Article

Physics as Generalized Number Theory III: Infinite Primes

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Abstract

Physics as a generalized number theory program involves three threads: various p-adic physics and their fusion together with real number based physics to a larger structure, the attempt to understand basic physics in terms of classical number fields, and infinite primes discussed in this article.

The construction of infinite primes is formally analogous to a repeated second quantization of an arithmetic quantum field theory by taking the many particle states of previous level elementary particles at the new level. Besides free many particle states also the analogs of bound states appear. In the representation in terms of polynomials the free states correspond to products of first order polynomials with rational zeros. Bound states correspond to n^{th} order polynomials with non-rational but algebraic zeros.

The construction can be generalized to classical number fields and their complexifications obtained by adding a commuting imaginary unit. Special class corresponds to hyper-octonionic primes for which the imaginary part of ordinary octonion is multiplied by the commuting imaginary unit so that one obtains a sub-space M^8 with Minkowski signature of metric. Also in this case the basic construction reduces to that for rational or complex rational primes and more complex primes are obtained by acting using elements of the octonionic automorphism group which preserve the complex octonionic integer property.

Can one map infinite primes/integers/rationals to quantum states? Do they have space-time surfaces as correlates? Quantum classical correspondence realized in terms of modified Dirac operator implies that if infinite rationals can be mapped to quantum states then the mapping of quantum states to space-time surfaces automatically gives the map to space-time surfaces. The question is therefore whether the mapping to quantum states defined by WCW spinor fields is possible. A natural hypothesis is that number theoretic fermions can be mapped to real fermions and number theoretic bosons to WCW ("world of classical worlds") Hamiltonians. The crucial observation is that one can construct infinite hierarchy of hyper-octonionic units by forming ratios of infinite integers such that their ratio equals to one in real sense: the integers have interpretation as positive and negative energy parts of zero energy states. One can construct also sums of these units with complex coefficients using commuting imaginary unit and these sums can be normalized to unity and have interpretation as states in Hilbert space. These units can be assumed to possess well defined standard model quantum numbers. It is possible to map the quantum number combinations of WCW spinor fields to these states. Hence the points of M^8 can be said to have infinitely complex number theoretic anatomy so that quantum states of the universe can be mapped to this anatomy. One could talk about algebraic holography or number theoretic Brahman=Atman identity.

One can also ask how infinite primes relate to the p-adicization program and to the hierarchy of Planck constants. The key observation is that infinite primes are in one-one correspondence with rational numbers at the lower level of hierarchy. At the first level of hierarchy the p-adic norm with respect to p-adic prime for this rational gives power p^{-n} so that one has two powers of p - p^{n_+} andp^{n_-}- since two infinite primes corresponding to fermionic vacua $X \pm 1$, where X is the product of all primes at given level of hierarchy, characterize the partonic 2-surface. The proposal inspired by the p-adicization program is that $\Delta\phi = 2\pi/p^n$ defines angle measurement resolution crucial in the construction of p-adic variants of WCW ("world of classical world") as a union of symmetric coset spaces by starting from discrete variants of the real counterpart of symmetric space having common points with tis p-adic variant. The two measurement resolutions correspond to CD and CP_2 degrees of freedom. The hierarchy of Planck constants generalizes imbedding space to a book like structure with pages identified in terms of singular coverings and factor spaces of CD and CP_2. There are good arguments suggesting that only coverings characterized by integers n_a and n_b are realized. The identifications $n_a = n_+$ and $n_b = n_-$ lead to highly non-trivial physical predictions and allow sharpen the view about the hierarchy of Planck constants. Therefore the notion of finite measurement resolution becomes the common denominator for the three threads of the number theoretic vision and give also a connection with the physics as infinite-dimensional geometry program and with the inclusions of hyper-finite factors defined by WCW spinor fields and proposed to characterize finite measurement resolution at quantum level.

Keywords: Infinite primes, arithmetic quantum field theory, second qantization, octonions, associativity, holography.

1 Introduction

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also for their complexifications and one can speak about infinite primes in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors.

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1.1 The notion of infinite prime

The original motivation for the notion of infinite prime came from the first attempts to construct TGD inspired theory of consciousness (around 1995) [5]. Suppose very naively that the 4-surfaces in a given sector of the "world of classical worlds" (WCW) are labelled by a fixed p-adic prime. The natural expectation is that evolution by quantum jumps means dispersion in the space of these sectors and leads to the increase of the p-adic prime characterizing the Universe. As one moves backwards in subjective time (sequence of quantum jumps) one ends up to the situation in which the prime characterizing the universe was $p = 2$. Should one assume that there was the first quantum jump when everything began? If not, then it would seem that the p-adic prime characterizing the Universe must be infinite. Second problem is that the p-adic length scales are finite and if the size scale of Universe is given by p-adic length scale the Universe has finite sized: this does not make sense in TGD framework. The only way out of the problems is the assumption that the p-adic prime characterizing the entire Universe is literally infinite and that p-adic primes characterizing space-time sheets are finite.

These argument, which are by no means central for the recent view about p-adic primes, motivated the attempt to construct a theory of infinite primes and to extend quantum TGD accordingly. This turns out to be possible. The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. It must be also emphasized that the notion of infinity is relativistic. With respect to the p-adic norm infinite primes have unit norm for all finite and infinite primes so that there is nothing to become scared of!

Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave fuctions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of imbedding space and space-time surface are subject to a number theoretic evolution. In philosophical mood one can of course also ask whether there exists a hierarchy of imbedding spaces in which the imbedding space at the lower level represents something with infinitesimal size in the sense of real topology and whether this hierarchy is accompanied also by a hierarchy of conscious entities.

This picture suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [26] providing a rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively. Same generalization could make sense for all classical number fields [22, 23, 24].

1.2 Infinite primes and physics in TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1.2.1 Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations [29] (for super-conformal invariance see [29] could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this manner.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and this cheated me to believe that this supersymmetry is completely absent in TGD Universe. The progress in the understanding of the properties of the modified Dirac action however led to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [10].

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [13] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [8].

1.2.2 Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers [25] suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

1.2.3 The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution[13], the dark matter hierarchy characterized by increasing values of $\hbar$ [12], the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory with finite measurement resolution.

1.2.4 Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. This conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

Quantum classical correspondence requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The modified Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries. The notion of finite measurement resolution allows to deduce much more detailed about this correspondence. In particular, the rational defined by the infinite prime classifies the finite sub-manifold geometry defined by the discretization of the partonic 2-surface implied by the finite measurement resolution. Also a direct correlation between integers defining Planck constant and the "fermionic" part of the infinite prime emerges.

1.3 Infinite primes, cognition, and intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. One can define the notion of prime also for the algebraic extensions of rationals. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum of infinitesimals (real zeros) is replaced by multiplication of real units meaning that the set of real and also more general units becomes infinitely degenerate.

3. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

4. In zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and could fix to a high degree the quantum number spectrum and correlations between various quantum numbers. It is not however clear whether the original identification of M^2 momenta as projections of four-momenta is right: Chern-Simons Dirac equation involves M^2 pseudo-momenta having identification in terms of hyper-octonionic primes [7].

5. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

2 Infinite primes, integers, and rationals

The definition of the infinite integers and rationals is a straightforward procedure and structurally similar to a repeated second quantization of a super-symmetric quantum field theory but including also the number theoretic counterparts of bound states.

