Article

Parallel Curves of General Helices in the Sol Space Sol^3

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Abstract

In this paper, we study parallel curves of general helices in **Sol^3.** We characterize parallel curves of general helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations of this new curve in **Sol^3**.

Key Words: general helix, Sol Space, parallel curve.

1. General Helices in Sol Space Sol^3

Assume that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{1.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{T})=1, g_{\mathrm{Sol}^3}(\mathbf{N},\mathbf{N})=1, g_{\mathrm{Sol}^3}(\mathbf{B},\mathbf{B})=1, \\ &g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{N})=g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{B})=g_{\mathrm{Sol}^3}(\mathbf{N},\mathbf{B})=0. \end{aligned} \tag{1.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3. \end{aligned} \tag{1.3}$$

Theorem 1.1. ([14]) *Let* $\gamma: I\to \mathrm{Sol}^3$ *be a unit speed non-geodesic general helix. Then, the parametric equations of* γ *are*

$$x(s)=\frac{\sin\mathrm{P}e^{-\cos\mathrm{P}s-\mathrm{C}_3}}{\mathrm{C}_1^2+\cos^2\mathrm{P}}[-\cos\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_4,$$

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$$y(s)=\frac{\sin \mathrm{P} e^{\cos \mathrm{P} s+\mathrm{C}_3}}{\mathrm{C}_1^2+\cos^2 \mathrm{P}}[-\mathrm{C}_1\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\cos \mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_5,3.4 \qquad (1.4)$$

$$z(s)=\cos \mathrm{P} s+\mathrm{C}_3,$$

where $\mathrm{C}_1,\mathrm{C}_2,\mathrm{C}_3,\mathrm{C}_4,\mathrm{C}_5$ *are constants of integration*.

The obtained parametric equations for Eq.(1.4) is illustrated in Fig.1:

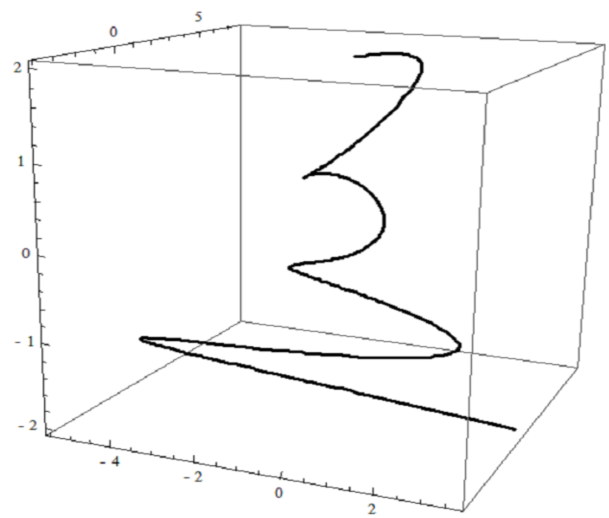

Fig.1

2. Parallel Curves of General Helices in Sol Space Sol^3

We introduce a new curve as parallel curve. Firstly, a spherical wave consisting of points R at the distance r from the moving point $\gamma(s)$, we have

$$g_{\mathrm{Sol}^3}(\gamma(s)-\mathrm{R},\gamma(s)-\mathrm{R})=r^2. \qquad (2.1)$$

Then above equation implies

$$\gamma(s)-\mathrm{R}=a\mathbf{N}+b\mathbf{B}$$

for appropriate coefficients $a,b.$

Theorem 2.1. *Let* $\gamma: I\to \mathrm{Sol}^3$ *be a unit speed non-geodesic general helix and* R_γ *its parallel curve on* Sol^3. *Then,*

$$\mathrm{R}_\gamma=[\frac{\sin \mathrm{P}}{\mathrm{C}_1^2+\cos^2 \mathrm{P}}[-\cos \mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_4 e^{\cos \mathrm{P} s+\mathrm{C}_3}$$

$$+\frac{1}{\kappa^2}[-\frac{1}{C_1}\sin P\sin[C_1s+C_2]+\cos P\sin P\cos[C_1s+C_2]]$$
$$+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin P\sin[C_1s+C_2][\sin^2 P\sin^2[C_1s+C_2]-\sin^2 P\cos^2[C_1s+C_2]]$$
$$-\frac{1}{\kappa}\cos P[\frac{1}{C_1}\sin P\cos[C_1s+C_2]-\cos P\sin P\sin[C_1s+C_2]]]]\mathbf{e}_1 \qquad (2.2)$$

