Article

Commutator Curves & Exponential Maps on Generalized Heisenberg Group

Talat Körpinar[1] & Essin Turhan*[2]

[1] Muş Alparslan University, Department of Mathematics 49100, Muş, Turkey
[2] Fırat University, Department of Mathematics 23119, Elazığ, Turkey

Abstract
In this paper, we study comutator curves in terms of exponential maps in the generalized Lorentzian Heisenberg group.

Key Words: commutator curves, exponential maps, Lorentzian Heisenberg group.

1. Introduction

In computer vision, the exponential map is the natural generalisation of the ordinary exponential function to matrix elements. The technique is based on generating a manifold embedding of the geometric features of the scene on which to estimate trajectories primarily of motion or invariance. An advantage of using the exponential map is the existence of a closed form time-update equation for the state.

2. Lorentzian Heisenberg Group $\mathbb{H}_{2n+1}$

We begin with a well-known description of the Heisenberg group of dimension $2n+1$. Let $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the Euclidean space with coordinates (x, y, t) where $x = (x_1, ..., x_n) \in \mathbb{R}^n$; $y = (y_1, ..., y_n) \in \mathbb{R}^n$; $t \in \mathbb{R}$. Then the Heisenberg group $\mathbb{H}_{2n+1}$ is this space with the following multiplication rule:

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \left(\langle x, y' \rangle - \langle x', y \rangle\right)\right),$$

where $\langle , \rangle$ is a scalar product in $\mathbb{R}^n$. The element zero, *0 = (0,...,0)*, is the unit of this group structure.

Let $\mathbb{H}_{2n+1} = (\mathbb{R}^{2n+1}, g)$ be the Lorentzian Heisenberg group endowed with the Lorentzian metric g which is defined by

---

* Correspondence: Essin Turhan, Fırat University, Department of Mathematics 23119, Elazığ, Turkey. E-Mail: essin.turhan@gmail.com

$$g = \sum_{i=1}^{n} dx_i^2 + \sum_{i=1}^{n} dy_i^2 - (dt + \frac{1}{2}\sum_{i=1}^{n}(y_i dx_i - x_i dy_i))^2 \quad (2.1)$$

Note that the metric g is left invariant.

The vector fields

$$\mathbf{X}_i = \frac{\partial}{\partial x_i} - \frac{1}{2} y_i \frac{\partial}{\partial t},\ \mathbf{Y}_i = \frac{\partial}{\partial y_i} + \frac{1}{2} x_i \frac{\partial}{\partial t},\ \mathbf{T} = \frac{\partial}{\partial t} \quad (2.2)$$

are then left-invariant vector fields. We define the left-invariant metric on H_{2n+1} by taking $\{\mathbf{X}_i, \mathbf{Y}_i, \mathbf{T}\}$ as the orthonormal frame.

The bracket relations for our left-invariant fields

$$[\mathbf{X}_i, \mathbf{Y}_i] = 2\mathbf{T},\ [\mathbf{X}_i, \mathbf{Y}_j] = [\mathbf{X}_i, \mathbf{T}] = [\mathbf{Y}_i, \mathbf{T}] = 0,\ i \neq j. \quad (2.2)$$

3. Exponential maps in H_{2n+1}

The mapping

$$\exp : \underset{X}{\mathrm{h}_{2n+1}} \underset{\to}{\to} \underset{\exp X}{\mathrm{H}_{2n+1}}$$

is called exponential mapping and for $\forall s,t \in \mathrm{R}$,

$$\exp(t+s)X = \exp tX \exp sX.$$

Let $p \in \mathrm{H}_{2n+1}$ and $f \in C^{\infty}(\mathrm{H}_{2n+1})$. Since the homomorphism $\theta(t) = \exp tX$ satisfies $\theta'(0) = X$, we obtain

$$\tilde{X}_p f = X(f \circ L_p) = \left\{\frac{d}{dt} f(p \exp tX)\right\}_{t=0}.$$

, [20].

