equation for the scale factor in terms of the cosmological term Λ alone as

$$\frac{\ddot{a}}{a} = \left(1 - \frac{3}{2}\gamma\right)\left(\frac{\dot{a}}{a}\right)^2 + \frac{\gamma}{2}\Lambda\,. \tag{12}$$

FRW Cosmological Model

The system of equations (5) and (6) supply only two equations in five unknowns (a, P, ρ, G and Λ). Three extra equations are needed to solve the system completely. For this purpose, we take a cosmological term is proportional to the square of Hubble parameter. Recently Borges and Carnerio [44] have considered a cosmological term proportional to H. The variation law for vacuum density has initially proposed by Schützhold [45, 46]. We assume the Decay Laws containing the relations of $G\rho$ and Λ as

$$G\rho \propto \frac{\Lambda}{8\pi} \text{ and } \Lambda \propto H^2, \tag{13}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter which measures the rate of expansion of the universe.

From equations (8), (9) and (13), we find

$$\frac{\dot{\rho}}{\rho} + 3\gamma\left(\frac{\dot{a}}{a}\right) = 0, \tag{14}$$

$$\frac{\dot{G}}{G} - \frac{2}{ka_0}\left(\frac{\dot{a}}{a}\right) = 0. \tag{15}$$

and

$$\frac{\dot{\Lambda}}{\Lambda} + \frac{2}{a_0}\left(\frac{\dot{a}}{a}\right) = 0. \tag{16}$$

Integrating equations (14- 16), we obtain

$$\rho = \rho_0 a^{-3\gamma}, \tag{17}$$

$$G = G_0 a^{\frac{2}{ka_0}} \tag{18}$$

and

$$\Lambda = \Lambda_0 a^{-\frac{2}{a_0}}\,. \tag{19}$$

where ρ_0, a_0 and Λ_0 are constants of integration.

We define the deceleration parameter $q(t)$ as

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2} = \left[\frac{d}{dt}\left(\frac{1}{H}\right) - 1\right]. \tag{20}$$

This gives

$$\frac{\ddot{a}}{a} + q(t)\left(\frac{\dot{a}}{a}\right) = 0. \tag{21}$$

The general solution of (21) yield

$$\int f(a)da = t + t_0 \ . \tag{22}$$

where t_0 is the constant of integration and

$$lnf(a) = \int \frac{b}{a} da \ ,$$

where b is a variable, that is, $b = b(t) = b(a(t))$.

The choice of $f(a)$ is arbitrary. So we choose $f(a) = [a(t)]^{m-1}$, where *m* is a positive arbitrary constant. We consider the following cases:

Case I: If $m \neq 0$, then integrating (22) with the value of $f(a)$, after simplification, we find the scale factor as

$$a(t) = (\alpha t + \beta)^{\frac{1}{m}} \ . \tag{23}$$

where α and β are constants.

Thus the geometry of FRW metric with time-dependant gravitational and cosmological constants for fluid matter is as follows:

$$ds^2 = dt^2 - (\alpha t + \beta)^{\frac{2}{m}}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{24}$$

The model (24) represents the FRW cosmological model with time-dependant gravitational and cosmological constants containing the matter in the form of perfect fluid. The FRW model has no initial singularity.

The physical and kinematical parameters of the FRW model (24) are given by the following expressions:

Spatial Volume $V = a^3 = (\alpha t + \beta)^{\frac{3}{m}}$, Energy density of the cosmic matter $\rho = \rho_0(\alpha t + \beta)^{-\frac{3\gamma}{m}}$,

Pressure $P = \rho_0(\gamma - 1)(\alpha t + \beta)^{-\frac{3\gamma}{m}}$,

The Hubble parameter $H = \dfrac{\alpha}{m(\alpha t+\beta)}$,

The expansion scalar $\theta = \dfrac{3\alpha}{m(\alpha t+\beta)}$,

The cosmological constant $\Lambda(t) = \Lambda_0(\alpha t + \beta)^{-\frac{2}{ma_0}}$

and the gravitational constant $G(t) = G_0(\alpha t + \beta)^{\frac{2}{mka_0}}$, where $k,\ a_0 > 0$.

Case II: If $m = 0$, then integrating (22) with the value of $f(a)$, after simplification, we find the scale factor as

$$a(t) = Ae^t \,, \tag{25}$$

where A is constants.