$$[\frac{\sin P}{C_1^2+\cos^2 P}[-C_1\cos[C_1s+C_2]+\cos P\sin[C_1s+C_2]]+C_5e^{-\cos Ps-C_3}$$
$$+\frac{1}{\kappa^2}[\frac{1}{C_1}\sin P\cos[C_1s+C_2]-\cos P\sin P\sin[C_1s+C_2]]$$
$$-(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin P\cos[C_1s+C_2][\sin^2 P\sin^2[C_1s+C_2]-\sin^2 P\cos^2[C_1s+C_2]]$$
$$-\frac{1}{\kappa}\cos P[-\frac{1}{C_1}\sin P\sin[C_1s+C_2]+\cos P\sin P\cos[C_1s+C_2]]]]\mathbf{e}_2$$

$$+[\cos Ps+C_3+\frac{1}{\kappa^2}[\sin^2 P\sin^2[C_1s+C_2]-\sin^2 P\cos^2[C_1s+C_2]]$$

$$+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin P\cos[C_1s+C_2][\frac{1}{C_1}\sin P\cos[C_1s+C_2]-\cos P\sin P\sin[C_1s+C_2]]$$
$$-\frac{1}{\kappa}\sin P\sin[C_1s+C_2][-\frac{1}{C_1}\sin P\sin[C_1s+C_2]+\cos P\sin P\cos[C_1s+C_2]]]]\mathbf{e}_3.$$

where C_1, C_2, C_3, C_4, C_5 *are constants of integration*.

Proof. Assume that γ be a unit speed non-geodesic general helix.

From (2.1), we have

$$a\kappa+1=0 \text{ and } a^2+b^2=r^2.$$

Since

$$a=-\frac{1}{\kappa} \text{ and } b=(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}.$$

Thus we easily obtain that

$$R_\gamma=\gamma+\frac{1}{\kappa}\mathbf{N}+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}\mathbf{B}.$$

On the other hand, from Theorem 1.1, we get

$$\mathbf{T} = \sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2]\mathbf{e}_1 + \sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2]\mathbf{e}_2 + \cos \mathrm{P}\mathbf{e}_3.$$

Using first equation of Eq.(1.3), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_1 T_3)\mathbf{e}_1 + (T_2' - T_2 T_3)\mathbf{e}_2 + (T_3' - T_1^2 + T_2^2)\mathbf{e}_3.$$

Here, ∇ is covariant derivative.

By the use of Frenet formulas and above equation, we get

$$\begin{aligned}\mathbf{N} &= \frac{1}{\kappa}[-\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2] + \cos \mathrm{P} \sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2]]\mathbf{e}_1 \\ &+ \frac{1}{\kappa}[\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2] - \cos \mathrm{P} \sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2]]\mathbf{e}_2 \\ &+ \frac{1}{\kappa}[\sin^2 \mathrm{P} \sin^2[\mathrm{C}_1 s + \mathrm{C}_2] - \sin^2 \mathrm{P} \cos^2[\mathrm{C}_1 s + \mathrm{C}_2]]\mathbf{e}_3.\end{aligned} \tag{2.3}$$

Also, we immediately arrive at

$$\begin{aligned}\mathbf{B} &= [\frac{1}{\kappa}\sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2][\sin^2 \mathrm{P} \sin^2[\mathrm{C}_1 s + \mathrm{C}_2] - \sin^2 \mathrm{P} \cos^2[\mathrm{C}_1 s + \mathrm{C}_2] \\ &- \frac{1}{\kappa}\cos \mathrm{P}[\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2] - \cos \mathrm{P} \sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2]]]\mathbf{e}_1 \\ &- [\frac{1}{\kappa}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2][\sin^2 \mathrm{P} \sin^2[\mathrm{C}_1 s + \mathrm{C}_2] - \sin^2 \mathrm{P} \cos^2[\mathrm{C}_1 s + \mathrm{C}_2] \\ &- \frac{1}{\kappa}\cos \mathrm{P}[-\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2] + \cos \mathrm{P} \sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2]]]\mathbf{e}_2 \\ &+ [\frac{1}{\kappa}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2] - \cos \mathrm{P} \sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2] \\ &- \frac{1}{\kappa}\sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2] + \cos \mathrm{P} \sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2]]]\mathbf{e}_3.\end{aligned} \tag{2.4}$$

Combining Eq.(2.3) and Eq.(2.1), we obtain Eq.(2.2). Hence the proof is completed.