It follows that the value of $\tilde{X}f$ at $(p \exp uX)$ is

$$[\tilde{X}f](p \exp uX) = \left\{\frac{d}{dt} f(p \exp uX \exp tX)\right\}_{t=0} = \frac{d}{du} f(p \exp uX).$$

By induction

$$[\tilde{X}^m f](p \exp uX) = \frac{d^m}{du^m} f(p \exp uX).$$

The Taylor formula is given by

$$f(p \exp tX) = \sum_{m=0}^{\infty} \frac{1}{m!}[\tilde{X}^m f](g). \quad (3.1)$$

Theorem 3.2.

$$\exp(t\mathbf{E})\exp(t\mathbf{K}) = \exp\{t\sum_{i=1}^{n}[(\alpha_i+\lambda_i)\mathbf{X}_i + (\beta_i+\mu_i)\mathbf{Y}_i + (\gamma+\xi)\mathbf{T}] + \frac{t^2}{2}\gamma\mathbf{T}(\xi)\mathbf{T} - \frac{t^2}{2}\xi\mathbf{T}(\gamma)\mathbf{T} + \frac{t^2}{2}\mathbf{\Pi}\} \tag{3.2}$$

where $\mathbf{E}$ *and* $\mathbf{K}$ *are arbitrary vector field and* $\alpha_i,\beta_i,\lambda_i,\mu_i,\gamma,\xi \in C^\infty(\mathrm{H}_{2n+1})$,

$$\begin{aligned}\mathbf{\Pi} = [\sum_{i=1,j=1}^{n}[&\alpha_i\mathbf{X}_i(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\alpha_i)\mathbf{X}_i + \alpha_i\mu_i\mathbf{T} - \mu_j\mathbf{Y}_j(\alpha_i)\mathbf{X}_i\\ &+\alpha_i\mathbf{X}_i(\xi)\mathbf{T} - \xi\mathbf{T}(\alpha_i)\mathbf{X}_i - \beta_i\lambda_i\mathbf{T} + \beta_i\mathbf{Y}_i(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\beta_i)\mathbf{Y}_i\\ &+\beta_i\mathbf{Y}_i(\mu_j)\mathbf{Y}_j - \mu_j\mathbf{Y}_j(\beta_i)\mathbf{Y}_i + \beta_i\mathbf{Y}_i(\xi)\mathbf{T} - \xi\mathbf{T}(\beta_i)\mathbf{Y}_i\\ &+\gamma\mathbf{T}(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\gamma)\mathbf{T} + \gamma\mathbf{T}(\mu_j)\mathbf{Y}_j + \alpha_i\mathbf{X}_i(\mu_j)\mathbf{Y}_j - \mu_j\mathbf{Y}_j(\gamma)\mathbf{T}].\end{aligned}$$

Proof. Assume that

$$\mathbf{E} = \sum_{i=1}^{n}\alpha_i\mathbf{X}_i + \beta_i\mathbf{Y}_i + \gamma\mathbf{T},$$

$$\mathbf{K} = \sum_{j=1}^{n}\lambda_j\mathbf{X}_j + \mu_j\mathbf{Y}_j + \xi\mathbf{T}$$

where $\alpha_i,\beta_i,\lambda_i,\mu_i,\gamma,\xi \in C^\infty(\mathrm{H}_{2n+1})$.

Then we can write

$$\begin{aligned}[\mathbf{E},\mathbf{K}] &= \left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i + \beta_i\mathbf{Y}_i + \gamma\mathbf{T}, \sum_{j=1}^{n}\lambda_j\mathbf{X}_j + \mu_j\mathbf{Y}_j + \xi\mathbf{T}\right]\\ &= \left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i, \sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right] + \left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i, \sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right] + \left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i, \xi\mathbf{T}\right]\\ &+ \left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i, \sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right] + \left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i, \sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right] + \left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i, \xi\mathbf{T}\right]\\ &+ \left[\gamma\mathbf{T}, \sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right] + \left[\gamma\mathbf{T}, \sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right] + [\gamma\mathbf{T}, \xi\mathbf{T}].\end{aligned}$$

