Article

Physics as Generalized Number Theory I: p-Adic Physics and Number Theoretic Universality

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Abstract

Physics as a generalized number theory program involves three threads: various p-adic physics and their fusion together with real number based physics to a larger structure, the attempt to understand basic physics in terms of classical number fields (in particular, identifying associativity condition as the basic dynamical principle), and infinite primes whose construction is formally analogous to a repeated second quantization of an arithmetic quantum field theory. In this article p-adic physics and the technical problems relates to the fusion of p-adic physics and real physics to a larger structure are discussed.

The basic technical problems relate to the notion of definite integral both at space-time level, imbedding space level and the level of WCW (the "world of classical worlds"). The expressibility of WCW as a union of symmetric spaces leads to a proposal that harmonic analysis of symmetric spaces can be used to define various integrals as sums over Fourier components. This leads to the proposal the p-adic variant of symmetric space is obtained by a algebraic continuation through a common intersection of these spaces, which basically reduces to an algebraic variant of coset space involving algebraic extension of rationals by roots of unity. This brings in the notion of angle measurement resolution coming as $\Delta\phi = 2\pi/p^n$ for given p-adic prime p. Also a proposal how one can complete the discrete version of symmetric space to a continuous p-adic versions emerges and means that each point is effectively replaced with the p-adic variant of the symmetric space identifiable as a p-adic counterpart of the real discretization volume so that a fractal p-adic variant of symmetric space results.

If the Kähler geometry of WCW is expressible in terms of rational or algebraic functions, it can in principle be continued the p-adic context. One can however consider the possibility that that the integrals over partonic 2-surfaces defining flux Hamiltonians exist p-adically as Riemann sums. This requires that the geometries of the partonic 2-surfaces effectively reduce to finite sub-manifold geometries in the discretized version of $\delta M^4_+ \times CP_2$. If Kähler action is required to exist p-adically same kind of condition applies to the space-time surfaces themselves. These strong conditions might make sense in the intersection of the real and p-adic worlds assumed to characterized living matter.

Keywords: p-Adic numbers, Kähler metric, p-adic integration, symmetric space, harmonic analysis, measurement resolution.

1 Introduction

In this article basic facts about p-adic numbers [42] and the question about their relation to real numbers are discussed. Also the basic technicalities related to the notion of p-adic physics are discussed. Also included is a section about the physics in the intersection of real and p-adic worlds relevant to living systems in TGD Universe.

1.1 Problems

It is far from obvious what the p-adic counterpart of real physics could mean and how one could fuse together real and p-adic physics. Therefore it is good to list the basic problems and proposals for their solution.

The first problem concerns the correspondence between real and p-adic numbers.

1. The success of p-adic mass calculations involves the notions of p-adic probability, thermodynamics, and the mapping of p-adic propapabilities to the real ones by a continuous correspondence $x = \sum x_n p^n \rightarrow Id(x) = \sum x_n p^{-n}$ that I have christened canonical identification. The problem is that I n does not respect symmetries defined by isometries and also general coordinate invariance is possible only if one can identify preferred imbedding space coordinates. The reason is that I does not commute with the basic arithmetic operations. I allows several variants and it is possible to have correspondence which respects symmetries in arbitrary accuracy in preferred coordinates. Thus I can play a role at space-time level only if one defines symmetries modulo measurement resolution. I would make sense only in the interval defining the measurement resolution for a given coordinate variable and the p-adic effective topology would make sense just because the finite measurement resolution does not allow to well-order the points.

2. The identification of real and p-adic numbers via rationals common to all number fields - or more generally along algebraic extension of rationals- respects symmetries and algebra but is not continuous. At the imbedding space level preferred coordinates are required also now. The maximal symmetries of the imbedding space allow identification of this kind of coordinates. They are not unique. For instance, M^4 linear coordinates look very natural but for CP_2 trigonometric functions of angle like coordinates look more suitablel and Fourier analysis suggests strongly the introduction of algebraic extensions involving roots of unity. Partly the non-uniqueness has an interpretation as an imbedding space correlate for the selection of the quantization axes. The symmetric space [30] property of WCW gives hopes that general coordinate invariance in quantal sense can be realized. The existence of p-adic harmonic analysis suggests a discretization of the p-adic variant of imbedding space and WCW based on roots of unity.

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3. One can consider a compromise between the two correspondences. Discretization via common algebraic points can be completed to a p-adic continuum by assigning to each real discretizaton interval (say angle increment $2\pi/N$) p-adic numbers with norm smaller than one.

Second problem relates to integration and Fourier analysis. Both these procedures are fundamental for physics -be it classical or quantum. The p-adic variant of definite integral does not exist in the sense required by the action principles of physics although classical partial differential equations assigned to a particular variational principle make perfect sense. Fourier analysis is also possible only if one allows algebraic extension of p-adic numbers allowing a sufficient number of roots of unity correlating with the measurement resolution of angle. The finite number of them has interpretation in terms of finite angle resolution. Fourier analysis provides also an algebraic realization of definite integral when one integrates over the entire manifold as one indeed does in the case of WCW. If the space in question allows maximal symmetries as WCW and imbedding space do, there are excellent hopes of having p-adic variants of both integration and harmonic analysis and the above described procedure allows a precise completion of the discretized variant of real manifold to its continuous p-adic variant.

The third problem relates to the definitions of the p-adic variants of Riemannian, symplectic [34], and Kähler [32] geometries. It is possible to generalize formally the notion of Riemann metric although non-local quanttiies like areas and total curvatures do not make sense if defined in terms of integrals. If all relevant quantities assignable to the geometry (family of Hamiltonians defining isometries, Killing vector fields, components of metric and Kähler form, Kähler function, etc...) are expressible in terms of rational functions involving only rational numbers as coefficients of polynomials, they allow an algebraic continuation to the p-adic context and the p-adic variant of the geometry makes sense.

The fourth problem relates to the question what one means with p-adic quantum mechanics. In TGD framework p-adic quantum theory utilizes p-adic Hilbert space. The motivation is that the notions of p-adic probability and unitarity are well defined. From the beginning it was clear that the straightforward generalization of Schrödinger equation is not very interesting physically and gradually the conviction has developed that the most realistic approach must rely on the attempt to find the p-adic variant of the TGD inspired quantum physics in all its complexity. The recent approach starts from a rather concrete view about generalized Feynman diagrams defining the points of WCW and leads to a rather detailed view about what the p-adic variants of QM could be and how they could be fused with real QM to a larger structure. Even more, just the requirement that this p-adicization exists, gives very powerful constraints on the real variant of the quantum TGD.

