Article

Abel Resummation, Regularization, Renormalization & Infinite Series

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Abstract

In this paper, Abel summation method is applied to evaluate infinite series and divergent integrals. Several examples of how one can obtain regularizations are given.

Key Words: Abel sum formula, Abel-Plana formula, poles, infinities, renormalization,, multiple integrals, regularization, Casimir effect.

Abel summation for divergent series

Given a power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which is convergent on the region $|x|<1$, we define the Abel resummation of the series $\sum_{n=0}^{\infty} a_n$ as the limit $\lim_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n = A(S)$

If the previous limit exists, then the series $\sum_{n=0}^{\infty} a_n$ is said to be 'Abel-summable' to the value A(s).

As an example, let be the series [6]

$$\sum_{n=0}^{\infty}(-1)n^k = 1-2^k+3^k-....=\left(x\frac{d}{dx}\right)^k\frac{1}{1+x}=\frac{2^{k+1}-1}{k+1}B_{k+1} \tag{1}$$

Unfortunately, the series $\sum_{n=0}^{\infty} n^k$ is NOT Abel-summable due to the pole at x=1 of the function $(1-x)^{-1}$.

However, Guo [5], using an exponential regulator, studied this series and gave the following identity

$$\sum_{n=0}^{\infty} n^k e^{-\varepsilon n} = \left(-\frac{d}{d\varepsilon}\right)^k \frac{1}{1-e^{-\varepsilon}} = \frac{\Gamma(k+1)}{\varepsilon^{k+1}} + \sum_{j=0}^{\infty}\frac{Z(-k-j)}{j!}(-\varepsilon)^k \qquad k\neq -1 \tag{2}$$

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where both , the Taylor expansion involving Bernoulli's number $\frac{x}{e^x-1}=\sum_{j=0}^{\infty}B_j\frac{x^j}{j!}$ and the expression for negative values of the Riemann zeta function $\zeta(1-k)=-\frac{B_k}{k}$ were used.

To evaluate the Riemann zeta inside (2) for negative values, one needs the Riemann's functional equation defined by $\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$, with $\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}$.

Guo introduces an small parameter 'epsilon' and after calculations take the limit $\varepsilon\to 0$, Unfortunately for k= -1 Guo's method gives only an infinite answer $\sum_{n=1}^{\infty}n^{-1}e^{-\varepsilon n}=\log\left(\frac{1}{\varepsilon}\right)$, this all is because the following expressions for the n-th Harmonic number and the Laplace transform for the logarithm

$$H_n=1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}\approx\gamma+\log n \qquad \int_0^{\infty}dte^{-\varepsilon t}\log t=-\frac{\gamma+\log\varepsilon}{\varepsilon} \tag{3}$$

where $\gamma=0.57721..$ is the Euler-Mascheroni constant.

If one ignores the pole part in (2), one has $f.p\left(\sum_{n=0}^{\infty}n^k e^{-\varepsilon n}\right)=\zeta(-k)$ for every k except k=-1.

This is precisely the value of the series obtained via Zeta regularization.
So, Abel resummation and Zeta regularization are related and give the same answer for the divergent series provided that one ignores the pole part proportional to ε^{-k-1}.

As an example, we will study the Casimir Effect to see how the regularization and renormalization of the divergent sum is made.

- ***Casimir effect:***

The Casimir effect is a physical force due to the quantization of Electromagnetic fields (see, e.g., [7]). In the simplest version of the Casimir effect, the vacuum Energy of the system per unit of Area 'A' is given by

$$\frac{\langle E\rangle}{A}=\frac{\hbar c}{4\pi^2}\sum_n\int_0^{\infty}2\pi r dr\left|r^2+\frac{n^2\pi^2}{a^2}\right|^{1/2}=-\frac{\hbar c\pi^2}{6a^3}\sum_n|n|^3 \tag{4}$$

where $\hbar=\frac{h}{2\pi}=1.054\times10^{-34}J.s$ is the reduced Planck's constant and $c=3\times10^8 m/s$ is the speed of light in the vacuum.

If we use Zeta regularization [3], we find the value $\sum_{n=1}^{\infty} n^3 = \frac{1}{120}$. After we insert this value inside (4), we get the correct experimental value for Casimir effect $\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3}$.

So $\frac{F_c}{A} = -\frac{d}{da}\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{240 a^4}$.