Using (2.2), we get

$$\begin{aligned}\left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i, \sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right] &= \sum_{i=1,j=1}^{n}\alpha_i\lambda_j[\mathbf{X}_i,\mathbf{X}_j] + \sum_{i=1,j=1}^{n}\alpha_i\mathbf{X}_i(\lambda_j)\mathbf{X}_j - \sum_{i=1,j=1}^{n}\lambda_j\mathbf{X}_j(\alpha_i)\mathbf{X}_i\\ &= \sum_{i=1,j=1}^{n}\alpha_i\mathbf{X}_i(\lambda_j)\mathbf{X}_j - \sum_{i=1,j=1}^{n}\lambda_j\mathbf{X}_j(\alpha_i)\mathbf{X}_i,\end{aligned}$$

$$\left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i,\sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right]=\sum_{i=1,\,j=1}^{n}\alpha_i\mu_j\left[\mathbf{X}_i,\mathbf{Y}_j\right]+\sum_{i=1,\,j=1}^{n}\alpha_i\mathbf{X}_i\left(\mu_j\right)\mathbf{Y}_j-\sum_{i=1,\,j=1}^{n}\mu_j\mathbf{Y}_j\left(\alpha_i\right)\mathbf{X}_i$$
$$=\sum_{i=1}^{n}\alpha_i\mu_i\mathbf{T}+\sum_{i=1,\,j=1}^{n}\alpha_i\mathbf{X}_i\left(\mu_j\right)\mathbf{Y}_j-\sum_{i=1,\,j=1}^{n}\mu_j\mathbf{Y}_j\left(\alpha_i\right)\mathbf{X}_i,$$

$$\left[\sum_{i=1}^{n}\alpha_i\mathbf{X}_i,\xi\mathbf{T}\right]=\sum_{i=1}^{n}\alpha_i\xi\left[\mathbf{X}_i,\mathbf{T}\right]+\sum_{i=1}^{n}\alpha_i\mathbf{X}_i\left(\xi\right)\mathbf{T}-\sum_{i=1}^{n}\xi\mathbf{T}\left(\alpha_i\right)\mathbf{X}_i$$
$$=\sum_{i=1}^{n}\alpha_i\mathbf{X}_i\left(\xi\right)\mathbf{T}-\sum_{i=1}^{n}\xi\mathbf{T}\left(\alpha_i\right)\mathbf{X}_i,$$

$$\left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i,\sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right]=\sum_{i=1,\,j=1}^{n}\beta_i\lambda_j\left[\mathbf{Y}_i,\mathbf{X}_j\right]+\sum_{i=1,\,j=1}^{n}\beta_i\mathbf{Y}_i\left(\lambda_j\right)\mathbf{X}_j-\sum_{i=1,\,j=1}^{n}\lambda_j\mathbf{X}_j\left(\beta_i\right)\mathbf{Y}_i$$
$$=-\sum_{i=1}^{n}\beta_i\lambda_i\mathbf{T}+\sum_{i=1,\,j=1}^{n}\beta_i\mathbf{Y}_i\left(\lambda_j\right)\mathbf{X}_j-\sum_{i=1,\,j=1}^{n}\lambda_j\mathbf{X}_j\left(\beta_i\right)\mathbf{Y}_i,$$

$$\left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i,\sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right]=\sum_{i=1,\,j=1}^{n}\beta_i\mu_j\left[\mathbf{Y}_i,\mathbf{Y}_j\right]+\sum_{i=1,\,j=1}^{n}\beta_i\mathbf{Y}_i\left(\mu_j\right)\mathbf{Y}_j-\sum_{i=1,\,j=1}^{n}\mu_j\mathbf{Y}_j\left(\beta_i\right)\mathbf{Y}_i$$
$$=\sum_{i=1,\,j=1}^{n}\beta_i\mathbf{Y}_i\left(\mu_j\right)\mathbf{Y}_j-\sum_{i=1,\,j=1}^{n}\mu_j\mathbf{Y}_j\left(\beta_i\right)\mathbf{Y}_i,$$