The fifth problem relates to the notion of information in p-adic context. p-Adic thermodynamics leads naturally to the question what p-adic entropy might mean and this in turn leads to the realization that for rational or even algebraic probabilities p-adic variant of Shannon entropy can be negative and has minimum for a unique prime. One can say that the entanglement in the intersection of real and p-adic worlds is negentropic. This leads to rather fascinating vision about how negentropic entanglement makes it possible for living systems to overcome the second law of thermodynamics. The formulation of quantum theory in the intersection of real and living worlds becomes the basic challenge.

The proposed solutions to the technical problems could be rephrased in terms of the notion of algebraic universality. Various p-adic physics are obtained as algebraic continuation of real physics through the common algebraic points of real and p-adic worlds and by performing completion in the sense that the interval corresponding to finite measurement resolution are replaced with their p-adic counterpart via canonical identification. This allows to have exact symmetries as their discrete variants and also a continuous correspondence if desired. Particular p-adicization is characterized by a choice for preferred imbedding space coordinates, which has interpretation in terms of a particular cognitive representation. Hence one is forced to refine the view about general coordinate invariance. Different coordinate choices correspond to different cognitive representations having delicate effects on physics if it is assumed to include also the effects of cognition.

1.2 Program

These ideas lead to a reasonably well defined p-adicization program. Try to define precisely the concepts of the p-adic space-time and configuration space (WCW), formulate the finite-p p-adic versions of quantum TGD. Try to fuse together real and various p-adic quantum TGDs are to form a full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, and the notions of probability and unitarity to the p-adic context. Also new physical thinking and philosophy is needed. The notions of zero energy ontology, hierarchy of Planck constants and the generalization of the notion of imbedding space required by it are essential but not discussed in detail in this article.

In the following I try to describe the most central problems and ideas of the p-adicization program. Page number of a readable article must be finite and this has forced to leave away a lot of topics. p-Adic mass calculations, which form the corner stone of the entire approach would require entire article series. The vision about how to define generalized Feynman diagrams and their p-adic variants by utilizing the assumption that WCW is symmetric space allowing algebraization of integration crucial for the entire approach is discussed in the May issue of this Journal [29]. Negentropy Maximization Principle [25] relevant for understanding the profound implications of the negentropic entanglement is not discussed. The applications of p-adic length scale hypothesis to the physics of living matter [28] and the model of cognition and intentionality based on p-adic numbers [26] have been also left out.

2 Summary of the basic physical ideas

In the following various manners to end up with p-adic physics and with the idea about p-adic topology as an effective topology of space-time surface are described.

2.1 p-Adic mass calculations briefly

p-Adic mass calculations based on p-adic thermodynamics with energy replaced with the generator $L_0 = zd/dz$ of infinitesimal scaling are described in the first part of [5].

1. p-Adic thermodynamics is justified by the randomness of the motion of partonic 2-surfaces restricted only by the light-likeness of the orbit.

2. It is essential that the conformal symmetries associated with the light-like coordinates of parton and light-cone boundary are not gauge symmetries but dynamical symmetries. The point is that there are two kinds of super-conformal symmetries [49, 51]: the super-symplectic conformal symmetries assignable to the light-like boundaries of $CD \times CP_2$ and super Kac-Moody symmetries [52] assignable to light-like 3-surfaces defining fundamental dynamical objects. In so called coset construction [53] the differences of super-conformal generators of these algebras annihilate the physical states. This leads to a generalization of Equivalence Principle since one can assign four-momentum to the generators of both algebras identifiable as inertial *resp.* gravitational four-momentum. A second important consequence is that the generators of either algebra do not act like gauge transformations so that it makes sense to construct p-adic thermodynamics for them.

3. In p-adic thermodynamics scaling generator L_0 having conformal weights as its eigen values replaces energy and Boltzmann weight $exp(H/T)$ is replaced by p^{L_0/T_p}. The quantization $T_p = 1/n$ of conformal temperature and thus quantization of mass squared scale is implied by number theoretical existence of Boltzmann weights. p-Adic length scale hypothesis states that primes $p \simeq 2^k$, k integer. A stronger hypothesis is that k is prime (in particular Mersenne prime or Gaussian Mersenne) makes the model very predictive and fine tuning is not possible.

   Mersenne primes are very special number theoretically because bit as the unit of information unit corresponds to $log(2)$ and can be said to exists for M_n-adic topology. The reason is that $log(1+p)$ existing always p-adically corresponds for $M_n = 2^n - 1$ to $log(2^n) \equiv nlog(2)$ so that one has $log(2 \equiv log(1 + M_n)/n$. Since the powers of 2 modulo p give all integers $n \in \{1, p-1\}$ by Fermat's theorem, one can say that the logarithms of all integers modulo M_n exist in this sense and therefore the logarithms of all p-adic integers not divisible by p exist. For other primes one must introduce a transcendental extension containing $log(a)$ where are is so called primitive root. One could criticize the identification since $log(1 + M_n)$ corresponding in the real sense to n bits corresponds in p-adic sense to to a very small information content since the p-adic norm of the p-adic bit is $1/M_n$.

The basic mystery number of elementary particle physics defined by the ratio of Planck mass and proton mass follows thus from number theory once CP_2 radius is fixed to about 10^4 Planck lengths. Mass scale becomes additional discrete variable of particle physics so that there is not more need to force top quark and neutrinos with mass scales differing by 12 orders of magnitude to the same multiplet of gauge group. Electron, muon, and τ correspond to Mersenne prime $k = 127$ (the largest non-super-astrophysical Mersenne), and Mersenne primes $k = 113, 107$. Intermediate gauge bosons and photon correspond to Mersenne M_{89}, and graviton to M_{127}.