2.1 The first level of hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1 + X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{2.1}$$

If P were divisible by finite prime then $P - X = 1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N = P - k$, $k > 1$, are certainly not primes since k can be taken as a factor. The number $P' = P - 2 = -1 + X$ could however be prime. P is certainly not divisible by $P - 2$. It seems that one cannot express P and $P - 2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p \in U} p + q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P-2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{aligned} &P(\pm,n) = \pm 1 + nX \ , \\ &k(p) = 0,1,..... \ , \\ &n = \prod_p p^{k(p)} \ , \\ &X = \prod_p p \ , \end{aligned} \tag{2.2}$$

where $k(p) \neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good canditates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{aligned} &P(\pm,m,n,s) = n\frac{X}{s} \pm ms \ , \\ &m = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{2.3}$$

Here $x|y$ means 'x divides y'. To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm,m,n,s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+,1,1,1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{aligned} &P(\pm,m,n,s|r) = nY^r \pm ms \ , \\ &Y = \frac{X}{s} \ , \\ &m = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{2.4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r=2$ case is provided by the following unsuccessful ansatz

$$\begin{aligned} &N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\ &Y = \frac{X}{s} \ , \\ &n_1m_2 - n_2m_1 = 0 \ . \end{aligned}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1,m_1) and (n_2,m_2) are linearly dependent as vectors. This encourages the guess that all other $r=2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n,m,r)(Y) = nY^r + m$ with integer valued coefficients $(n > 0)$ defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{array}{l} P(\pm, m, n, s|r_1, r_2) = nY^{r_1} \pm ms \ , \\ m = P_{r_2}(Y)Y + m_0 \ , \\ Y = \frac{X}{s} \ , \\ m_0 = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{2.5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1:th or r_2:th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm, m, n, s)$. $P(\pm, 1, 1, 1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm, m, n, s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm, m, n, s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+, m, n, s)$ or $P(-, m, n, s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

2.2 Infinite primes form a hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm, m, n, s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction receipe. In present case 'vacuum primes' at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm, m, n, s)} P(\pm, m, n, s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{2.6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm, m, n, s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type 'bosons' respectively. The non-negative integer-valued function $K(P) = K(\pm, m, n, s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type 'bosons'. More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm, m_1, n_1, s_1 K_1, S_1}{P_2(\pm, m_2, n_2, s_2, K, S_2)} = \frac{n_1 s_2}{n_2 s_1} \prod_{\pm, m, n, s} (\frac{n}{s})^{K_1^+(\pm, n, m, s) - K_2^+(\pm, n, m, s)} \ . \tag{2.7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1 P_2 \to P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

2.3 Construction of infinite primes as a repeated quantization of a super-symmetric arithmetic quantum field theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actually use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the 'vacuum' X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ 'p-bosons' in mode of type X/s and multiplication of the 'vacuum' s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ 'p-bosons'. in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{2.8}$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P \, mod\, 4 = 3$ and $P \, mod\, 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,\ldots,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about ...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

2.4 Construction in the case of an arbitrary commutative number field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [14] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [14]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [14]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

2.5 Mapping of infinite primes to polynomials and geometric objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \ . \tag{2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

2.6 How to order infinite primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the 'large' and the 'small' part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

2.7 What is the cardinality of infinite primes at given level?

The basic problem is to decide whether Nature allows also integers S , $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

2.8 How to generalize the concepts of infinite integer, rational and real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

2.8.1 Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N \;=\; \prod_k p_k^{n_k} = nM \;\;,\;\; n_k \geq 0 \;, \tag{2.10}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_iM_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N \;=\; m_0 + \sum m_iM_i \;. \tag{2.11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \quad (2.12)$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \quad (2.13)$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultrametricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

2.8.2 Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \quad (2.14)$$

2.8.3 Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \quad (2.15)$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \quad (2.16)$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part.* Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N> \ . \quad (2.17)$$

The natural inner product is

$$\langle X, Y\rangle = x_0 y_0 + \sum_N x_N y_N \ . \quad (2.18)$$

The inner product is well defined if the number of N:s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X + Y = x_0 + y_0 + \sum_N (x_N + y_N) p^{-N} \ , \tag{2.19}$$

The product XY is expressible in the form

$$XY = x_0 y_0 + x_0 Y + X y_0 + \sum_{N_1, N_2} x_{N_1} y_{N_2} p^{-N_1 - N_2} \ , \tag{2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums $N_1 + N_2$ by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \ \rightarrow \ x_p = \sum_k x_k p^k \ ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \ \rightarrow \ (x_0)_p + \sum_N (x_N)_p p^N \ , \tag{2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \ \rightarrow \ x_0 + \sum_N x_N p_2^{-N} \ . \tag{2.22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional configuration space of 3-surfaces, or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

2.9 Comparison with the approach of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement 'Set is Many allowing to regard itself as One' really means and to the fact that there is no obvious connection with physics.

The proposed approach is based on the introduction of the concept of prime as a basic concept whereas partial ordering is based on the use of ratios: using these one can recursively define partial ordering and get precise quantitative information based on finite reals. The ordering is only partial and there is infinite number of ratios of infinite integers giving rise to same real unit which in turn leads to the idea about number theoretic anatomy of real point.

The 'Set is Many allowing to regard itself as One' is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as 'One' and its decomposition to a product of primes corresponds to the set as 'Many'. The concept of prime, the ultimate 'One', has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

3 Can one generalize the notion of infinite prime to the non-commutative and non-associative context?

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [16, A5] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [16, A5]. Also the hierarchy of infinite primes should generalize as also the representation of infinite primes as polynomials although associativity is expected to pose technical problems.

3.1 Quaternionic and octonionic primes and their hyper counterparts

The loss of commutativity and associativity implies that the definitions of quaternionic and octonionic primes are not completely straightforward.

3.1.1 Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many manners and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio 3 : 1 [30]. For other Lie-groups this ratio is either 2:1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

3.1.2 Quaternionic and octonionic primes

Quaternionic primes with $p\ mod\ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p\ mod\ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p\ mod\ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p\ mod\ 4 = 3$ with norm $z\bar{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

3.1.3 Hyper primes

The basic objection against the notion of hyper-octonionic (and also -quaternionic) primes is that hyper-octonions define only a sub-space of the algebra defined by complexified octonions and that their products do not belong to this sub-space. This difficulty can be circumvented by defining hyper-octonionic primes as orbits of hyper-complex primes under a discrete subgroup of the automorphism group G_2 of octonions respecting the complexified octonionic integer property. Hyper-complex numbers define a commutative algebra and the product $(x+iIy)(x-iIy) = x^2-y^2$ has norm property and equals to the the square of Minkowski metric. Non-commutativity however prevents the generalization of this norm so that it would make sense for complexified octonions. Same argument applies to quaternions. Therefore the approach of a mathematician based on the notion of prime ideal does not work at the level of complexified octonions and quaternions.

Accepting this definition, all reduces to the case of hyper-complex primes. The hyper-complex primes come in two types depending on whether the imaginary part is vanishing or not.

1. For $n_3 = 0$ primeness property reduces to $n_0 = p$, where p is an ordinary prime. These primes are analogous to momenta for particles at rest. It is safer to talk about pseudo-momenta rather than actual momenta and it indeed turns out that the identification as a real momentum is not plausible [7, A3].

2. For $n_3 \neq 0$ the factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic prime has one particular representative as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...) = ((p + 1)/2, (p - 1)/2$, for $p > 2$. These primes are analogous to momenta of particles, which cannot be brought to rest although they are massive. For large values of p these particles are highly relativistic.

3. These two kinds of primes correspond to the primary and secondary p-adic length scales. The primary p-adic length scales are proportional to $\sqrt{p}$ and are assigned by p-adic mass calculations to elementary particles whereas secondary p-adic length scales are are proportional to p and define in excellent approximation size scales for CDs for $p \simeq 2^k$.

4. $p = 2$ is exceptional in the sense that it does not reduce to hyper-complex prime with $n_3 \neq 0$. The representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. $p = 2$ however can correspond prime $(2, 0)$ and thus to a particle at rest.

The introduction of a preferred hyper-complex plane necessary for several reasons- in particular for the possibility to identify standard model quantum numbers in terms of infinite primes- allows to identify the momentum of particle in the preferred plane as the first two components of the hyper prime in fixed coordinate frame. Note that this leads to a universal spectrum for what might have interpretation as mass squared. In the following I will mostly talk about mass and momentum but one must keep in mind that pseudo-mass and pseudo-momentum could be in question. The generalized eigenstates of Chern-Simons Dirac operator indeed lead to M^2-valued pseudo-momentum spectrum having interpretation in terms of hyper-octonionic primes [7].

For time like hyper-primes the momentum is always time like for hyper-primes. In this case it is possible to find a rest frame by applying a hyper-primeness preserving G_2 transformation so that the resulting momentum has no component in the preferred frame. As a matter fact, SU(3) rotation is enough for a suitable choice of SU(3). These transformations form a discrete subgroup of SU(3) since hyper-integer property must be preserved. Massless states correspond to a null norm for the corresponding hyper integer unless one allows also tachyonic hyper primes with minimal representatives $(n_3, n_3 - 1, 0, ...)$, $n_3 = (p - 1)/2$. Note that Gaussian primes with $p \ mod 4 \ = 1$ are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$ and would correspond to genuine tachyons. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

The notion of "irreducible" (see Appendix of [14]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7, 1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

3.2 Hyper-octonionic infinite primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance. It might be however not possible to interpret them as as 8-momenta with mass squared equal to prime. It is however far from clear with this momentum corresponds to the ordinary 4-momentum. The proper identification of standard model quantum numbers will be discussed later.