$$\left[\sum_{i=1}^{n}\beta_i\mathbf{Y}_i,\xi\mathbf{T}\right]=\sum_{i=1}^{n}\beta_i\xi\left[\mathbf{Y}_i,\mathbf{T}\right]+\sum_{i=1}^{n}\beta_i\mathbf{Y}_i\left(\xi\right)\mathbf{T}-\sum_{i=1}^{n}\xi\mathbf{T}\left(\beta_i\right)\mathbf{Y}_i$$
$$=\sum_{i=1}^{n}\beta_i\mathbf{Y}_i\left(\xi\right)\mathbf{T}-\sum_{i=1}^{n}\xi\mathbf{T}\left(\beta_i\right)\mathbf{Y}_i,$$

$$\left[\gamma\mathbf{T},\sum_{j=1}^{n}\lambda_j\mathbf{X}_j\right]=\sum_{j=1}^{n}\gamma\lambda_j\left[\mathbf{T},\mathbf{X}_j\right]+\sum_{j=1}^{n}\gamma\mathbf{T}\left(\lambda_j\right)\mathbf{X}_j-\sum_{j=1}^{n}\lambda_j\mathbf{X}_j\left(\gamma\right)\mathbf{T}$$
$$=\sum_{j=1}^{n}\gamma\mathbf{T}\left(\lambda_j\right)\mathbf{X}_j-\sum_{j=1}^{n}\lambda_j\mathbf{X}_j\left(\gamma\right)\mathbf{T},$$

$$\left[\gamma\mathbf{T},\sum_{j=1}^{n}\mu_j\mathbf{Y}_j\right]=\sum_{j=1}^{n}\gamma\mu_j\left[\mathbf{T},\mathbf{Y}_j\right]+\sum_{j=1}^{n}\gamma\mathbf{T}\left(\mu_j\right)\mathbf{Y}_j-\sum_{j=1}^{n}\mu_j\mathbf{Y}_j\left(\gamma\right)\mathbf{T}$$

$$= \sum_{j=1}^{n} \gamma \mathbf{T}(\mu_j)\mathbf{Y}_j - \sum_{j=1}^{n} \mu_j \mathbf{Y}_j(\gamma)\mathbf{T},$$

$$[\gamma \mathbf{T}, \xi \mathbf{T}] = \gamma \xi [\mathbf{T}, \mathbf{T}] + \gamma \mathbf{T}(\xi)\mathbf{T} - \xi \mathbf{T}(\gamma)\mathbf{T}$$

By straightforward calculation we have

$$\begin{aligned}
[\mathbf{E},\mathbf{K}] = & \sum_{i=1,j=1}^{n} \alpha_i \mathbf{X}_i(\lambda_j)\mathbf{X}_j - \sum_{i=1,j=1}^{n} \lambda_j \mathbf{X}_j(\alpha_i)\mathbf{X}_i + \sum_{i=1}^{n} \alpha_i \mu_i \mathbf{T} + \sum_{i=1,j=1}^{n} \alpha_i \mathbf{X}_i(\mu_j)\mathbf{Y}_j \\
& - \sum_{i=1,j=1}^{n} \mu_j \mathbf{Y}_j(\alpha_i)\mathbf{X}_i + \sum_{i=1}^{n} \alpha_i \mathbf{X}_i(\xi)\mathbf{T} - \sum_{i=1}^{n} \xi \mathbf{T}(\alpha_i)\mathbf{X}_i - \sum_{i=1}^{n} \beta_i \lambda_i \mathbf{T} \\
& + \sum_{i=1,j=1}^{n} \beta_i \mathbf{Y}_i(\lambda_j)\mathbf{X}_j - \sum_{i=1,j=1}^{n} \lambda_j \mathbf{X}_j(\beta_i)\mathbf{Y}_i + \sum_{i=1,j=1}^{n} \beta_i \mathbf{Y}_i(\mu_j)\mathbf{Y}_j \\
& - \sum_{i=1,j=1}^{n} \mu_j \mathbf{Y}_j(\beta_i)\mathbf{Y}_i + \sum_{i=1}^{n} \beta_i \mathbf{Y}_i(\xi)\mathbf{T} - \sum_{i=1}^{n} \xi \mathbf{T}(\beta_i)\mathbf{Y}_i + \sum_{j=1}^{n} \gamma \mathbf{T}(\lambda_j)\mathbf{X}_j \\
& - \sum_{j=1}^{n} \lambda_j \mathbf{X}_j(\gamma)\mathbf{T} + \sum_{j=1}^{n} \gamma \mathbf{T}(\mu_j)\mathbf{Y}_j - \sum_{j=1}^{n} \mu_j \mathbf{Y}_j(\gamma)\mathbf{T} + \gamma \mathbf{T}(\xi)\mathbf{T} - \xi \mathbf{T}(\gamma)\mathbf{T}.
\end{aligned} \tag{3.3}$$