The value of k for quark can depend on hadronic environment [23] and this would produce precise mass formulas for low energy hadrons. This kind of dependence conforms also with the indications that neutrino mass scale depends on environment [66]. Amazingly, the biologically most relevant length scale range between 10 nm and 4 μm contains four Gaussian Mersennes $(1 + i)^n - 1$, $n = 151, 157, 163, 167$ and scaled copies of standard model physics in cell length scale could be an essential aspect of macroscopic quantum coherence prevailing in cell length scale.

p-Adic mass thermodynamics is not quite enough: also Higgs boson is needed and wormhole contact carrying fermion and anti-fermion quantum numbers at the light-like wormhole throats is excellent candidate for Higgs [22]. The coupling of Higgs to fermions can be small and induce only a small shift of fermion mass: this could explain why Higgs has not been observed. Also the Higgs contribution to mass squared can be understood thermodynamically if identified as absolute value for the thermal expectation value of the eigenvalues of the modified Dirac operator having interpretation as complex square root of conformal weight.

The original belief was that only Higgs corresponds to wormhole contact. The assumption that fermion fields are free in the conformal field theory applying at parton level forces to identify all gauge bosons as wormhole contacts connecting positive and negative energy space-time sheets [22]. Fermions correspond to topologically condensed CP_2 type extremals with single light-like wormhole throat. Gravitons are identified as string like structures involving pair of fermions or gauge bosons connected by a flux tube. Partonic 2-surfaces are characterized by genus which explains family replication phenomenon and an explanation for why their number is three emerges [21]. Gauge bosons are labeled by pairs (g_1, g_2) of handle numbers and can be arranged to octet and singlet representations of the resulting dynamical SU(3) symmetry. Ordinary gauge bosons are SU(3) singlets and the heaviness of octet bosons explains why higher boson families are effectively absent. The different character of bosons could also explain why the p-adic temperature for bosons is $T_p = 1/n < 1$ so that Higgs contribution to the mass dominates.

2.2 p-Adic length scale hypothesis, zero energy ontology, and hierarchy of Planck constants

Zero energy ontology and the hierarchy of Planck constants realized in terms of the generalization of the imbedding space lead to a deeper understanding of the origin of the p-adic length scale hypothesis.

2.2.1 Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the light-like boundaries of CD. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe?, Is theory building completely useless if only single solution of field equations is realized?). At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events.

2.2.2 Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [11] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra [37] spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [60] known as hyperfinite factor of type II_1 (HFF) [13, 12, 11]. HFF [61, 59] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [57]. The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [62], anyons [65], quantum groups and conformal field theories[64], and knots and topological quantum field theories [54, 55].

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

One can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of CD corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is .1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [11]. The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics. Connes tensor product [59] provides a mathematical description of the finite measurement resolution but does not fix the M-matrix as was the original hope. The remaining challenge is the calculation of M-matrix and the progress induced by zero energy ontology during last years has led to rather concrete proposal for the construction of M-matrix.

It turns out however that the mathematical representation for the notion of finite resolution for angle measurement serves as a common denominator for all basic approaches to quantum TGD: the Kähler geometry [32] of WCW identified as a union of infinite-dimensional symmetric spaces, inclusions of hyper finite factors as representation of finite measurement resolution, p-adicization program, the role of classical number fields [39, 40, 41], and infinite primes so that it is fair to say that all approaches to TGD which originally seemed almost independent, converge to a coherent mathematical structure.

2.2.3 How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

Zero energy ontology in which zero energy states have as imbedding space correlates CDs for which the distance between the tips of future and past directed light-cones are power of 2 multiples of fundamental time scale ($T_n = 2^nT_0$) implies in a natural manner coupling constant evolution. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^nT_0$ (or $T_p = pT_0$) induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section

$X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5$ μm (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of configuration space.

2.2.4 Mersenne primes and Gaussian Mersennes

The generalization of the imbedding space required by the postulated hierarchy of Planck constants [13] means a book like structure for which the pages are products of singular coverings or factor spaces of CD (causal diamond defined as intersection of future and past directed light-cones) and of CP_2 [13]. This predicts that Planck constants are rationals and that a given value of Planck constant corresponds to an infinite number of different pages of the Big Book, which might be seen as a drawback. If only singular covering spaces are allowed the values of Planck constant are products of integers and given value of Planck constant corresponds to a finite number of pages given by the number of decompositions of the integer to two different integers. The definition of the book like structure assigns to a given CD preferred quantization axes and so that quantum measurement has direct correlate at the level of moduli space of CDs.

TGD inspired quantum biology and number theoretical considerations suggest preferred values for $r = \hbar/\hbar_0$. For the most general option the values of $\hbar$ are products and ratios of two integers n_a and n_b. Ruler and compass integers defined by the products of distinct Fermat primes and power of two are number theoretically favored values for these integers because the phases $exp(i2\pi/n_i)$, $i \in \{a, b\}$, in this case are number theoretically very simple and should have emerged first in the number theoretical evolution via algebraic extensions of p-adics and of rationals. p-Adic length scale hypothesis favors powers of two as values of r.

One can however ask whether a more precise characterization of preferred Mersennes could exist and whether there could exists a stronger correlation between hierarchies of p-adic length scales and Planck constants. Mersenne primes $M_k = 2^k - 1$, $k \in \{89, 107, 127\}$, and Gaussian Mersennes $M_{G,k} = (1+i)k - 1$, $k \in \{113, 151, 157, 163, 167, 239, 241..\}$ are expected to be physically highly interesting and up to $k = 127$ indeed correspond to elementary particles. The number theoretical miracle is that all the four p-adic length scales with $k \in \{151, 157, 163, 167\}$ are in the biologically highly interesting range 10 nm-2.5 μm). The question has been whether these define scaled up copies of electro-weak and QCD type physics with ordinary value of $\hbar$. The proposal that this is the case and that these physics are in a well-defined sense induced by the dark scaled up variants of corresponding lower level physics leads to a prediction for the preferred values of $r = 2^{k_d}$, $k_d = k_i - k_j$.

What induction means is that dark variant of exotic nuclear physics induces exotic physics with ordinary value of Planck constant in the new scale in a resonant manner: dark gauge bosons transform to their ordinary variants with the same Compton length. This transformation is natural since in length scales below the Compton length the gauge bosons behave as massless and free particles. As a consequence, lighter variants of weak bosons emerge and QCD confinement scale becomes longer.