3.2.1 Should infinite primes be commutative and associative?

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes. Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

The original idea was that infinite hyper-octonionic primes could be mapped to polynomials and one could assign to these space-time surfaces in analogy with the identification of surfaces as zero locii of polynomals. Although this idea has been given up, it is good to make clear its problematic aspects.

1. The sums of products of monomials of generating infinite primes define higher level infinite primes and also here non-commutativity and associativity cause potential technical difficulties. The assignment of a monomial to a quaternionic or octonionic infinite prime is not unique since the rational obtained by dividing the finite part mr with the integer n associated with infinite part can be defined either as $(1/n) \times mr$ or $mr \times (1/n)$ and the resulting non-commuting rationals are different.

2. If the polynomial associated with infinite prime has real-rational coefficients, these difficulties do not appear. The problem is that the polynomials as such would not contain information about the number field in question.

3. Commutativity requirement for infinite primes allows real-rationals or possibly algebraic extensions of them as the coefficients of the polynomials formed from hyper-octonionic infinite primes. If only infinite primes with complex rational coefficients are allowed and only the vacuum state $V_{\pm} = X \pm 1$ involving product over all primes of the number field, would reveal the number field. One could thus construct the generating infinite primes using the notion of hyper-octonionic prime for any algebraic extension of rationals.

The idea about mapping of infinite primes to polynomials in turn defining space-time surfaces is non-realistic. The recent view is more abstract and based on the mapping of wave functions in the space of hyper-octonion units assignable to single imbedding space point by its number-theoretic anatomy and a further mapping of quantum numbers to the geometry of space-time surface by the coupling of the modified Dirac action to the quantum numbers via measurement interaction. In this approach one cannot assume commutatitivity of hyper-octonionic primes at any level. The problems due to non-commutativity and non-associativity are however circumvented by assuming that permutations and associations of are represented as phase factors and therefore do not change the quantum state. This means the introduction of association statistics besides permutation statistics. Besides Fermi and Bose statistics one can consider braid statistics. Note that Fermi statistics makes sense only when the fermionic finite primes appearing in the state do not commute.

3.2.2 The construction recipe for hyper-octonionic infinite primes

The following argument represents the construction recipe for the first level hyper-octonionic primes without the restriction to rational infinite primes. If the reduction is possible always by a suitable G_2 rotation then the construction of the infinite primes analogous to bound states is obtained in trivial manner from that for rational variants of these primes. The recipe generalizes to the higher levels in trivial manner.

Each hyper-octonionic prime has a number of conjugates obtained by applying transformations of G_2 respecting the property of being hyper-octonionic integer.

1. The number of conjugates of given finite prime depends on the number of non-vanishing components of the the prime with norm p in the minimal representation having minimal energy. Several primes with a given norm p not related by a multiplication with unit or by automorphism are in principle possible. The degeneracy is determined by the number of elements of a subgroup of Galois group acting non-trivially on the prime.

   Galois group contains the permutations of 7 imaginary units and 7 conjugations of units consistent with the octonionic product. X is proportional to $p^{N(p)}$ where $N(p)$ in principle depends on p.

   There could exist also G_2 transformations which change the number of components of the infinite prime. They satisfy tight number theoretical constraints since the quantity $\sum_{i=1}^{7} n_i^2$ must be preserved. For instance, for the transformation from standard form with two components to that with more than two components one has $n_1^2(i) = \sum_k n_k^2(f)$. For the transformation from 2-component prime to 3-component prime one has a condition characterizing Pythagorean triangle. One can however consider also a situation when no such G_2 transformation exist so that one has several G_2 orbits corresponding to the same rational prime.

The construction itself would be relatively straightforward. Consider first the construction of the "vacuum" primes.

1. In the case of ordinary infinite primes there are two different vacuum primes $X \pm 1$. This is the case also now. I turns out that this degeneracy corresponds to the spin and orbital degrees of freedom for the spinor fields of WCW.

2. The product X of all hyper-octonionic irreducibles can be regarded as the counterpart of Dirac vacuum in a rather concrete sense. Moreover, in the hyper-quaternionic and octonionic case the norm of X is analogous to the Dirac determinant of a fermionic field theory with prime valued (pseudo-)mass spectrum and integer valued momentum components. The inclusion of only irreducible eliminates from the infinite product defining Dirac determinant product over various Lorentz boosts of $p^k\gamma_k - m$.

3. Infinite prime property requires that X must be defined by taking one representative from each G_2 equivalence class representing irreducible and forming the product of all its G_2 conjugates. The standard representative for the hyper-octonionic primes can be taken to be time-like positive energy prime unless one allows also tachyonic primes in which case a natural representative has a vanishing real component. The conjugates of each irreducible appear in X so for a given norm p the net result is real for each rational prime p.

The construction of non-vacuum primes is equally straighforward.

1. If the conjectured effective 2-dimensionality holds true, it is enough to construct hyper-complex primes first. To the finite hyper-complex primes appearing in these infinite primes one can apply transformations of G_2 mapping hyper-octonionic integers to hyper-octonionic integers. The infinite prime would have degeneracy defined by the product of G_2 orbits of finite primes involved. Every finite prime would be like particle possessing finite number of quantum states. If there are several G_2 orbits corresponding to the same finite prime exist they must be also included and the conjectured effective 2-dimensionality fails.

2. An interesting question is what happens when the finite part of an infinite prime is multiplied by light like integer k. The first guess is that k describes the presence of a massless particle. If the resulting infinite integer is multiplied with conjugates $k_{c,i}$ of k an integer of form $\prod_i k_{c,i} mX/n$ having formally zero norm results. It would thus seem that there is a kind of gauge invariance in the sense that infinite primes for which both finite and infinite part are multiplied with the same light-like primes, are divisors of zero and correspond to gauge degrees of freedom. This conclusion is supported by the interpretation of the projection of infinite prime to the preferred hyper-complex plane as momentum of particle in a preferred M^2 plane assigned by the hierarchy of Planck constants to each CD and also required by the p-adicization.

3. More complex infinite hyper-octonionic primes can be constructed from rational hyper-complex and complex infinite primes using a representation in terms of polynomials and then acting on the finite primes appering in their expression by elements of G_2 preserving integer property. This construction works at all levels of the hierarchy and one might hope that it is all that is needed. If there are several G_2 orbits for given finite prime p one encounters a problem since hyper-octonionic primes with more than 2 components do not allow associative and commutative polynomial representations. The interpretation as bound states is suggestive.

4 How to interpret the infinite hierarchy of infinite primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

4.1 Infinite primes and hierarchy of super-symmetric arithmetic quantum field theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. It turns also possible to assign to infinite prime space-time surface as a geometric correlate although the original proposal for how to achieve this failed. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-symplectic representations (for instance, ordinary particles correspond to states of this kind).

4.1.1 Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ .$$

2. Form the vacuum states

$$V_\pm = X \pm 1 \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \to X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \to mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

4.1.2 More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed. As found, the conjectured effective 2-dimensionality for hyper-octonionic primes allows the reduction of polynomial representation of hyper-octonionic primes to that for hyper-complex primes. This would be in accordance with the effective 2-dimensionality of the basic objects of quantum TGD.

The physical counterpart of n:th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

4.1.3 How infinite rationals correspond to quantum states and space-time surfaces?

The most promising answer to the question how infinite rationals correspond to space-time surfaces is discussed in detail in the next section. Here it is enough to give only the basic idea.

1. In zero energy ontology hyper-octonionic units (in real sense) defined by ratios of infinite integers have an nterpretation as representations for pairs of positive and negative energy states. Suppose that the quantum number combinations characterizing positive and negative energy quantum states are representable as superpositions of real units defined by ratios of infinite integers at each point of the space-time surface. If this is true, the quantum classical correspondence coded by the measurement interaction term of the modified Dirac action maps the quantum numbers also to space-time geometry and implies a correspondence between infinite rationals and space-time surfaces.