Theorem 2.2. *Let* $\gamma : I \to \mathrm{Sol}^3$ *be a unit speed non-geodesic general helix and* R_γ *its parallel curve on* Sol^3. *Then, the parametric equations of* R_γ *are*

$$x_{\mathrm{R}_\gamma} = \exp[-[\cos \mathrm{P} s + \mathrm{C}_3 + \frac{1}{\kappa^2}[\sin^2 \mathrm{P} \sin^2[\mathrm{C}_1 s + \mathrm{C}_2] - \sin^2 \mathrm{P} \cos^2[\mathrm{C}_1 s + \mathrm{C}_2]]$$

$$+ (r^2 - \frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin \mathrm{P} \cos[\mathrm{C}_1 s + \mathrm{C}_2] - \cos \mathrm{P} \sin \mathrm{P} \sin[\mathrm{C}_1 s + \mathrm{C}_2]$$

$$-\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]]]$$

$$[\frac{\sin\mathrm{P}}{\mathrm{C}_1^2+\cos^2\mathrm{P}}[-\cos\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_4 e^{\cos\mathrm{P}s+\mathrm{C}_3}$$

$$+\frac{1}{\kappa^2}[-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]] \tag{2.5}$$

$$+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]$$

$$-\frac{1}{\kappa}\cos\mathrm{P}[\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]]],$$

$$y_{\mathrm{R}_\gamma}=\exp[-[\cos\mathrm{P}s+\mathrm{C}_3+\frac{1}{\kappa^2}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]]$$

$$+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]$$

$$-\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]]]$$

$$[\frac{\sin\mathrm{P}}{\mathrm{C}_1^2+\cos^2\mathrm{P}}[-\mathrm{C}_1\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_5 e^{-\cos\mathrm{P}s-\mathrm{C}_3}$$

$$+\frac{1}{\kappa^2}[\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]$$

$$-(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]$$

$$-\frac{1}{\kappa}\cos\mathrm{P}[-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]],$$

$$z_{\mathrm{R}_\gamma}=[\cos\mathrm{P}s+\mathrm{C}_3+\frac{1}{\kappa^2}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]]$$

$$+(r^2-\frac{1}{\kappa^2})^{\frac{1}{2}}[\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]$$

$$-\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]],$$

where $\mathrm{C}_1,\mathrm{C}_2,\mathrm{C}_3,\mathrm{C}_4,\mathrm{C}_5$ *are constants of integration*.

Similarly, the obtained parametric equations for Eq.(2.5) is illustrated in Fig.2:

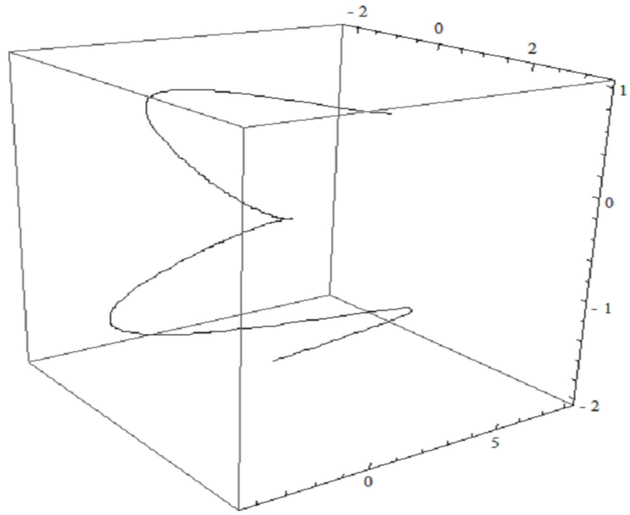

Fig.2.

Finally, the obtained parametric equations for Eqs.(1.4) and (2.5) is illustrated in Fig.3:

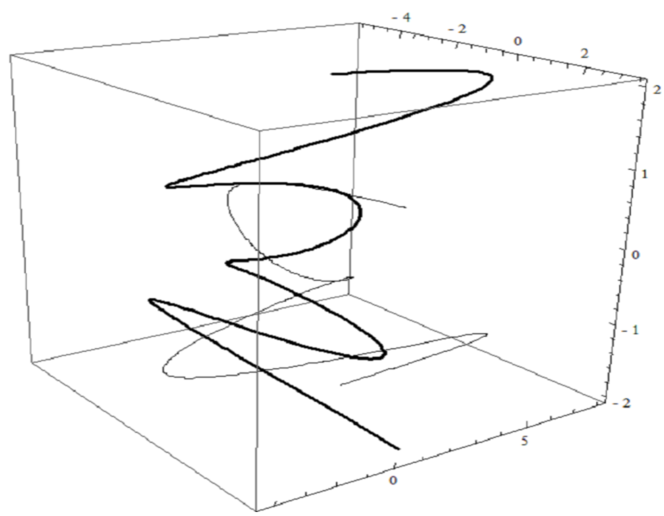

Fig.3

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