The physicists's approach to 'Casimir effect' is a bit more complicated. For example, they use renormalization and compute the quantity (difference)

$$\langle \delta E \rangle = \langle E_{discrete} \rangle - \langle E \rangle = -\frac{\hbar c \pi^2}{6a^3}\left(\sum_{n=0}^{\infty} n^3 e^{-n\varepsilon} - \int_0^{\infty} dt t^3 e^{-\varepsilon t} \right) \quad (5)$$

This difference can be computed with the aid of the Euler-Maclaurin sum formula

$$\sum_{n=0}^{\infty} f(n) - \int_a^{\infty} f(x)dx = -\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) \quad f(x) = x^3 e^{-\varepsilon x} \quad (6)$$

Or with the Abel-Plana sum formula ,with $\varepsilon \to 0$

$$\sum_{n=0}^{\infty} f(n) - \int_a^{\infty} f(x)dx = \frac{f(0)}{2} + i\int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \quad f(x) = x^3 e^{-\varepsilon x} \quad (7)$$

If we return to Guo's formula (2) and use the identity $\int_0^{\infty} dt e^{-t\varepsilon} t^k = \frac{\Gamma(k+1)}{\varepsilon^{k+1}}$, we find the following result

$$\sum_{n=0}^{\infty} n^k e^{-\varepsilon n} == \sum_{j=0}^{\infty} \frac{Z(-k-j)}{j!}(-\varepsilon)^k + \int_0^{\infty} dt e^{-t\varepsilon} t^k = \frac{\Gamma(k+1)}{\varepsilon^{k+1}} \quad (8)$$

So, although the Abel regularization is not valid for the series $\sum_{n=0}^{\infty} n^k$, the difference

$$\Delta = \sum_{n=0}^{\infty} n^k e^{-n\varepsilon} - \int_0^{\infty} dt e^{-t\varepsilon} t^k = \zeta(-k) \quad \varepsilon \to 0 \quad (9)$$

makes perfect sense and is always FINITE. Also, for the case k=-1 we find that the Harmonic series is 'summable' and its sum is equal to Euler-Mascheroni constant $\sum_{n=0}^{\infty} n^{-1} = \gamma$ after removing the regulator $e^{-\varepsilon}$.

So, both methods 'renormalization' and zeta regularization gives the same finite answer. However, Zeta regularization is an easier and faster method and can be generalized to the case of more general operators. For example,

$$E = \frac{1}{2}\hbar c Trace\left(\sqrt{-\Delta}\right) \qquad \Delta = \frac{1}{\sqrt{|g|}}\sum_{i,j}\partial_i\left(\sqrt{|g|}g^{i,j}\partial_j\right) \tag{10}$$

where the operator is the Laplace-Beltrami operator and $g = \det\begin{vmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{2,2} \end{vmatrix}$ is a determinant of a 2x2 matrix, the quantity (10) is the Vacuum energy for the Laplacian operator in two dimensions.

Abel summation and divergent integrals

Abel summation formula can be extended to obtain finite results for divergent integrals too, first we need the formula

$$\begin{aligned} \int_a^{\infty} x^m dx = \frac{m}{2}\int_a^{\infty} x^{m-1}dx + \left(\sum_{i=1}^{\infty} i^m\right) - \sum_{i=1}^{a} i^m + a^m \\ -\sum_{r=1}^{\infty}\frac{B_{2r}\Gamma(m+1)}{(2r)!\Gamma(m-2r+2)}(m-2r+1)\int_a^{\infty} x^{m-2r}dx \end{aligned} \tag{11}$$

where 'a' is a positive integer and the infinite sum inside (11) must be understood in the Abel regularization sense $\sum_{i=1}^{\infty} i^k \to \sum_{n=0}^{\infty} n^k e^{-\varepsilon n}$

Also, the recurrence (11) is finite if 'k' is a positive integer due to the poles of the Gamma function at the negative integers.
In case 'k' is a positive and real number, the recurrence (11) is infinite and it must be truncated , we can also use inside (11) the identity $\int_a^{\infty}\frac{dx}{x^m} = \frac{1}{a^{m-1}}\frac{1}{m-1}$,which is valid for $\mathrm{Re}(m) > 1$.