2. The space-time surface associated with the infinite rational is in general not a union of the space-time surfaces associated with the primes composing the integers defining the rational. There the classical description of interactions emerges automatically. The description of classical states in terms of infinite integers would be analogous to the description of many particle states as finite integers in arithmetic quantum field theory. This mapping could in principle make sense both in real and p-adic sectors of WCW.

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime.

4.1.4 What is the interpretation of the higher level infinite primes?

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

4.2 Infinite primes, the structure of many-sheeted space-time, and the notion of finite measurement resolution

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in a rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface. It turns out that the notion of finite measurement resolution is the key concept: infinite prime characterizes angle measurement resolution. This gives a direct connection with the p-adicization program relying also on angle measurement resolution as well as a connection with the hierarchy of Planck constants. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory.

4.2.1 The first intuitions

The concrete prediction of the general vision is that the hierarchy of infinite primes should somehow correspond to the hierarchy of space-time sheets or partonic 2-surfacse if one accepts the effective 2-dimensionality. The challenge is to find space-time counterparts for infinite primes at the lowest level of the hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by join along boundaries bonds to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at the lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on... . A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles. The concrete realization of multi-p p-adicity would be in terms of infinite integers coming as power series $\sum x_n N^n$ and having interpretation as p-adic numbers for any prime dividing N.

2. Effective 2-dimensionality would suggest that the individual p-adic topologies could be assigned with the 2-dimensional partonic surfaces. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. There are however reasons to think that even single partonic 2-surface corresponds to a multi-p p-adic topology.

4.2.2 Do infinite primes code for the finite measurement resolution?

The above describe heuristic picture is not yet satisfactory. In order to proceed, it is good to ask what determines the finite prime or set of them associated with a given partonic 2-surface. It is good to recall first the recent view about the p-adicization program relying crucially on the notion of finite measurement resolution.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M and measurement resolution does not depend on on the value of M. Situation is different if one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer. For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$ if modular arithetmics is used to define the the measurement resolution. This works at the first level of the hierarcy but not at higher levels. The alternative manner to assign a finite measurement resolution to M/N for given p is as $\Delta\phi = 2\pi|N/M|_p = 2\pi/p^n$. In this case the small fermionic part of the infinite prime would fix the measurement resolution. The argument below shows that only this option works also at the higher levels of hierarchy and is therefore more plausible.

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3. p-Adicization conditions in their strong form require that the notion of integration based on harmonic analysis [20] in symmetric spaces [19] makes sense even at the level of partonic 2-surfaces. These conditions are satisfied if the partonic 2-surfaces in a given measurement resolution can be regarded as algebraic continuations of discrete surfaces whose points belong to the discrete variant of the $\delta M^4_\pm \times CP_2$. This condition is extremely powerful since it effectively allows to code the geometry of partonic 2-surfaces by the geometry of finite sub-manifold geometries for a given measurement resolution. This condition assigns the integer N to a given partonic surface and all primes appearing as factors of N define possible effective p-adic topologies assignable to the partonic 2-surface.

How infinite primes could then code for the finite measurement resolution? Can one identify the measurement resolution for $M/N = M/(Rp^n)$ as $\Delta\phi = ((M/R)\ mod\ p^n) \times 2\pi/p^n$ or as $\Delta\phi = 2\pi/p^n$? The following argument allows only the latter option.

1. Suppose that p-adic topology makes sense also for infinite primes and that state function reduction selects power of infinite prime P from the product of lower level infinite primes defining the integer N in M/N. Suppose that the rational defined by infinite integer defines measurement resolution also at the higher levels of the hierarchy.

2. The infinite primes at the first level of hierarchy representing Fock states are in one-one correspondence with finite rationals M/N for which integers M and N can be chosen to characterize the infinite bosonic part and finite fermionic part of the infinite prime. This correspondence makes sense also at higher levels of the hierarchy but M and N are infinite integers. Also other option obtained by exchanging "bosonic" and "fermionic" but later it will be found that only the first identification makes sense.

3. The first guess is that the rational M/N characterizing the infinite prime characterizes the measurement resolution for angles and therefore partially classifies also the finite sub-manifold geometry assignable to the partonic 2-surface. One should define what $M/N = ((M/R)\ mod\ P^n) \times P^{-n}$ is for infinite primes. This would require expression of M/R in modular arithmetics modulo P^n. This does not make sense.

4. For the second option the measurement resolution defined as $\Delta\phi = 2\pi|N/M|_P = 2\pi/P^n$ makes sense. The Fourier basis obtained in this manner would be infinite but all states $exp(ik/P^n)$ would correspond in real sense to real unity unless one allows k to be infinite P-adic integer smaller than P^n and thus expressible as $k = \sum_{m<n} k_m P^m$, where k_m are infinite integers smaller than P. In real sense one obtains all roots $exp(iq2\pi)$ of unity with $q < 1$ rational. For instance, for $n = 1$ one can have $0 < k/P < 1$ for a suitably chosen infinite prime k. Thus one would have essentially continuum theory at higher levels of the hierarchy. The purely fermionic part N of the infinite prime would code for both the number of Fourier components in discretization for each power of prime involved and the ratio characterize the angle resolution.

The proposed relation between infinite prime and finite measurement resolution implies very strong number theoretic selection rules on the reaction vertices.

1. The point is that the vertices of generalized Feyman diagrams correspond to partonic 2-surfaces at which the ends of light-like 3-surfaces describing the orbits of partonic 2-surfaces join together. Suppose that the partonic 2-surfaces appearing a both ends of the propagator lines correspond to same rational as finite sub-manifold geometries. If so, then for a given p-adic effective topology the integers assignable to all lines entering the vertex must contain this p-adic prime as a factor. Particles would correspond to integers and only the particles having common prime factors could appear in the same vertex.

2. In fact, already the work with modelling dark matter [12] led to ask whether particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions. It also seemed natural to assume that that only the space-time sheets containing common primes in this collection can interact. This inspired the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The possibility of multi-p p-adicity raises the question about how to fix the p-adic prime characterizing the mass of the particle. The mass scale of the contribution of a given throat to the mass squared is given by $p^{-n/2}$, where $T = 1/n$ corresponds to the p-adic temperature of throat. Hence the dominating contribution to the mass squared corresponds to the smallest prime power p^n associated with the throats of the particle. This works if the integers characterizing other particles than graviton are divisible by the gravitonic p-adic prime or a product of p-adic primes assignable to graviton. If the smallest power p^n assignable to the graviton is large enough, the mass of graviton is consistent with the empirical bounds on it. The same consideration applies in the case of photons. Recall that the number theoretically very natural condition that in zero energy ontology the number of generalized Feynman graphs contributing to a given process is finite is satisfied if all particles have a non-vanishing but arbitrarily small p-adic thermal mass [7].

4.2.3 Interpretational problem

The identification of infinite prime as a characterizer of finite measurement resolution looks nice but there is an interpretational problem.

1. The model characterizing the quantum numbers of WCW spinor fields to be discussed in the next section involves a pair of infinite primes P_+ and P_- corresponding to the two vacuum primes $X \pm 1$. Do they correspond to two different measurement resolutions perhaps assignable to CD and CP_2 degrees of freedom?

2. Different measurement resolutions in CD and CP_2 degrees of freedom need not be not a problem as long as one considers only the discrete variants of symmetric spaces involved. What might be a problem is that in the general case the p-adic primes associated with CD and CP_2 degrees of freedom would not be same unless the integers N_+ and N_- are assumed to have have same prime factors (they indeed do if $p^0 = 1$ is formally counted as prime power factors).

3. The idea of assigning different p-adic effective topologies to CD and CP_2 does not look attractive. Both CD and CP_2 and thus also partonic 2-surface could however possess simultaneously both p-adic effective topologies. This kind of option might make sense since the integers representable as infinite powers series of integer N can be regarded as p-adic integers for all prime factors of N. As a matter fact, this kind of multi-p p-adicity could make sense also for the partonic 2-surfaces characterized by a measurement resolution $\Delta\phi = 2\pi M/N$. One would have what might be interpreted as N_+N_--adicity.

4. It will be found that quantum measurement means also the measurement of the p-adic prime selecting same p-adic prime from N_+ and N_-. If $N_\pm$ is divisible only by $p^0 = 1$, the corresponding angle measurement resolution is trivial. From the point of view of consciousness state function reduction selects also the p-adic prime characterizing the cognitive representation which is very natural since quantum superpositions of different p-adic topologies are not natural physically.