Since H_{2n+1} have 2-step nilpotent, we have

$$O(t^3) = 0. \tag{3.4}$$

Using (3.3) and (3.4) we get (3.2). This completes the proof.

By applying Theorem 3.2. we get

Corollary 3.3

i) $\exp(t\mathbf{X}_i)\exp(t\mathbf{Y}_i) = \exp\left\{ t(\mathbf{X}_i + \mathbf{Y}_i) + \frac{t^2}{2}\mathbf{T} \right\},$

ii) $\exp(t\mathbf{X}_i)\exp(t\mathbf{T}) = \exp t(\mathbf{X}_i + \mathbf{T}),$

iii) $\exp(t\mathbf{Y}_i)\exp(t\mathbf{T}) = \exp t(\mathbf{Y}_i + \mathbf{T}).$

Theorem 3.4

$$\exp(-t\mathbf{E})\exp(-t\mathbf{K})\exp(t\mathbf{E})\exp(t\mathbf{K}) = \exp\{t^2 \gamma \mathbf{T}(\xi)\mathbf{T} - t^2 \xi \mathbf{T}(\gamma)\mathbf{T} + t^2 \Pi\}$$

where $\mathbf{E}$ *and* $\mathbf{K}$ *are arbitrary vector field and* $\alpha_i, \beta_i, \lambda_i, \mu_i, \gamma, \xi \in C^\infty(\mathsf{H}_{2n+1}),$

$$\begin{aligned}
\mathbf{\Pi} = [& \sum_{i=1,j=1}^{n} [\alpha_i \mathbf{X}_i(\lambda_j)\mathbf{X}_j - \lambda_j \mathbf{X}_j(\alpha_i)\mathbf{X}_i + \alpha_i \mu_i \mathbf{T} - \mu_j \mathbf{Y}_j(\alpha_i)\mathbf{X}_i \\
& + \alpha_i \mathbf{X}_i(\xi)\mathbf{T} - \xi \mathbf{T}(\alpha_i)\mathbf{X}_i - \beta_i \lambda_i \mathbf{T} + \beta_i \mathbf{Y}_i(\lambda_j)\mathbf{X}_j - \lambda_j \mathbf{X}_j(\beta_i)\mathbf{Y}_i \\
& + \beta_i \mathbf{Y}_i(\mu_j)\mathbf{Y}_j - \mu_j \mathbf{Y}_j(\beta_i)\mathbf{Y}_i + \beta_i \mathbf{Y}_i(\xi)\mathbf{T} - \xi \mathbf{T}(\beta_i)\mathbf{Y}_i \\
& + \gamma \mathbf{T}(\lambda_j)\mathbf{X}_j - \lambda_j \mathbf{X}_j(\gamma)\mathbf{T} + \gamma \mathbf{T}(\mu_j)\mathbf{Y}_j + \alpha_i \mathbf{X}_i(\mu_j)\mathbf{Y}_j - \mu_j \mathbf{Y}_j(\gamma)\mathbf{T}].
\end{aligned}$$