The case m=-1 must be considered separately. If we take the finite part then $f.p\left(\sum_{n=0}^{\infty} n^{-1}e^{-n\varepsilon}\right)=\gamma$, or if we use the expression $f(x)=\frac{e^{-x\varepsilon}}{x+1}$ inside the Euler-Maclaurin summation formula

$$\sum_{n=a+1}^{\infty} f(n)=\int_a^{\infty} f(x)dx-\frac{f(a)+f(\infty)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(\infty)-f^{(2k-1)}(a)\right) \quad (12)$$

and take into account the following series expansion for the Digamma function

$$\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\log x+\frac{1}{2x}-\sum_{r=1}^{\infty}\frac{B_{2r}}{2n}\frac{1}{x^{2n}} \qquad \Psi(1)=-\gamma \quad (13)$$

We get the renormalized result for the integral with a logarithmic divergence $\int_a^{\infty}\frac{dx}{x+a}_{renorm}=-\log a$,which means that in a regularized/renormalized sense the 3 integrals $\int_1^{\infty}\frac{dx}{x+1}$ $\int_0^{1}\frac{dx}{x}$ and $\int_0^{\infty}\frac{dx}{x}$ are equal to 0 .

For the case k=0, we find that $\int_0^{\infty}dx=\frac{f(a)+f(\infty)}{2}+\left(\sum_{n=1}^{\infty}n^0e^{-n\varepsilon}\right)=\frac{1}{2}$ because the value $\zeta(0)=-\frac{1}{2}$ for the Riemann zeta function, this means that this series has the finite part $f.p\left(\sum_{n=0}^{\infty}n^0e^{-\varepsilon n}\right)=\zeta(0)$

- ***Renormalization/regularization theory from divergent series***

Using Abel-summation and formula (11), we can give an easy method to regularize divergent integrals of the form $\int_0^{\infty} f(x)dx$, which is easy to understand.

This method of renormalization and regularization is based on the resummation of divergent series of power of positive integers and a relationship in the form of a recurrence equation between the divergent integral $\int_0^{a} x^k dx$ and its discrete divergent series counterpart $\sum_{n=1}^{\infty} n^k$, the method to regularize divergent integrals would be then the following

- Split the integral above into a finite part $\int_0^a f(x)dx$ plus a divergent part $\int_a^\infty f(x)dx$, this can always be made;
- Expand the integrand inside $\int_a^\infty f(x)dx$ into a Laurent series of the form $\sum_{n=-\infty}^{k} a_n x^n$ the coefficients of this expansion are given by an integral over the complex plane (Cauchy's theorem [1]) $a_n = \frac{1}{2\pi i}\int_C dz \frac{f(z)}{z^{n+1}}$;
- Apply integration on each term of the form $\sum_{n=-\infty}^{-2} a_n x^n$ and use the formula $\int_a^\infty \frac{dx}{x^m} = \frac{a^{-m+1}}{m-1}$ which is valid and well defined for $m \ge 2$;
- Use the regularization for the Harmonic series $\sum_{n=0}^{\infty} n^{-1} = \gamma$ and the regularization of the logarithmic integral $\int_a^\infty \frac{dx}{x} = -\log a$ to regularize and give a finite meaning to the logarithmic divergence
- Use formula (11) to regularize the divergent integrals $\int_a^\infty dxx^m$ for every m=0,1,2,...,k , for the series $\sum_{n=1}^{\infty} n^m e^{-\varepsilon n}$, the 'renormalized' value for every 'm' of these is just $\left.\sum_{n=1}^{\infty} n^m e^{-\varepsilon n}\right|_{renorm} = \zeta(-m)$ so Abel and Zeta regularization give both the same results, except for the harmonic series which is not zeta regularizable
- Another definition of the renormalized infinite series is made with the Abel-plana sum formula, use Abel-Plana formula to compute the renormalized value of the series $\sum_{n=0}^{\infty} e^{-n\varepsilon} - \int_a^\infty x^n e^{-x\varepsilon}dx$, when the regulator 'epsilon' is taken to 0, this results is analogue to zeta regularization.

As an example, let the divergent integral be $\int_0^\infty \frac{x^2}{x+c}dx$, with c >0, the renormalized value of this integral using formula (11) would be

$$\int_0^\infty \frac{x^2}{x+c}dx = \int_0^\infty xdx - c\int_0^\infty dx + c^2\int_0^\infty \frac{dx}{x+c} \to_{reg} \int_0^\infty \frac{x^2}{x+c}dx = -c^2\log c - \frac{c}{2} + \frac{1}{6} \qquad (14)$$

A more complicated 2-loop integral $\int_0^\infty dx \int_0^\infty dy \frac{xy}{x+y+1}$ can be computed within our renormalization method based on the regularization and study of divergent series.
In this case, the integral has a subdivergence in the variable 'x' which should be renormalized first, the renormalized value of this integral is

$$\int_0^\infty dx \int_0^\infty dy \frac{x}{x+y+1} = -\int_0^\infty dy y^2 (y+1) \int_0^\infty \frac{dx}{(x+y+1)(x+1)} + \frac{1}{2}\int_0^\infty y dy \qquad (15)$$

The integral inside (15) $\int_0^\infty \frac{dx}{(x+y+1)(x+1)} = f(x)$ is finite for every positive 'x'.
To simplify the calculations, we can replace (approximate) this integral by a quadrature formula with n-points so the sum (quadrature) is easier to work with.