4.3 How the hierarchy of Planck constants could relate to infinite primes and p-adic hierarchy?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution[13], the dark matter hierarchy characterized by increasing values of $\hbar$ [12], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations give hopes about developing a more quantitative vision about the relationship between these hierarchies, in particular between the hierarchy of infinite primes, p-adic length scale hierarchy, and the hierarchy if Planck constants.

If infinite primes code for the hierarchy of measurement resolutions, the correlations between the p-adic hierarchy and the hierarchy of Planck constants indeed suggest themselves and allow also to select between two interpretations for the fact that two infinite primes N_+ and N_- are needed to characterize elementary particles (see the next section).

Recall that the hierarchy of Planck constants in the most general situation corresponds to a replacement M^4 and CP_2 factors of the imbedding space with singular coverings and factor spaces. The condition that Planck constant is integer valued allows only singular coverings characterized by two integers n_a *resp.* n_b assignable to CD *resp.* CP_2. This condition also guarantees that a given value of Planck constant corresponds to only a finite number of pages of the "Big Book" and therefore looks rather attractive mathematically. This option also forces evolution as a dispersion to the pages of the books characterized by increasing values of Planck constant.

Concerning the correspondence between the hierarchy of Planck constants and p-adic length scale hierarchy there seems to be only single working option. The following assumptions make precise the relationship between finite measurement resolution, infinite primes and hierarchy of Planck constants.

1. Measurement resolution CD *resp.* CP_2 degrees of freedom is assumed to correspond to the rational M_+/N_+ *resp.* M_-/N_-. $N_\pm$ is identified as the integer assigned to the fermionic part of the infinite integer..

2. One must always fix the consideration to a fixed p-adic prime. This process could be regarded as analogous to fixing the quantization axes and p would also characterize the p-adic cognitive space-time sheets involved. The p-adic prime is therefore same for CD and CP_2 degrees of freedom as required by internal consistency.

3. The relationship to the hierarchy of Planck constants is fixed by the identifications $n_a = n_+(p)$ and $n_b = n_-(p)$ so that the number of sheets of the covering equals to the number of bosons in the fermionic mode p of the quantum state defined by infinite prime.

4. A physically attractive hypothesis is that number theoretical bosons *resp.* fermions correspond to WCW orbital *resp.* spin degrees of freedom. The first ones correspond to the symplectic algebra [21] of WCW and the latter one to purely fermionic degrees of freedom.

Consider now the basic consequences of these assumptions from the point of view of physics and cognition.

1. Finite measurement resolution reduces for a given value of p to

$$\Delta\phi = \frac{2\pi}{p^{n_\pm(p)+1}} = \frac{2\pi}{p^{n_{a/b}}} ,$$

where $n_\pm(p) = n_{a/b} - 1$ is the number of bosons in the mode p in the fermionic part of the state. The number theoretical fermions and bosons and also their probably existing physical counterparts are necessary for a non-trivial angle measurement resolution. The value of Planck constant given by

$$\frac{\hbar}{\hbar_0} = n_a n_b = (n_+(p) + 1) \times (n_-(p) + 1)$$

tells the total number of bosons added to the fermionic mode p assigned to the infinite prime.

2. The presence of $\hbar > \hbar_0$ partonic 2-surfaces is absolutely essential for a Universe able to measure its own state. This is in accordance with the interpretation of hierarchy of Planck constants in TGD inspired theory of consciousness. One can also say that $\hbar = 0$ sector does not allow cognition at all since $N_\pm = 1$ holds true. For given p $\hbar = n_a n_b = 0$ means that given fermionic prime corresponds to a fermion in the Dirac sea meaning $n_\pm(p) = -1$. Kicking out of fermions from Direac sea makes possible cognition. For purely bosonic vacuum primes one has $\hbar = 0$ meaning trivial measurement resolution so that the physics is purely classical and would correspond to the purely bosonic sector of the quantum TGD.

3. For $\hbar = \hbar_0$ the number of bosons in the fermionic state vanishes and the general expression for the measurement resolution reduces to $\Delta\phi = 2\pi/p$. When one adds $n_\pm(p)$ bosons to the fermionic part of the infinite prime, the measurement resolution increases from $\Delta\phi = 2\pi/p$ to $\Delta\phi = 2\pi/p^{n_\pm(p)+1}$. Adding a sheet to the covering means addition of a number theoretic boson to the fermionic part of infinite prime. The presence of both number theoretic bosons and fermions with the values of p-adic prime $p_1 \neq p$ does not affect the measurement resolution $\Delta\phi = 2\pi/p^n$ for a given prime p.

4. The resolutions in CD and CP_2 degrees of freedom correspond to the same value of the p-adic prime p so that one has dicretizations based on $\Delta\phi = 2\pi/p^{n_a}$ in CD degrees of freedom and $\Delta\phi = 2\pi/p^{n_b}$ in CP_2 degrees of freedom. The finite sub-manifold geometries make sense in this case and since the effective p-adic topology is same, the continuation to continuous p-adic partonic 2-surface is possible.

p-Adic thermodynamics involves the p-adic temperature $T = 1/n$ as basic parameter and the p-adic mass scale of the particle comes as $p^{-(n+1)/2}$. The natural question is whether one could assume the relation $T_\pm = 1/(n_\pm(p) + 1)$ between p-adic temperature and infinite prime and thus the relations $T_a = 1/n_a(p)$ and $T_b = 1/n_b(p)$. This identification is not consistent with the recent physical interpretation of the p-adic thermodynamics nor with the view about dark matter hierarchy and must be given up.

1. The minimal non-trivial measurement resolution with $n_i = 1$ and $\hbar = \hbar_0$ corresponds to the p-adic temperature $T_i = 1$. p-Adic mass calculations indeed predict $T = 1$ for fermions for $\hbar = \hbar_0$. In the case of gauge bosons $T \geq 2$ is favored so that gauge bosons would be dark. This would require that gauge bosons propagate along dark pages of the Big Book and become "visible" before entering to the interaction vertex.

2. p-Adic thermodynamics also assumes same p-adic temperature in CD and CP_2 degrees of freedom but the proposed identification allows also different temperatures. In principle the separation of the super-conformal degrees of freedom of CD and CP_2 might allow different p-adic temperatures. This would assign to different p-adic mass scales to the particles and the larger mass scale should give the dominant contribution.

3. For dark particles the p-adic mass scale would be by a factor $1/\sqrt{p}^{n_i(p)-1}$ lower than for ordinary particles. This is in conflict with the assumption that the mass of the particle does not depend on $\hbar$. This prediction would kill completely the recent vision about the dark matter.

5 How infinite primes could correspond to quantum states and space-time surfaces?

The hierarchy of infinite primes is in one-one correspondence with a hierarchy of second quantizations of an arithmetic quantum field theory. The additive quantum number in question is energy like quantity for ordinary primes and given by the logarithm of prime whereas p-adic length scale hypothesis suggests that the conserved quantity is proportional to the inverse of prime or its square root. For infinite primes at the first level of hierarchy these quantum numbers label single particles states having interpretation as ordinary elementary particles. For octonionic and hyper-octonionic primes the quantum number is analogous to a momentum with 8 components. The question is whether these number theoretic quantum numbers could have interpretation as genuine quantum numbers. Quantum classical correspondence raises another question. Is it possible to label space-time surfaces by infinite primes? Could this correspondence be even one-to-one?

I have considered these questions already more than decade ago. The discussion at that time was necessarily highly speculative and just a mathematical exercise. After that time however a lot of progress has taken place in quantum TGD and it is highly interaction to see what comes out from the interaction of the notion of infinite prime with the notions of zero energy ontology and generalized imbedding space, and with the recent vision about how measurement interaction in the modified Dirac action allows to code information about quantum numbers to the space-time geometry. The possibility of this coding allows to simplify the discussion dramatically. If one can map infinite hyper-octonionic primes to quantum numbers of the standard model naturally, then the their map of to the geometry of space-time surfaces realizes the coding of space-time surfaces by infinite primes (and more generally by integers and rationals). Also a detailed realization of number theoretic Brahman=Atman identity emerges as an outcome.