Thus the geometry of FRW metric with time-dependant gravitational and cosmological constants for fluid matter is as follows:

$$ds^2 = dt^2 - A^2e^{2t}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{26}$$

The model (26) shows the FRW cosmological model in time-dependant gravitational and cosmological constants containing the matter in the form of perfect fluid. The FRW model has no initial singularity.

The physical and kinematical parameters of the FRW model (26) are given by the following expressions:

Spatial Volume $V = a^3 = A^3e^{3t}$,

Energy density of the cosmic matter $\rho = Be^{-3\gamma t}$,

Pressure $P = B(\gamma - 1)e^{-3\gamma t}$,

The Hubble parameter $H = 1$,

The expansion scalar $\theta = 3H = 3$,

The cosmological constant $\Lambda(t) = Ce^{-\frac{2t}{a_0}}$

and the gravitational constant $G(t) = De^{\frac{2t}{ka_0}}$, where $k,\ a_0 > 0$, $B = \rho_0 A^{-3\gamma}$, $C = \Lambda_0 A^{-\frac{2}{a_0}}$ and $D = G_0 A^{\frac{2}{ka_0}}$ are arbitrary constants.

Discussions

For case-I : We observed that at $t = 0$, all the physical and kinematical parameters like spatial volume, Hubble parameter, expansion scalar, energy density, pressure, cosmological constant and gravitational constant are constant. This shows that universe starts evolving with constant volume with an finite rate of expansion. Hence the model (24) has a point type singularity at initial epoch. The spatial volume increases with increase of time but the expansion scalar decreases with increase of time. Thus the rate of expansion slow down with increase of time and then identically zero at $t \to \infty$. At $t \to \infty$, the Hubble parameter become zero and hence the universe is static. At $t \to \infty$, an energy density and pressure decreases.

It is also seen that the gravitational constant is infinitely large at $t \to \infty$ but cosmological constant Λ is decreasing. This has been investigated by Abdel-Rahman [5], Chow [47], Levitt [48] and Milne [49]. This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe. A decreasing functional form permits Λ to be large in the early universe. For radiation like case ($\gamma = \frac{4}{3}$), the energy density of the matter is $\rho = \rho_0(\alpha t + \beta)^{-\frac{4}{m}}$, and its pressure is $P = \frac{1}{3}\rho$. For dust like case ($\gamma = 1$), the energy density of the cosmic matter $\rho = \rho_0(\alpha t + \beta)^{-\frac{3}{m}}$ and its pressure become zero. For stiff matter ($\gamma = 2$), the energy density of the matter is equal to its pressure, that is, $P = \rho = \rho_0(\alpha t + \beta)^{-\frac{6}{m}}$. For all above cases, it is observed that at $t \to \infty$, P and ρ approaches identically zero where as $t \to 0$, P and ρ becomes constant except the dust-like case.

For case II, we observed that from equation (25), the scale factor cannot be negative and hence $a(t) > 0$, $A > 0$. At $= 0$, all the physical and kinematical parameters like spatial volume, Hubble parameter, expansion scalar, energy density, pressure, cosmological constant and gravitational constant are constant. The spatial volume increases with increase of time but expansion scalar and Hubble parameters are constant throughout the evolution of universe at any time. At $t \to \infty$, an energy density and pressure decreases. It is also seen that the gravitational constant is infinitely large at $t \to \infty$ but cosmological constant Λ is decreasing. For radiation like case ($\gamma = \frac{4}{3}$), the energy density of the matter is $\rho = Be^{-4t}$,, and its pressure is $P = \frac{1}{3}\rho$. For dust like case ($\gamma = 1$), the energy density of the cosmic matter $\rho = Be^{-3t}$, and its pressure become zero. For stiff matter ($\gamma = 2$), the energy density of the matter is equal to its pressure, that is, $P = \rho$ = Be^{-6t},. For all above cases, it is observed that at $t \to \infty$, P and ρ approaches identically zero where as $t \to 0$, P and ρ becomes constant except the dust-like case.

Conclusion

In this paper, we have studied the solutions of Einstein's field equations for FRW metric in time-dependant gravitational and cosmological constants containing the matter in the form of perfect fluid. FRW model admits exact solutions by choosing time dependent deceleration parameter q. It is seen that gravitational constant G no longer varies with time, it is just constant but the cosmological constant term Λ becomes zero initially. It is remarkable that the rate of expansion and the expansion scalar are constants initially and then grow with time through out the evolution of the universe.

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