This proposal will be referred to as Mersenne hypothesis. It leads to strong predictions about EEG [28] since it predicts a spectrum of preferred Josephson frequencies for a given value of membrane potential and also assigns to a given value of $\hbar$ a fixed size scale having interpretation as the size scale of the body part or magnetic body. Also a vision about evolution of life emerges. Mersenne hypothesis is especially interesting as far as new physics in condensed matter length scales is considered: this includes exotic scaled up variants of the ordinary nuclear physics and their dark variants. Even dark nucleons are possible and this gives justification for the model of dark nucleons predicting the counterparts of DNA,RNA, tRNa, and aminoacids as well as realization of vertebrate genetic code [27].

These exotic nuclear physics with ordinary value of Planck constant could correspond to ground states that are almost vacuum extremals corresponding to homologically trivial geodesic sphere of CP_2 near criticality to a phase transition changing Planck constant. Ordinary nuclear physics would correspond to homologically non-trivial geodesic sphere and far from vacuum extremal property. For vacuum extremals of this kind classical Z^0 field proportional to electromagnetic field is present and this modifies dramatically the view about cell membrane as Josephson junction. The model for cell membrane as almost vacuum extremal indeed led to a quantitative breakthrough in TGD inspired model of EEG and is therefore something to be taken seriously. The safest option concerning empirical facts is that the copies of electro-weak and color physics with ordinary value of Planck constant are possible only for almost vacuum extremals - that is at criticality against phase transition changing Planck constant.

2.3 p-Adic physics and the notion of finite measurement resolution

Canonical identification mapping p-adic numbers to reals in a continuous manner plays a key role in some applications of TGD and together with the discretization necessary to define the p-adic variants of integration and harmonic analysis suggests that p-adic topology identified as an effective topology could provide an elegant manner to characterize finite measurement resolution.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modeling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with a pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

This interpretation would suggest that there is an infinite hierarchy of measurement resolutions characterized by the value of the p-adic prime. This would mean quite interesting refinement of the notion of finite measurement resolution. At the level of quantum theory it could be interpreted as a maximization of p-adic entanglement negentropy as a function of the p-adic prime. Perhaps one might say that there is a unique p-adic effective topology allowing to maximize the information content of the theory relying on finite measurement resolution.

2.4 p-Adic numbers and the analogy of TGD with spin-glass

The vacuum degeneracy of the Kähler action leads to a precise spin glass analogy at the level of the configuration space geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity could be realized at the level of the configuration space. Also the concept of p-adic space-time surface emerges rather naturally.

2.4.1 Spin glass briefly

The basic characteristic of the spin glass phase [47] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines 'energy landscape' and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [46].

2.4.2 Vacuum degeneracy of Kähler action

The Kähler action defining configuration space geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrangian manifold [36] of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new Lagrangian sub-manifold [36]. An explicit representation for Lagrangian sub-manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ , \ \ i = 1, 2 \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by the classical gravitational interaction there are good reasons to expect large ground state degeneracy, when the system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass but is 4-dimensional.

2.4.3 Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Lagrangian sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum extremal and deforming them.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function become possible. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe [56]. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multifurcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate preferred extremals. This non-determinism is expected to make the preferred extremals of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

2.4.4 p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{2.1}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the preferred extremals are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible 'engineering' at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible 'engineering' at the level of the real world.

2.5 Life as islands of rational/algebraic numbers in the seas of real and p-adic continua?

The possibility to define entropy differently for rational/algebraic entanglement and the fact that number theoretic entanglement entropy can be negative raises the question about which kind of systems can possess this kind of entanglement. I have considered several identifications but the most elegant interpretation is based on the idea that living matter resides in the intersection of real and p-adic worlds, somewhat like rational numbers live in the intersection of real and p-adic number fields.

The observation that Shannon entropy allows an infinite number of number theoretic variants for which the entropy can be negative in the case that probabilities are algebraic numbers leads to the idea that living matter in a well-defined sense corresponds to the intersection of real and p-adic worlds. This would mean that the mathematical expressions for the space-time surfaces (or at least 3-surfaces or partonic 2-surfaces and their 4-D tangent planes) make sense in both real and p-adic sense for some primes p. Same would apply to the expressions defining quantum states. In particular, entanglement probabilities would be rationals or algebraic numbers so that entanglement can be negentropic and the formation of bound states in the intersection of real and p-adic worlds generates information and is thus favored by NMP.

This picture has also a direct connection with consciousness.

1. Algebraic entanglement is a prerequisite for the realization of intentions as transformations of p-adic space-time sheets to real space-time sheets representing actions. Essentially a leakage between p-adic and real worlds is in question and makes sense only in zero energy ontology. since various quantum numbers in real and p-adic sectors are not in general comparable in positive energy ontology so that conservation laws would be broken. Algebraic entanglement could be also called cognitive. The transformation can occur if the partonic 2-surfaces and their 4-D tangent space-distributions are representable using rational functions with rational coefficients in preferred coordinates for the imbedding space dictated by symmetry considerations. Intentional systems must live in the intersection of real and p-adic worlds. For the minimal option life would be also effectively 2-dimensional phenomenon and essentially a boundary phenomenon as also number theoretical criticality suggests.

2. The generation of non-rational (non-algebraic) bound state entanglement between the system and external world means that the system loses consciousness during the state function reduction process following the U-process generating the entanglement. What happens that the Universe corresponding to given CD decomposes to two un-entangled subsystems, which in turn decompose, and the process continues until all subsystems have only entropic bound state entanglement or negentropic algebraic entanglement with the external world.

3. If the sub-system generates entropic bound state entanglement in the the process, it loses consciousness. Note that the entanglement entropy of the sub-system is a sum over entanglement entropies over all subsystems involved. This hierarchy of subsystems corresponds to the hierarchy if sub-CDs so that the survival without a loss of consciousness depends on what happens at all levels below the highest level for a given self. In more concrete terms, ability to stay conscious depends on what happens at cellular level too. The stable evolution of systems having algebraic entanglement is expected to be a process proceeding from short to long length scales as the evolution of life indeed is.

4. U-process generates a superposition of states in which any sub-system can have both real and algebraic entanglement with the external world. This would suggest that the choice of the type of entanglement is a volitional selection. A possible interpretation is as a choice between good and evil. The hedonistic complete freedom resulting as the entanglement entropy is reduced to zero on one hand, and the algebraic bound state entanglement implying correlations with the external world and meaning giving up the maximal freedom on the other hand. The hedonistic option is risky since it can lead to non-algebraic bound state entanglement implying a loss of consciousness. The second option means expansion of consciousness - a fusion to the ocean of consciousness as described by spiritual practices.