5.1 A brief summary about various moduli spaces and their symmetries

It is good to sum up the number theoretic symmetries before trying to construct an overall view about the situation. Several kinds of number theoretical symmetry groups are involved corresponding to symmetries in the moduli spaces of hyper-octonionic and hyper-quaternionic structures, symmetries mapping hyper-octonionic primes to hyper-octonionic primes, and translations acting in the space of causal diamonds (CDs) and shifting. The moduli space for CDs labeled by pairs of its tips that its pairs of points of $M^4 \times CP_2$ is also in important role.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit. $SO(7,1)$ acts as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for hyper-quaternionic structures.

2. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

3. Color group $SU(3)$ is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. For given hyper-octonionic prime one can identify a subgroup of SU(3) generating a finite set of hyper-octonionic primes for it at sphere S^7. This suggests wave function at the orbit of given hyper-octonionic prime in turn generalizing to wave functions in the space of infinite primes.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

These symmetries deserve a more detailed discussion.

1. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures. $SO(7)$ respects the choice of the real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as sub-structures of hyper-octonionic structure. The choice of global hyper-octonionic structures involves also a choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of CD. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3)\times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture. The choice of both tips of CD reduces $SO(1,3)$ to $SO(3)$.

2. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

3. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit meaning a choice of preferred hyper-complex plane $M^4 \subset M^4$. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement in number theoretical and as it turns also in physical sense. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

5.2 Associativity and commutativity or only their quantum variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Hyper-quaternionicity formulated in terms of the modified gamma matrices defined by Kähler action fixes classical space-time dynamics and a very beautiful algebra formulation of quantum TGD in terms of the complexified local Clifford algebra of imbedding space emerges.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to complex rational infinite primes. Since one can decompose complex rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative and commutative. In case of associativity this means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

5.3 The correspondence between infinite primes and standard model quantum numbers

I have considered several candidates for the correspondence between infinite primes and standard model quantum numbers. The confusing aspect has been the dual nature of hyper-octonionic primes. One one hand they could be interpreted as components of 8-D momentum representing perhaps momentum and other quantum numbers. On the other hand, they transform like representations of $SU(3) \subset G_2$ and behave like color singlets and triplets so that the idea about quantum superpositions of infinite primes related by $SU(3)$ action is attractive. The second puzzling feature is that there are two kinds of infinite primes corresponding to two signs for the "small" part of the infinite prime. The following proposal leads to an interpretation for these aspects.

1. The number of components of hyper-octonionic prime is 8 as is the dimension of the Cartan algebra of the product of Poincare group, color group SU(3) and electro-weak gauge group $SU(2)_L \times U(1)$ defining the quantum numbers of particles. One might therefore dream about a number theoretic interpretation of elementary particle quantum numbers by intepreting hyper-octonionic prime as 8-momentum. This form of the big idea fails. The point is that complexified basis for octonions consists of two color singlets and color triplet and its conjugate. For a given hyper-octonionic prime one can construct new primes by using a subgroup G of SU(3) by definition respecting the property that the values of the components of prime as integers and as a consquence also the modulus squared so that the primes are at sphere S^7. This group is analogous to Galois group. Identifying prime as an element of basis of quantum states, one can form wave functions at the discrete orbit of given prime transforming according to irreducible representations of color group. Triality $t{\pm}1$ states correspond to color partial waves associated with quarks and antiquarks and triality $t = 0$ states to gluons and leptons and their color excitations. The states can be chosen to be eigenstates of the preferred hyper-octonionic imaginary unit ie_1. Additive four-momentum could be assigned the M^2 part of the hyper-octonion as will be found. Therefore the construction applies in special but natural coordinates assignable to the particle required also by zero energy ontology and hierarchy of Planck constants as well as by p-adicization program.

2. This construction gives only the quantum numbers assignable to color partial waves in configuration space degrees of freedom. Also the quantum numbers assignable to imbedding space spinors are wanted. Luckily, there are two kinds of infinite primes, which might be denoted by $P_{\pm}$ because the sign of the "small" part of the infinite prime can be chosen freely. Super-conformal symmetry [27] suggests that quantum numbers associated with spinorial and configuration space degrees freedom can be assigned to the infinite primes of these two types.

   (a) In the case of spinor degrees of freedom one can restrict the multiplets to those generated by $SU(2)$ subgroup of SU(3) identified as rotation group. The interpretation is in terms of automorphism group of quaternions. Discrete subgroups of SU(2) generate the orbit of given hyper-octonionic prime and one obtains finite number of SU(2) multiplets having interpretation in terms of rotational degrees of freedom associated with the light-cone boundary. In the case of fermions (bosons) only half odd integer (integer) spins are allowed.

   (b) Remarkably, four of the hyper-octonionic units remain invariant under $SU(2)$. Also now only the hyper-complex projection in $M^2 \subset M^4$ can be interpreted as four-momentum in the preferred frame and the interpretation as a counterpart of Dirac equation eliminating four complex non-physical helicities of the imbedding spinor of given chirality. The states of same spin associated with the two spin doublets have interpretation as electro-weak doublets. As a representation of $SU(3)$ electro-weak doublets would correspond to quark and antiquark in color isospin doublet. This leaves two additional quantum numbers assignable to the color isospin singlets. The natural interpretation is in terms of electromagnetic charge and weak isospin. An analogous picture emerges also in the description of super-symmetric QFT limit of TGD [10] replacing massless particles identified as light-like geodesics of M^4 with light like geodesics of $M^4 \times CP_2$ and assigning to them two quantum numbers in the Cartan algebra of $SU(3)$ and identified as electro-weak charges. Also conformal weight expressible in terms of stringy mass formula allows a description in terms of infinite primes. What is not achieved is the number theoretical description of genus of the partonic 2-surface and wave functions in the moduli space of the partonic 2-surfaces.

3. In this picture leptons, gauge bosons, and gluons correspond to an infinite prime of type P_+ or P_- whereas quarks as well as color excitations of leptons correspond to a pair of primes of type P_+ and P_-. One can fix the notations by assigning color quantum numbers to P_+ and and spinorial quantum numbers to P_-. Both P_+ and P_- contribute to four-momentum. Each pair of infinite primes of this kind defines a finite-dimensional space of quantum states assignable to the subgroups of $SU(3)$ and $SU(2)$ respecting the prime property. Needless to say, this prediction is extremely powerful and fixes the spectrum of the quantum numbers almost completely!

4. An interesting question is whether one can require number theoretical color confinement in the sense that the physical states resulting as tensor products of states assignable to a given infinite prime in P_+ are color singlets. This might be necessary to guarantee associativity. G_2 singletness would be even stronger condition but not possible for massless states. What is interesting is that spin and color in well-defined sense separate

from each other. One can wonder whether this relates somehow to the spin puzzle of proton meaning that quarks do not seem to contribute to baryonic spin.

5. The appearance of discrete subgroups of $SU(3)$ and $SU(2)$ strongly suggests a connection with the inclusions of the hyper-finite factors of type II_1 characterized by these subgroups, which are expected to play a fundamental role in quantum TGD. An interesting question is whether also infinite subgroups could be involved. For instance, one can consider the subgroups generated by discrete subgroup and infinite cyclic group and these might be involved with the inclusions for which the index is equal to four. The appearance of these groups suggests also a connection with the hierarchy of Planck constants and one can ask how the singular coverings defining the pages of the book like structure relate to the moduli space of causal diamonds.

The rather unexpected conclusion is that the wave functions in the discrete space defined by infinite primes are able to code for the quantum numbers of configuration space spinor fields and thus for configuration space spinor fields. A fascinating possibility is that even M-matrix- which is nothing but a characterization of zero energy state- could find an elegant formulation as entanglement coefficients associated with the pair of the integer and inverse integer characterizing the positive and negative energy states.

1. The great vision is that associativity and commutativity conditions fix the number theoretical quantum dynamics completely. Quantum associativity states that the wave functions in the space of infinite primes, integers, and rationals are invariant under associations of finite hyper-octonionic primes ($A(BC)$ and $(AB)C$ are the basic associations), physics requires associativity only apart from a phase factor, in the simplest situation $+1/-1$ but in more general case phase factor. The condition of commutativity poses a more familiar condition implying that permutations induce only a phase factor which is +/- 1 for boson and fermion statistics and a more general phase for quantum group statistics for the anyonic phases, which correspond to nonstandard values of Planck constant in TGD framework. These symmetries induce time-like entanglement for zero energy stats and perhaps non-trivial enough M-matrix.