Proof. Applying $\exp(-t\mathbf{E})$ ve $\exp(-t\mathbf{K})$ in (3.2), we have

$$\exp(-t\mathbf{E})\exp(-t\mathbf{K})\exp(t\mathbf{E})\exp(t\mathbf{K}) = \exp\{t\sum_{i=1}^{n}[(\alpha_i+\lambda_i)\mathbf{X}_i+(\beta_i+\mu_i)\mathbf{Y}_i+(\gamma+\xi)\mathbf{T}]$$
$$-t\sum_{i=1}^{n}[(\alpha_i+\lambda_i)\mathbf{X}_i+(\beta_i+\mu_i)\mathbf{Y}_i+(\gamma+\xi)\mathbf{T}]+\frac{t^2}{2}[\mathbf{E},\mathbf{K}]+\frac{t^2}{2}[\mathbf{E},\mathbf{K}]\}.$$

Corollary 3.5

i) $\exp(-t\mathbf{X}_i)\exp(-t\mathbf{Y}_i)\exp(t\mathbf{X}_i)\exp(t\mathbf{Y}_i) = \exp\{t^2\mathbf{T}\},$

ii) $\exp(-t\mathbf{X}_i)\exp(-t\mathbf{T})\exp(t\mathbf{X}_i)\exp(t\mathbf{T}) = \mathbf{1}_{\mathrm{H}_{2n+1}},$

iii) $\exp(-t\mathbf{Y}_i)\exp(-t\mathbf{T})\exp(t\mathbf{Y}_i)\exp(t\mathbf{T}) = \mathbf{1}_{\mathrm{H}_{2n+1}}.$

Similarly, we may prove

Theorem 3.6

$$\exp(t\mathbf{E})\exp(t\mathbf{K})\exp(-t\mathbf{E}) = \exp\{t\sum_{j=1}^{n}[\lambda_j\mathbf{X}_j+\mu_j\mathbf{Y}_j]+t\xi\mathbf{T}+\frac{t^2}{2}\gamma\mathbf{T}(\xi)\mathbf{T}-\frac{t^2}{2}\xi\mathbf{T}(\gamma)\mathbf{T}+\frac{t^2}{2}\Pi\}$$

where $\mathbf{E}$ *and* $\mathbf{K}$ *are arbitrary vector field and* $\alpha_i,\beta_i,\lambda_i,\mu_i,\gamma,\xi\in C^\infty(\mathrm{H}_{2n+1}),$

$$\begin{aligned}\mathbf{\Pi} = [\sum_{i=1,j=1}^{n}[&\alpha_i\mathbf{X}_i(\lambda_j)\mathbf{X}_j-\lambda_j\mathbf{X}_j(\alpha_i)\mathbf{X}_i+\alpha_i\mu_i\mathbf{T}-\mu_j\mathbf{Y}_j(\alpha_i)\mathbf{X}_i\\&+\alpha_i\mathbf{X}_i(\xi)\mathbf{T}-\xi\mathbf{T}(\alpha_i)\mathbf{X}_i-\beta_i\lambda_i\mathbf{T}+\beta_i\mathbf{Y}_i(\lambda_j)\mathbf{X}_j-\lambda_j\mathbf{X}_j(\beta_i)\mathbf{Y}_i\\&+\beta_i\mathbf{Y}_i(\mu_j)\mathbf{Y}_j-\mu_j\mathbf{Y}_j(\beta_i)\mathbf{Y}_i+\beta_i\mathbf{Y}_i(\xi)\mathbf{T}-\xi\mathbf{T}(\beta_i)\mathbf{Y}_i\\&+\gamma\mathbf{T}(\lambda_j)\mathbf{X}_j-\lambda_j\mathbf{X}_j(\gamma)\mathbf{T}+\gamma\mathbf{T}(\mu_j)\mathbf{Y}_j+\alpha_i\mathbf{X}_i(\mu_j)\mathbf{Y}_j-\mu_j\mathbf{Y}_j(\gamma)\mathbf{T}].\end{aligned}$$

Corollary 3.7

i) $\exp(t\mathbf{X}_i)\exp(t\mathbf{Y}_i)\exp(-t\mathbf{X}_i) = \exp\left\{t\mathbf{Y}_i+\frac{t^2}{2}\mathbf{T}\right\},$

ii) $\exp(t\mathbf{X}_i)\exp(t\mathbf{T})\exp(-t\mathbf{X}_i) = \exp(t\mathbf{Y}_i)\exp(t\mathbf{T})\exp(-t\mathbf{Y}_i) = \exp\{t\mathbf{T}\}.$