5. This formulation means a sharpening of the earlier statement "Everything is conscious and consciousness can be only lost" with the additional statement "This happens when non-algebraic bound state entanglement is generated and the system does not remain in the intersection of real and p-adic worlds anymore". Clearly, the quantum criticality of TGD Universe seems has very many aspects and life as a critical phenomenon in the number theoretical sense is only one of them besides the criticality of the space-time dynamics and the criticality with respect to phase transitions changing the value of Planck constant and other more familiar criticalities. How closely these criticalities relate remains an open question.

A good guess is that algebraic entanglement is essential for quantum computation, which therefore might correspond to a conscious process. Hence cognition could be seen as a quantum computation like process, a more approriate term being quantum problem solving. Living-dead dichotomy could correspond to rational-irrational or to algebraic-transcendental dichotomy: this at least when life is interpreted as intelligent life. Life would in a well defined sense correspond to islands of rationality/algebraicity in the seas of real and p-adic continua.

The view about the crucial role of rational and algebraic numbers as far as intelligent life is considered, could have been guessed on very general grounds from the analogy with the orbits of a dynamical system. Rational numbers allow a predictable periodic decimal/pinary expansion and are analogous to one-dimensional periodic orbits. Algebraic numbers are related to rationals by a finite number of algebraic operations and are intermediate between periodic and chaotic orbits allowing an interpretation as an element in an algebraic extension of any p-adic number field. The projections of the orbit to various coordinate directions of the algebraic extension represent now periodic orbits. The decimal/pinary expansions of transcendentals are un-predictable being analogous to chaotic orbits. The special role of rational and algebraic numbers was realized already by Pythagoras, and the fact that the ratios for the frequencies of the musical scale are rationals supports the special nature of rational and algebraic numbers. The special nature of the Golden Mean, which involves $\sqrt{5}$, conforms the view that algebraic numbers rather than only rationals are essential for life.

2.6 p-Adic physics as physics of cognition and intention

The vision about p-adic physics as physics of cognition has gradually established itself as one of the key idea of TGD inspired theory of consciousness. There are several motivations for this idea.

The strongest motivation is the vision about living matter as something residing in the intersection of real and p-adic worlds. One of the earliest motivations was p-adic non-determinism identified tentatively as a space-time correlate for the non-determinism of imagination. p-Adic non-determinism follows from the fact that functions with vanishing derivatives are piecewise constant functions in the p-adic context. More precisely, p-adic pseudo constants depend on the pinary cutoff of their arguments and replace integration constants in p-adic differential equations. In the case of field equations this means roughly that the initial data are replaced with initial data given for a discrete set of time values chosen in such a manner that unique solution of field equations results. Solution can be fixed also in a discrete subset of rational points of the imbedding space. Presumably the uniqueness requirement implies some unique pinary cutoff. Thus the space-time surfaces representing solutions of p-adic field equations are analogous to space-time surfaces consisting of pieces of solutions of the real field equations. p-Adic reality is much like the dream reality consisting of rational fragments glued together in illogical manner or pieces of child's drawing of body containing body parts in more or less chaotic order.

The obvious looking interpretation for the solutions of the p-adic field equations is as a geometric correlate of imagination. Plans, intentions, expectations, dreams, and cognition in general are expected to have p-adic space-time sheets as their geometric correlates. This in the sense that p-adic spacetime sheets somehow initiate the real neural processes providing symbolic counterparts for the cognitive representations provided by p-adic spacetime sheets and p-adic fermions. A deep principle seems to be involved: incompleteness is characteristic feature of p-adic physics but the flexibility made possible by this incompleteness is absolutely essential for imagination and cognitive consciousness in general.

p-Adic space-time regions can suffer topological phase transitions to real topology and vice versa in quantum jumps replacing space-time surface with a new one. This process has interpretation as a topological correlate for the mind-matter interaction in the sense of transformation of intention to action and symbolic representation to cognitive representation. p-Adic cognitive representations could provide the physical correlates for the notions of memes [67] and morphic fields [68]. p-Adic real entanglement makes possible makes possible cognitive measurements and cognitive quantum computation like processes, and provides correlates for the experiences of understanding and confusion.

At the level of brain the fundamental sensory-motor loop could be seen as a loop in which real-to-p-adic phase transition occurs at the sensory step and its reverse at the motor step. Nerve pulse patterns would correspond to temporal sequences of quark like sub-CDs of duration 1 millisecond inside electronic sub-CD of duration .1 s with the states of sub-CDs allowing interpretation as a bit (this would give rise to memetic code). The real space-time sheets assignable to these sub-CDs are transformed to p-adic ones as sensory input transforms to thought. Intention in transforms to action in the reverse process in motor action. One can speak about creation of matter from vacuum in these time scales.

Although p-adic space-time sheets as such are not conscious, p-adic physics would provide beautiful mathematical realization for the intuitions of Descartes. The formidable challenge is to develop experimental tests for p-adic physics. The basic problem is that we can perceive p-adic reality only as 'thoughts' unlike the 'real' reality which represents itself to us as sensory experiences. Thus it would seem that we should be able generalize the physics of sensory experiences to physics of cognitive experiences.

3 p-Adic numbers

3.1 Basic properties of p-adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [44]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k\geq k_0} x(k)p^k ,\ x(k) = 0, ..., p-1\ . \tag{3.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)}\ . \tag{3.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x)\ , \tag{3.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\}\ . \tag{3.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D . \tag{3.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

2. Distances of points x and y inside single class are smaller than distances between different classes.

3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [45]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

3.2 Extensions of p-adic numbers

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and might explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four. The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adic numbers are discussed in [15].

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving also transcendental numbers are possible: for instance, Neper number e defines a p-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimick real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime p and increasing dimension of algebraic extension appear in quantum jumps.

3.2.1 Recipe for constructing algebraic extensions

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible
[44]. The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreducibility means that the polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p,1)$, that is modulo p in R_p.

Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k \theta^k . \tag{3.6}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as a sum of lower powers of θ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d , \\ \epsilon &= \pm 1 . \end{aligned} \tag{3.7}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d , \tag{3.8}$$

where d is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n} exp(i\phi(m)),\ m = 0,1,....,n-1 , \\ \phi(m) &= \frac{m2\pi}{n} \text{ or } \frac{m\pi}{n} . \end{aligned} \tag{3.9}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

3.2.2 p-Adic valued norm for numbers in algebraic extension

The p-adic valued norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with x: $y \rightarrow xy$

$$N(x) = det(x)^{\alpha} \ , \ \alpha > 0 \ . \tag{3.10}$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k\theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..,)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in R_+^n and p-adic multiplication induces a multiplication for the vectors of R_+^n.

3.2.3 Algebraic extension allowing square root of ordinary p-adic numbers

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

1. The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the 'real' p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

2. The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

3. p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

4. Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length ds involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of imbedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

1. In $p > 2$ case the general form of extension is

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{3.11}$$

where the condition $\theta^2 = x$ for some p-adic number x not allowing square root as a p-adic number. For $p \ mod \ 4 = 3$ θ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Imbedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

2. In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z = x_0 + \sum_k x_k\theta_k + \sum_{k<l} x_{kl}\theta_{kl} + x_{123}\theta_1\theta_2\theta_3 \ . \tag{3.12}$$

Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and H are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in H vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

3.2.4 Is e an exceptional transcendental?

One can consider also the possibility of transcental extensions of p-adic numbers and an open problem is whether the infinite-dimensional extensions involving powers of π and logarithms of primes make sense and whether they should be allowed. For instance, it is not clear whether the allowance of powers of π is consistent with the extensions based on roots of unity. This question is not academic since Feynman amplitudes in real context involve powers of π and algebraic universality forces the consider that also they p-adic variants might involve powers of π.

Neper number obviously defines the simplest transcendental extension since only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$ so that the extension is finite-dimensional. In the case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

1. Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

2. The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

3. The differential equation allows to develop $f(x)$ in power series, say in origin

$$f(x) = \sum f_n \frac{x^n}{n!}$$

   such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

   The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

3.3 p-Adic Numbers and finite fields

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the elements of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \; mod \; p$, where p is prime, m an arbitrary integer and t is element of a field of characteristic p ($pt = 0$ for each t). The number of elements in a finite field is p^m, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p, 1)$ is just cyclic group Z_p with respect to addition and $G(p, m)$ is in rough sense m:th Cartesian power of $G(p, 1)$.

The elements of the finite field $G(p, 1)$ can be identified as the p-adic numbers $0, ..., p-1$ with p-adic arithmetics replaced with modulo p arithmetics. The finite fields $G(p, m)$ can be obtained from m-dimensional algebraic extensions of the p-adic numbers by replacing the p-adic arithmetics with the modulo p arithmetics. In TGD context only the finite fields $G(p > 2, 2)$, $p \; mod \; 4 = 3$ and $G(p = 2, 4)$ appear naturally. For $p > 2$, $p \; mod \; 4 = 3$ one has: $x + iy + \sqrt{p}(u + iv) \rightarrow x_0 + iy_0 \in G(p, 2)$.

An interesting observation is that the unitary representations of the p-adic scalings $x \rightarrow p^k x \; k \in Z$ lead naturally to finite field structures. These representations reduce to representations of a finite cyclic group Z_m if $x \rightarrow p^m x$ acts trivially on representation functions for some value of m, $m = 1, 2, ...$ Representation functions, or equivalently the scaling momenta $k = 0, 1, ..., m-1$ labelling them, have a structure of cyclic group. If $m \neq p$ is prime the scaling momenta form finite field $G(m, 1) = Z_m$ with respect to the summation and multiplication modulo m. Also the p-adic counterparts of the ordinary plane waves carrying p-adic momenta $k = 0, 1..., p-1$ can be given the structure of Finite Field $G(p, 1)$: one can also define complexified plane waves as square roots of the real p-adic plane waves to obtain Finite Field $G(p, 2)$.

4 What is the correspondence between p-adic and real numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification $x = \sum x_n p^n \rightarrow Id(x) = \sum x_n p^{-n}$ is a natural first guess. Also p-adic probabilities could be mapped to their real counterparts by a suitable normalization. One can wonder whether p-adic valued S-matrices have any physical meaning and whether they could be obtained as algebraic continuation from a number theoretically universal S-matrix whose matrix elements are algebraic numbers allowing an interpretation as real or p-adic numbers in suitable algebraic extension: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the imbedding space suggests itself here.

4.1 Generalization of the number concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals.

4.1.1 Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers appearing in the extension of p-adic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional algebraic and perhaps even transcendental extensions of p-adic numbers adds additional pages to this "Big Book".

This leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common points. The outcome of experimentation is that this generalization makes sense under very high symmetries and that it is safest to lean strongly on the physical picture provided by quantum TGD.

1. The most natural guess is that the coordinates of common points are rational or in some algebraic extension of rational numbers. General coordinate invariance and preservation of symmetries require preferred coordinates existing when the manifold has maximal number of isometries. This approach is especially natural in the case of linear spaces- in particular Minkowski space M^4. The natural coordinates are in this case linear Minkowski coordinates. The choice of coordinates is not completely unique and has interpretation as a geometric correlate for the choice of quantization axes for a given CD.

2. As will be found, the need to have a well-defined integration based on Fourier analysis (or its generalization to harmonic analysis [31] in symmetric spaces) poses very strong constraints and allows p-adicization only if the space has maximal symmetries. Fourier analysis requires the introduction of an algebraic extension of p-adic numbers containing sufficiently many roots of unity.

   (a) This approach is especially natural in the case of compact symmetric spaces such as CP_2 [33].

   (b) Also symmetric spaces such the 3-D proper time $a = constant$ hyperboloid of M^4-call it $H(a)$ -allowing Lorentz group as isometries allows a p-adic variant utilizing the hyperbolic counterparts for the roots of unity. $M^4 \times H(a = 2^n a_0)$ appears as a part of the moduli space of CDs.

   (c) For light-cone boundaries associated with CDs $SO(3)$ invariant radial coordinate r_M defining the radius of sphere S^2 defines the hyperbolic coordinate and angle coordinates of S^2 would correspond to phase angles and $M^4_\pm$ projections for the common points of real and p-adic variants of partonic 2-surfaces would be this kind of points. Same applies to CP_2 projections. In the "intersection of real and p-adic worlds" real and p-adic partonic 2-surfaces would obey same algebraic equations and would be obtained by an algebraic continuation from the corresponding equations making sense in the discrete variant of $M^4_\pm \times CP_2$. This connection with discrete sub-manifolds geometries means very powerful constraints on the partonic 2-surfaces in the intersection.