For example, if we use the Laguerre quadrature formula, valid for $[0,\infty)$ (see, e.g., [1]):

$$-y^2(y+1)\int_0^\infty \frac{dx}{(x+y+1)(x+1)} \approx -y^2(y+1)\sum_{j=0}^{n} \omega_j \frac{e^{x_j}}{\left(x_j+1+y\right)\left(x_j+1\right)} \qquad (16)$$

Since each term inside (16) depend on 'y', we have to renormalize the integrals $-\sum_{j=0}^{n} \omega_j \int_0^\infty dy \frac{e^{x_j}(y+1)y^2}{\left(x_j+1+y\right)\left(x_j+1\right)}$, which have all them a quartic divergence $\Lambda^4 \to \infty$. This can be seen if we introduce a cut-off term in the integral.

We have converted a 2-loop integral into an ordinary integral by using a numerical method and have also applied the Abel resummation and formula (11) to the original integral $\int_0^\infty dx \int_0^\infty dy \frac{xy}{x+y+1}$ to get a finite renormalized value for it.

- ***Understanding the Casimir effect renormalization and why the divergent series*** $\sum_{n=1}^{\infty} n^k = \zeta(-k)$ ***having a finite physical value***

Let be the boundary value problem

$$D_0 f = -\frac{d^2 f}{dx^2} \qquad f(0) = f(\pi) = 0 \qquad D_0 f = E_n f \qquad E_n = n^2 \qquad (17)$$

Now, if we define the operator $T=\sqrt{D_0}$, the sums $\sum_{n=0}^{\infty} n^k$ are then the traces of the powers of the operator 'T' in terms of the spectral zeta function of the Energies of the eigenvalue problem inside (17)

$$\sum_{n=0}^{\infty} n^k = Trace\left(T^k\right) = \zeta_T\left(-\frac{k}{2}, L=\pi\right) \qquad \zeta_T\left(s, L=\pi\right) = \sum_{n=1}^{\infty} E_n^{-s} = \zeta(2s) \tag{18}$$

The spectrum of problem (17) is discrete, since we have imposed the boundary conditions for the eigenfunctions $f(0)=f(L=\pi)=0$.

If we take the limit $L\to\infty$, the spectrum is no longer discrete and the traces are given by an integral instead of a discrete sum,

$$Trace\left(T^k\right)_{L\to\infty} = \int_0^{\infty} t^k dt = \zeta_T\left(-\frac{k}{2}, L\right) .$$

This integral is still divergent but, if we take the difference between the two (an exponential regulator is assumed), we can define a 'renormalized' value of the divergent series

$$\zeta_T\left(-\frac{k}{2}, L=\pi\right) - \zeta_T\left(-\frac{k}{2}, L=\infty\right) = \sum_{n=1}^{\infty} n^k e^{-n\varepsilon} - \int_0^{\infty} dt t^k e^{-t\varepsilon} = \zeta(-k) \tag{19}$$

For the case of the Harmonic series, the difference is $\zeta_T\left(\frac{1}{2}, L=\pi\right) - \zeta_T\left(\frac{1}{2}, L=\infty\right) = \gamma$,which is again a renormalization of the divergent Harmonic series.
So, in the end we have only a finite value for every divergent sum and integral.

When this method is used in the evaluation of the functional determinant of an operator with a discrete set of eigenvalues $\det(A)=\prod_n \lambda_n$, the expression $\sum_n \log\lambda_n$ is divergent in general.
But we can define the logarithm of the functional determinant as the finite difference (substraction of the divergence)

$$\log A - LogC = -\partial_s Z(0,x) + \partial_s Z(0,0) \qquad Z(s,x) = \sum_{n=0}^{\infty} \left(x+\lambda_n\right)^{-s} \tag{20}$$

where 'C' is a finite constant. For example, this method can be used to expand the Gamma function and the sine function into an infinite product over their zeros

$$\frac{\sqrt{2\pi}}{\Gamma(x+1)} = \prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right) \qquad \frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) \tag{21}$$

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