4 Commutator Curves in H_{2n+1}

Definition 4.1 *The curve*

$$\gamma: I \to \mathrm{H}_{2n+1}$$
$$t \to \gamma(t) = \exp\left(-t^{\frac{1}{2}}X\right)\exp\left(-t^{\frac{1}{2}}Y\right)\exp\left(t^{\frac{1}{2}}X\right)\exp\left(t^{\frac{1}{2}}Y\right), t\geq 0 \qquad (4.1)$$

is called commutator curve in H_{2n+1}, [27].

Theorem 4.2 *If we use basis vectors* $\mathbf{X}_i$ *and* $\mathbf{Y}_i$, *we have*

$$\gamma(t) = \exp t\mathbf{T}, \gamma'(0) = \mathbf{T}$$

Proof Combining (3.1), (3.2) and (4.1), we have

$$\gamma(t) = \exp\left(-t^{\frac{1}{2}}\mathbf{X}_i\right)\exp\left(-t^{\frac{1}{2}}\mathbf{Y}_i\right)\exp\left(t^{\frac{1}{2}}\mathbf{X}_i\right)\exp\left(t^{\frac{1}{2}}\mathbf{Y}_i\right)$$

Then, we have

$$\gamma(t) = \exp\{t\mathbf{T}\}.$$

Using Definition 4.1, we obtain

$$\gamma'(0) = \mathbf{T}.$$

We can give the following theorem:

Theorem 4.3 *In general, the commutator curve is given by*

$$\gamma(t) = \exp\{t\gamma\mathbf{T}(\xi)\mathbf{T} - t\xi\mathbf{T}(\gamma)\mathbf{T} + t\Pi\},$$

where $\mathbf{E}$ *and* $\mathbf{K}$ *are arbitrary vector field and* $\alpha_i, \beta_i, \lambda_i, \mu_i, \gamma, \xi \in C^\infty(\mathrm{H}_{2n+1})$,

$$\begin{aligned}\mathbf{\Pi} = [\sum_{i=1, j=1}^{n} [&\alpha_i\mathbf{X}_i(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\alpha_i)\mathbf{X}_i + \alpha_i\mu_i\mathbf{T} - \mu_j\mathbf{Y}_j(\alpha_i)\mathbf{X}_i \\ &+ \alpha_i\mathbf{X}_i(\xi)\mathbf{T} - \xi\mathbf{T}(\alpha_i)\mathbf{X}_i - \beta_i\lambda_i\mathbf{T} + \beta_i\mathbf{Y}_i(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\beta_i)\mathbf{Y}_i \\ &+ \beta_i\mathbf{Y}_i(\mu_j)\mathbf{Y}_j - \mu_j\mathbf{Y}_j(\beta_i)\mathbf{Y}_i + \beta_i\mathbf{Y}_i(\xi)\mathbf{T} - \xi\mathbf{T}(\beta_i)\mathbf{Y}_i \\ &+ \gamma\mathbf{T}(\lambda_j)\mathbf{X}_j - \lambda_j\mathbf{X}_j(\gamma)\mathbf{T} + \gamma\mathbf{T}(\mu_j)\mathbf{Y}_j + \alpha_i\mathbf{X}_i(\mu_j)\mathbf{Y}_j - \mu_j\mathbf{Y}_j(\gamma)\mathbf{T}].\end{aligned}$$

Proof. Using Theorem 3.4 ve Definition 4.1 we have theorem. W

Corollary 4.4 *In general, tangent vector of the commutator curve at 0 is given by*

$$\gamma'(0) = \gamma\mathbf{T}(\xi)\mathbf{T} - \xi\mathbf{T}(\gamma)\mathbf{T} + \Pi.$$

References

[1] E. Abbena, An example of an almost Kahler manifold which is not Kahlerian, Boll. Un. Mat. Ital. 6 (3) (1984), 383-392.
[2] L. Auslander, The structure of complete locally affine manifolds, Topology Suppll. 3(1), 131-139.
[3] W. Batat and S. Rahmani, Homogeneous Lorentzian structures on the generalized Heisenberg group, Differential Geometry - Dynamical Systems, 12 (2010), 12-17.