3. The common algebraic points of real and p-adic variant of the manifold form a discrete space but one could identify the p-adic counterpart of the real discretization intervals $(0, 2\pi/N)$ for angle like variables as p-adic numbers of norm smaller than 1 using canonical identification or some variant of it. Same applies to the the hyperbolic counterpart of this interval. The non-uniqueness of this map could be interpreted in terms of a finite measurement resolution. In particular, the condition that WCW allows Kähler geometry requires a decomposition to a union of symmetric spaces so that there are good hopes that p-adic counterpart is analogous to that assigned to CP_2.

4.1.2 How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n + y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

4.1.3 Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions for which polynomials have rational coefficients are obviously functions satisfying this constraint. Algebraic functions for which polynomials have rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed.

For instance, one can ask whether residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the back of the book like structure (in very metaphoral sense) having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense. Contrary to the first expectations the algebraically continued residue calculus does not seem to have obvious applications in quantum TGD.

4.2 Canonical identification

Canonical There exists a natural continuous map $Id : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0, 1, .., p-1\} \ . \end{aligned} \tag{4.1}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also desimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{4.2}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{4.3}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

/Users/mattipitkanen/Desktop/tgd/figures/norm.png

Figure 1: The real norm induced by canonical identification from 2-adic norm.

4.2.1 Canonical identification is a continuous map of non-negative reals to p-adics

The topology induced by the inverse of the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. 4.2.1) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when a length scale resolution L_p is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. There is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. This is not a problem if canonical identification applies only to the coordinate interval $(0, 2\pi/N)$ or its hyperbolic variant defining the finite measurement resolution. That p-adicization program works only for highly symmetric spaces is not a problem from the point of view of TGD.

4.2.2 Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [5]. The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization p^{L_0/T_p}, where L_0 is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. T_p is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzman weights the weights p^{-L_0/T_p}. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

Also p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

4.2.3 The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals or roots of unity is that it does not respect continuity. The restriction of S-matrix to a discrete intersection of real and p-adic worlds is one manner to solve this difficulty.

One can also consider alternative approach to achieve a compromise between algebra and topology achieved by using a modification of canonical identification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [24]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in R_p are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R\rightarrow R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n} / \sum s_n p^{-n}$ interpreted as a p-adic number:

$$q = \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \rightarrow q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \ . \tag{4.4}$$

R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R\rightarrow R_{p_2}} I_{R_{p_1}\rightarrow R}$.

This variant of canonical identification seems to be an excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [24]. The deviations of predictions from those for standard form of canonical identification are however small.

The cautious conclusion of this section is that symmetric space approach involving both the identification along common rationals of roots of unity in large and canonical identification below the measurement resolution provide the safest approach to the p-adicization of quantum TGD. The impossibility to well-order the points below measurement resolution explains why effective p-adic topology works in real context. The discussion of integration and Fourier analysis will provide further support for the conclusion.

5 p-Adic variants of the basic mathematical structures relevant to physics

The basic existential questions worrying a person planning to become a p-adic quantum physicist are rather obvious. How to define p-adic probabilities, p-adic thermodynamics, and p-adic unitarity and perhaps even p-adic Hilbert space? Is it possible to define the p-adic variant of the manifold concept? As already noticed for symmetric spaces p-adic variants might exist but what about space-time surfaces: could it be enough to consider only the p-adic variants of the partonic 2-surfaces in the manner already discussed? Can one somehow circumvent the difficulties related to the definition of the p-adic variant of definite integral? Perhaps by using Fourier analysis? How can one circumvent the fact that the basic variational principle involves integral over space-time surface which is p-adically notoriously difficult to define? Is all this just a waste of time or could it be that the enormous constraints from p-adicization could provide information about real physics not achievable otherwise (as in the case of p-adic mass calculations)?

5.1 p-Adic probabilities

p-Adic super conformal representations necessitate p-adic QM based on the p-adic unitarity and p-adic probability concepts. The concept of a p-adic probability indeed makes sense as shown by [43]. p-Adic probabilities can be defined as relative frequencies N_i/N in a long series consisting of total number N of observations and N_i outcomes of type i. Probability conservation corresponds to

$$\sum_i N_i = N \ , \tag{5.1}$$

and the only difference as compared to the usual probability is that the frequencies are interpreted as p-adic numbers.

The interpretation as p-adic numbers means that the relative frequencies converge to probabilities in a p-adic rather than real sense in the limit of a large number N of observations. If one requires that probabilities are limiting values of the frequency ratios in p-adic sense one must pose restrictions on the possible numbers of the observations N if N is larger than p. For N smaller than p, the situation is similar to the real case. This means that for $p = M_{127} \simeq 10^{38}$, appropriate for the particle physics experiments, p-adic probability differs in no observable manner from the ordinary probability.

If the number of observations is larger than p, the situation changes. If N_1 and N_2 are two numbers of observations they are near to each other in the p-adic sense if they differ by a large power of p. A possible interpretation of this restriction is that the observer at the p:th level of the condensate cannot choose the number of the observations freely. The restrictions to this freedom come from the requirement that the sensible statistical questions in a p-adically conformally invariant world must respect p-adic conformal invariance [49].

The most important application of the p-adic probability is the description of the particle massivation based on p-adic thermodynamics. Instead of energy, Virasoro generator l is thermalized and in the low temperature phase temperature is quantized in the sense that the counterpart of the Boltzmann weight $exp(H/T)$ is $p^{L_0/T}$, where $T = 1/n$ from the requirement that Boltzmann weight exists (L_0 has integer spectrum). The surprising success of the mass calculations shows that p-adic probability theory is much more than a formal possibility.

In particle physics context coupling constant evolution is replaced with a discrete p-adic coupling constant evolution and the renormalization is related to the the change of the reduction of the p-adic length scale L_p in the length scale hierarchy rather than p-adic fractality for a fixed value of p. In zero energy ontology the evolution corresponds to the hierarchy of CDs with scales coming as powers of 2 in accordance with p-adic length scale hypothesis.

5.1.1 p-Adic probabilities and p-adic fractals

p-Adic probalities are natural in the statistical description of the fractal structures, which can contain same structural detail with all possible sizes.