2. One must also remember that besides the infinite primes defining the counterparts of free Fock states of supersymmetric QFT, also infinite primes analogous to bound states are predicted. The analogy with polynomial primes illustrates what is involved. In the space of polynomials with integer coefficients polynomials of degree one correspond free single particle states and one can form free many particle states as their products. Higher degree polynomials with algebraic roots correspond to bound states being not decomposable to a product of polynomials of first degree in the field of rationals. Could also positive and negative energy parts of zero energy states form a analog of bound state giving rise to highly non-trivial M-matrix?

5.4 How space-time geometry could be coded by infinite primes

Second key question is whether space-time geometry could be characterized in terms of infinite primes (and integers and rationals in the most general case) and how this is achieved. This problem trivializes by quantum classical correspondence realized in terms of the measurement interaction term in the modified Dirac action.

1. The addition of the measurement interaction term to the modified Dirac action defined by Kähler action implies that space-time sheets carry information about four-momentum, color quantum numbers, and electroweak quantum numbers. One must assing assign to the space-time sheet assignable to a given collection of partonic 2-surfaces at least one pair of infinite primes or rather wave function at the orbits of these primes under the group respecting the prime property. Pairs of infinite-primes at the first level would characterize the quantum numbers assigned with the partonic surface X^2, that is the tangent space of the space-time surface at X^2 fixing the initial values for the preferred extremal of Kähler action.

2. Zero energy ontology implies a hierarchy of CDs within CDs and this hierarchy as well as the hierarchy of space-time sheets corresponds naturally to the hierarchy of infinite primes. One can assign standard model quantum numbers to various partonic 2-surfaces with positive and negative energy parts of the quantum state assignable to the light-like boundaries of CD. Also infinite integers and rationals are possible and the inverses of infinite primes would naturally correspond to elementary particles with negative energy. The condition that zero energy state has vanishing net quantum numbers implies that the ratio of infinite integers assignable to zero energy state equals to real unit in real sense and has has vanishing total quantum numbers.

3. Neither quantum numbers nor infinite primes coding them cannot characterize the partonic 2-surface itself completely since they say nothing about the deformation of the space-time surface but only about labels characterizing the WCW spinor field. Also the topology of partonic 2-surface fails to be coded. Quantum classical correspondence however suggests that this correspondence could be possible in a weaker sense. In the Gaussian approximation for functional integral over the world of classical worlds space-time surface and thus the collection of partonic 2-surfaces is effectively replaced with the one corresponding to the maximum of Kähler function, and in this sense one-one correspondence is possible unless the situation is non-perturbative. In this case the physics implied by the hierarchy of Planck constants could however guarantee uniqueness. One of the basic ideas behind the identification of the dark matter as phases with non-standard value of Planck constant is that when perturbative description of the system fails, a phase transition increasing the value of Planck constant takes place and makes perturbative description possible. Geometrically this phase transition means a leakage to another sector of the imbedding space realized as a book like structure with pages partially labeled by the values of Planck constant. Anyonic phases and fractionization of quantum numbers is one possible outcome of this phase transition. An interesting question is what the fractionization of the quantum numbers means number theoretically.

5.5 How to achieve consistency with p-adic mass formula

The first argument against the proposal that infinite primes could code for four-momentum in preferred coordinates is that the logarithms of finite primes and even less those of hyper-octonionic primes are natural from the point of view of p-adic mass calculations predicting that the mass squared of particle behaves as $1/p$ for $T_p = 1$ (fermions) and $1/p^2$ for $T_p = 1/2$ (gauge bosons). This difficulty might be circumvented.

5.5.1 Ordinary primes

Consider first ordinary primes for which the inverse always exists.

1. One can map finite primes p to phase factors $exp(i2\pi/p)$. The roots of unity play the role of primes in the decomposition of the roots of unity $exp(i2\pi/n)$, $n = \prod_i p_i^{n_i}$. $1/n$ is expressible as a sum of form

$$\begin{aligned} \frac{1}{n} &= \sum_i P_i \ , \\ P_i &= \frac{k_i}{p_i^{n_i}} \ . \end{aligned} \tag{5.1}$$

   giving

$$exp(\frac{i2\pi}{n}) \ = exp(i2\pi \textstyle\sum_i P_i) = exp(i2\pi \sum_i \frac{k_i}{p_i^{n_i}}) \ . \tag{5.2}$$

   Apart from a common normalization factor one can interpret the coefficients P_i as energy like quantities assigned to the single particle states. The power $p_i^{n_i}$ would correspond to various p-adic inverse temperature $1/T_p = 2n_i$ in this expansion.

2. The representation in terms of phase factors is not unique since P_i^k and $P_i^k + np_i^k$ define the same phase. This non-uniqueness is completely analogous to the non-uniqueness of momentum in the presence of a discrete translational symmetry and can be interpreted in terms of lattice momentum. Physically this corresponds to a finite measurement resolution. Also in the formulation of symplectic QFT defining one part of quantum TGD only phases defined by the roots of unity appear and similar non-uniqueness emerges and is due to the discretization serving as a space-time correlate for a finite measurement resolution implying UV cutoff.

3. Mass squared is proportional to $1/p_i^2$ so that only the p-adic temperatures $T_p = 1/2n_i$ are possible for rational primes. For more general primes one can however have also a situation in which the modulus square of prime is ordinary prime. For instance, Gaussian (complex) primes $P = m + in$ satisfy $|P|^2 = p$ for $p \ mod\ 4 = 1$ and $|P|^2 = p^2$ for $p\ mod\ 4 = 3$ (for example, rational prime 5 decomposes as $5 = (2 + i)(2 - i)$). Therefore it is possible to have states satisfying $M^2 \propto 1/p$, p ordinary prime for hyper-octonionic primes. These primes correspond to the rational primes decomposing to the products of ordinary primes and also also higher roots of p might be possible. The finite prime assignable to the hyper-octonionic prime has a natural interpretation as the p-adic prime assignable to an elementary particle. In zero energy ontology this assignment makes sense also for virtual particles having interpretation as pairs of positive and negative energy on mass shell particles assignable to the light-like throats of wormhole contact.

5.5.2 Hyper-octonionic primes with inverse

Consider next the situation for hyper-octonionic primes when the integers in question have inverse. We are interested only in the longitudinal part of infinite prime in M^2. The phase factor makes sense also in the case of hyper-octonionic primes if the condition $|P| > 0$ holds true so that one has massive particles in 8-D sense possibly resulting via p-adic thermodynamics. If the imaginary unit appearing in the exponent is the imaginary unit i appearing in the complexification of octonions, the exponent has the character of a phase factor for hyper-octonionic primes. The reason is that $1/P = P^*/|P|^2$ is hyper-octonionic number of form $O_0 + iO_1$, where O_1 is a purely imaginary octonion. The exponent in the phase factor is therefore $2\pi(iO_0 - O_1)$ and involves only imaginary units, and one can write $exp(i2\pi(O_0 + iO_1)) = exp(iO_0) \times exp(-O_1)$. Both factors are phase factors. This condition analogous to unitarity is one further good reason for hyper-octonions and Minkowskian signature.

5.5.3 Light-like hyper-octonionic primes

The proposed representation as a phase factor fails for massless particles since light-like hyper-primes do not possess an inverse. One must therefore define the notion of primeness differently to see what might be the physical interpretation of these primes. Since the multiplication of hyper-octonionic integer by light-like prime yields zero norm prime, the natural interpretation would be as a gauge transformation and one might consider gauge transformations obtained by exponentiating Lie algebra with light-like coefficients.

One can consider two options depending on whether one requires that the relevant algebra has unit or not.

1. For the first option hyper-octonionic light-like integers are of form $n(1 + e)$ and the product of two light-like integers $n_i(1 + e)$ is of form $2n_1n_2(1 + e)$. Here e could be arbitrary hyper-octonionic imaginary unit consistent with the prime property. This does not however allow unit light-like integer acting like unit since one has $(1 + e)^2 = 2(1 + e)$. All odd integers would be primes.

2. The number $E = (1+e)/2$ behaves as a unit. If one requires that unit is included in the algebra integers can be defined as numbers of form nE so that their product is n_1n_2E and equivalent with the ordinary product of integers so that primes correspond to ordinary primes.