[4] W. Batat, S. Rahmani, Isometries, Geodesics and Jacobi Fields of Lorentzian Heisenberg Group, Mediterr. J. Math. 8 (2011), 411-430.
[5] D.A. Berdinsky, I.A. Taimanov, Surfaces in three-dimensional Lie groups, Siberian Math. J. 46 (2005) 1005--1019.
[6] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces, Lecture Notes in Mathematics 1598, Springer- Verlag, 1995.
[7] D. E. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York, 1976.
[8] D.E. Blair, S. Ianus, Critical associated metrics on symplectic manifolds, Contemp. Math. 51 (1986) 23--29.
[9] N. Birnie, Exponential Maps for Computer Vision, (preprint)
[10] M. do Carmo, Riemannian geometry. Birkhauser, 1992.
[11] A.L. Cordero, M. Fernandez, M. de Leon, Examples of compact non-Kähler almost Kähler manifolds, Proc. Am. Math. Soc. 95 (1985) 282--286.
[12] G. Calvaruso, R. Marinosci, Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups Mediterranean, J. Math. 3 (3--4) (2006) 467--481.
[13] Q. Chen, H. Qui, Weierstrass Representation for Surfaces in the Three-Dimensional Heisenberg Group, Chin. Ann. Math.31B(1) (2010), 119--132
[14] T. Draghici, On some 4-dimensional almost Kähler manifolds, Kodai Math. J. 18 (1995) 156--163.
[15] T. Draghici, Almost Kähler 4-manifolds with J-invariant Ricci tensor, Houston J. Math. 25 (1999) 133--145.
[16] A. Gray, Curvature identities for Hermitian and almost Hermitian manifolds, Tohoku Math. J. 28 (1976) 601--612.
[17] A. Gray, Almost complex submanifolds of the six sphere, Proc. Amer. Math. Soc.20 (1969) 277--279.
[18] S.I. Goldberg, Integrability of almost Kähler manifolds, Proc. Am. Math. Soc. 21 (1969) 96--100.
[19] H. H. Hacısalihoğlu: Yüksek Diferensiyel geometriye Giriş, İstanbul, 1980.
[20] S. Helgason. Differential Geometry, Lie Groups and Symmetric Spaces. Academic Press, 1978.
[21] S. Kobayashi and K. Nomizu. Foundations of differential geometry, I. Wiley-- Interscience, 1963.
[22] S. Kobayashi and K. Nomizu. Foundations of differential geometry, II. Wiley-- Interscience, 1969.
[23] T. Körpınar, On The Curvatures of Pseudo-Complex Lie groups, F rat University (2013), PhD Thesis.
[24] B. O'Neill: Semi-Riemannian Geometry, Academic Press, New York (1983).
[25] N. Rahmani and S. Rahmani, Structures homogenes Lorentziennes sur le groupe de Heisenberg I, J. Geom. Phys. 13 (1994), 254-258.
[26] N. Rahmani and S. Rahmani, Lorentzian Geometry of the Heisenberg group, Geom. Dedicata 118 (2006), 133--140.
[27] A.A. Sagle, R.E. Walde, Introduction to Lie Groups and Lie Algebras, Academic Press, New York, 1973.
[28] F. Tricerri and L. Vanhecke, Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc. 267 (1981), 365--398.
[29] E. Turhan, The curvature properties of some complex Lie groups, Ph. D. Thesis, F rat University, 1997.
[30] L. Vezzoni, On the Hermitian curvature of symplectic manifolds, Adv. Geom. 7 (2007), 207--214.
[31] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Co., Glenview, Illinois, 1971.
[32] H. Weyl, The classical groups. Their invariants and representations, Princeton Univ. Press, 1939, printing 1997.
[33] K. Yano, Differential Geometry on Complex and Almost Complex Spaces, New York, Pergamon Press, 1965.