One can construct the first level infinite primes from these primes just as in the case of ordinary primes. Now however $X = \prod p_i$ is replaced with $X = \prod_n[(2n+1)(1+e)]$ for the first option and equal to the $X = E\prod p_i$ for the second option.

The multiplicative phase factor could be defined for both options as $exp(i2\pi E/N)$ where N is a light-like hyper-octonionic integer. This definition would eliminate the singular $1/E$ factor and the situation reduces essentially to that for ordinary primes in the case of massless states. If the infinite prime $P_\pm$ is such that one can assign to it non-trivial multiplets in color or rotational degrees of freedom (half odd integer spin for fermions) it must have a part in the complement of M^2. For standard model elementary particles this is always the case. The energy spectrum is of form $1/2(2m+1)$ or $1/p$. For light-like hyper-octonions the projection to M^2 is in general time-like and quantized. If one does not allow the unit E in exponent the phase factor is ill-defined and one must identify the light-like hyper-octonionic primes as gauge degrees of freedom.

M^2 momentum is light-like only for states which are spinless color and electro-weak singlets having no counterpart in standard model counterpart nor in quantum TGD. Therefore light-like hyper-octonionic primes reducing to M^2 could correspond to gauge degrees of freedom. M^2 momentum is of form $P = (1,1)/2(2m+1)$ for the first option and of form $P = (1,1)/p$ for the second option. Even for graviton, photon, gluons, and right handed neutrino either hyper-octonionic prime is space-like if the state is massless. Light-like hyper-octonions can however characterize massive states but the proposed interpretation in terms of gauge degrees of freedom is highly suggestive.

If one interprets hyper-octonionic prime as 8-D momentum, which is of course not necessary in the recent framework, one could worry about conflict with TGD variant of twistor program. In accordance with associativity the role of 8-momentum in fermionic propagator is however taken by its projection to the hyper-quaternionic sub-space defined by the modified gamma matrices at given point of space-time sheet and masslessness holds for this projection so that 8-D tachyons are possible [11]. This is highly analogous to the identification of the four-momentum as M^2 projection of hyperfinite prime.

5.5.4 The treatment of zero modes

There are also zero modes which are absolutely crucial for quantum measurement theory. They entangle with quantum fluctuating degrees of freedom in quantum measurement situation and thus map quantum numbers to positions of pointers. The interior degrees of freedom of space-time interior must correspond to zero modes and they represent space-time correlates for quantum states realized at light-like partonic 3-surfaces. Quantum measurement theory suggests 1-1 correspondence between zero modes and quantum fluctuating degrees of freedom so that also super-symmetry should have zero mode counterpart. The recent progress in understanding of the modified Dirac action [7] leads to a concrete identification of the super-conformal algebra of zero modes as related to the deformation of the space-time surface defining vanishing second variations of Kähler action.

5.6 Complexification of octonions in zero energy ontology

The complexification of octonions plays a crucial role in the number theoretical vision and could be regarded as its weakest point. It has however a natural physical interpretation in zero energy ontology.

1. CD has two tips, which correspond to the points of M^4. For M^4 the fixing of the quantization axes requires choosing a time-like direction fixing the rest system. This direction is naturally defined by the tips of CD. The moduli space for CDs is $M^4 \times M^4_+$. The realization of the hierarchy of Planck constants forces also a choice of a space-like direction fixing the quantization axes of spin.

2. In the case of CP_2 the choice of the quantization axes requires fixing of a preferred point of CP_2 remaining invariant under $U(2)$ subgroup of $SU(3)$ acting linearly on complex coordinates having origin at this point and containing also the Cartan subgroup. This fixes the quantization axes of color hyper-charge. If the preferred CP_2 points associated with the light-like boundaries of CD are different they fix a unique geodesic circle of CP_2 fixing the quantization axes for color isospin. The moduli space is therefore $(CP_2)^2$.

3. The full moduli space is $M^4 \times M^4_+ \times (CP_2)^2$. In M^8 description the moduli space would naturally correspond to pairs of points of M^4 and E^4 so that the moduli space for the choices CDs and quantization axes would be $M^4 \times M^4_+ \times (E^4)^2$. This space can be regarded locally as the space of complexified octonions.

4. p-Adic length scale hypothesis follows if the time-like distance between the tips of CDs is quantized in powers of two so that a union of 3-D proper-time constant hyperboloids of M^4_+ results. Hierarchy of Planck constants implies rational multiples of these basic distances. Hyperboloids are coset spaces of Lorentz group and this suggests even more general quantization in which one replaces the hyperboloids with spaces obtained by identifying the points related by the action of a discrete subgroup of Lorentz group. This would give the analog of lattice cell obtained and one would obtain a lattice like structure consisting of unit cells labeled by the elements of the sub-group of Lorentz group. The interpretation of the moduli space of CDs as a discrete momentum space dual to the configuration space is suggestive. In the case of CP_2 similar quantization could correspond to the replacement of CP_2 with equivalence classes of points of CP_2 under action of a discrete subgroup of $SU(3)$.

5. Could this discrete space be identified as the space of hyper-octonionic primes as looks natural? In other words, could the discrete points of the dual space $M_+^4 \times CP_2$ decompose to subsets in one-one corresponds with the orbits of G_+ and G_- appearing in the reductions $SO(7,1) \rightarrow SO(7) \rightarrow G_2 \rightarrow SU(3) \rightarrow G_+$ for primes in P_+ and $SO(7,1) \rightarrow SO(7) \rightarrow G_2 \rightarrow SU(3) \rightarrow SU(2) \rightarrow G_-$ in P_-? One can also consider the subgroups of G_2 respecting the hyperbolic prime property. This would allow to integrate $G_+ \times G_-$ multiplets to larger multiplets and get an over all view about multiplet structure. An interesting question is whether $SO(7,1)$ could contain non-compact discrete subgroups with infinite number of elements and respecting the property of being hyper-octonionic prime. If this idea is correct, the dual space $M_+^4 \times CP_2$ would play a role of heavenly sphere providing a representation for the quantum numbers labeling configuration space spinor fields.

5.7 The relation to number theoretic Brahman=Atman identity

Number theoretic Brahman=Atman identity -one might also use the term algebraic holography - states the number theoretic anatomy of single space-time point is enough to code for both WCW and and WCW spinors fields- the quantum states of entire Universe or at least the sub-Universe defined by CD. The entire quantum TGD could be represented in terms of 8-D imbedding space with the notion of number generalized to allow real units defined as ration of infinite integers and having number theoretical anatomy.

Before continuing it is perhaps good to represent the most obvious objection against the idea. The correspondence between WCW and WCW spinors with infinite rationals and their discreteness means that also WCW (world of classical worlds) and space of WCW spinors should be discrete. First this looks non-sensible but is indeed what one obtains if space-time surfaces correspond to light-like 3-surfaces expressible in terms of algebraic equations involving rational functions with rational coefficients.

By the above considerations it is indeed clear that zero energy states correspond to ratios of infinite integers boiling down to a hyper-octonionic unit with vanishing net four-momentum and electro-weak charges. Configuration space spinor fields can be mapped to wave functions in the space of these units and even the reduced configuration space consisting of the maxima of Kähler function could be coded by these wave functions. The wave functions in the space of hyper-octonion units would be induced by the discrete wave functions associated with the orbits of hyper-octonionic finite primes appearing in the decomposition of the infinite hyper-octonionic primes of type P_+ and P_-. The net color and quantum numbers and spin associated with the wave function in the space of hyper-octonionic units are vanishing. Clearly, a detailed realization of number theoretic Brahman=Atman identity emerges predicting reducing even the spectrum of possible quantum numbers to number theory.

In the original formulation of Brahman-Atman identity the description based on H was used. This leads to the conclusion that that the analog of a complex Schrödinger amplitude in the space of number-theoretic anatomies of a given imbedding space point represented by single point of H and represented as 8-tuples of real units should naturally represent the dependence of WCW spinors understood as ground states of super-conformal representations obtained as an 8-fold tensor power of a fundamental representation or product of representations perhaps differing somehow. The 8-tuples define a number theoretical analog of $U(1)^8$ group in terms of which all number theoretical symmetries are represented. This description should be equivalent with the use of single hyper-octonion unit.

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