1. The concept of a structural detail in a fractal seems to be reasonably well defined concept. The structural detail is clearly fixed by its topology and p-adic conformal invariants associated with it. Clearly, a finite resolution defined by some power of p of the p-adic cutoff scale must be present in the definition. For example, p-adic angles are conformal invariants in the p-adic case, too. The overall size of the detail doesn't matter. Let us therefore assume that it is possible to make a list, possibly infinite, of the structural details appearing in the p-adic fractal.

2. What kind of questions related to the structural details of the p-adic fractal one can ask? The first thing one can ask is how many times i:th structural detail appears in a finite region of the fractal structure: although this number is infinite as a real number it might possess (and probably does so!) finite norm as a p-adic number and provides a useful p-adic invariant of the fractal. If a complete list about the structural details of the fractal is at use one can calculate also the total number of structural details defined as $N = \sum_i N_i$. This means that one can also define p-adic probability for the appearance of i:th structural detail as a relative frequency $p_i = N_i/N$.

3. One can consider conditional probabilities, too. It is natural to ask what is the probability for the occurrence of the structural detail subject to the condition that part of the structural detail is fixed (apart from the p-adic conformal transformations). In order to evaluate these probalities as relative frequencies one needs to look only for those structural details containing the substructure in question.

4. The evaluation of the p-adic probalities of occurrence can be done by evaluating the required numbers N_i and N in a given resolution. A better estimate is obtained by increasing the resolution and counting the numbers of the hitherto unobserved structural details. The increase in the resolution greatly increases the number of the observations in case of p-adic fractal and the fluctuations in the values of N_i and N increase with the resolution so that N_i/N has no well defined limit as a real number although one can define the probabilities of occurrence as a resolution dependent concept. In the p-adic sense the increase in the values of N_i and fluctuations are small and the procedure should converge rapidly so that reliable estimates should result with quite a reasonable resolution. Notice that the increase of the fluctuations in the real sense, when resolution is increased is in accordance with the criticality of the system.

5. p-Adic frequencies and probabilities define via the canonical correspondence real valued invariants of the fractal structure.

p-Adic fractality in this sense could have practical applications only for small values of p. They could be important in the macroscopic length scales if the hierarchy of Planck constants meaning scaling up $L_p \to \sqrt{r}L_p$, $r = \hbar/\hbar_0$, of the p-adic length scales. In elementary particle physics L_p is of the order of the Compton length associated with the particle for $r = 1$ and already in the first downward step CP_2 length scale R is achieved whereas upward step gives astrophysical length scale in the case of electron ($p = M_{127} = 2^{127} - 1$) for instance. For large enough values of Planck constant and for small p-adic primes p the situation changes.

5.1.2 Relationship between p-adic and real probabilities

There are uniqueness problems related to the mapping of p-adic probabilities to real ones. These problems find a nice resolution from the requirement that the map respects probability conservation. The implied modification of the original mapping does not change measurably the predictions for the masses of light particles.

1. How unique the map of p-adic probabilities and mass squared values are mapped to real numbers is?

The mapping of p-adic thermodynamical probabilities and mass squared values to real numbers is not completely unique.

1. The canonical identification $Id : \sum x_n p^n \to \sum x_n p^{-n}$ takes care of this mapping but does not respect the sum of probabilities so that the real images $I(p_n)$ of the probabilities must be normalized. This is a somewhat alarming feature.

2. The modification of the canonical identification mapping rationals by the formula $I(r/s) = I(r)/I(s)$ has appeared naturally in various applications, in particular because it respects unitarity of unitary matrices with rational elements with $r < p, s < p$. In the case of p-adic thermodynamic the formula $I(g(n)p^n/Z) \to I(g(n)p^n)/I(Z)$ would be very natural although Z need not be rational anymore. For $g(n) < p$ the real counterparts of the p-adic probabilities would sum up to one automatically for this option. One cannot deny that this option is more convincing than the original one. The generalization of this formula to map p-adic mass squared to a real one is obvious.

3. Options 1) and 2) differ dramatically when the $n = 0$ massless ground state has ground state degeneracy $D > 1$. For option 1) the real mass is predicted to be of order CP_2 mass whereas for option 2) it would be by a factor $1/D$ smaller than the minimum mass predicted by the option 1). Thus option 2) would predict a large number of additional exotic states. For those states which are light for option 1), the two options make identical predictions as far as the significant two lowest order terms are considered. Hence this interpretation would not change the predictions of the p-adic mass calculations in this respect. Option 2) is definitely more in accord with the real physics based intuitions and the main role of p-adic thermodynamics would be to guarantee the quantization of the temperature and fix practically uniquely the spectrum of the "Hamiltonian".

2. Under what conditions the mapping of p-adic ensemble probabilities to real probabilities respects probability conservation?

One can consider also a more general situation. Assume that one has an ensemble consisting of independent elementary events such that the number of events of type i is N_i. The probabilities are given by $p_i = N_i/N$ and $N = \sum N_i$ is the total number of elementary events. Even in the case that N is infinite as a real number it is natural to map the p-adic probabilities to their real counterparts using the rational canonical identification $I(p_i) = I(N_i)/I(N)$. Of course, N_i and N exist as well defined p-adic numbers under very stringent conditions only.

The question is under what conditions this map respects probability conservation. The answer becomes obvious by looking at the pinary expansions of N_i and N. If the integers N_i (possibly infinite as real integers) have pinary expansions having no common pinary digits, the sum of probabilities is conserved in the map. Note that this condition can assign also to a finite ensemble with finite number of a unique value of p.

This means that the selection of a basis for independent events corresponds to a decomposition of the set of integers labelling pinary digits to disjoint sets and brings in mind the selection of orthonormalized basis of quantum states in quantum theory. What is physically highly non-trivial that this "orthogonalization" alone puts strong constraints on probabilities of the allowed elementary events. One can say that the probabilities define distributions of pinary digits analogous to non-negative probability amplitudes in the space of integers labelling pinary digits, and the probabilities of independent events must be orthogonal with respect to the inner product defined by point-wise multiplication in the space of pinary digits.

p-Adic thermodynamics for which Boltzman weights $g(E)exp(-E/T)$ are replaced by $g(E)p^{E/T}$ such that one has $g(E) < p$ and E/T is integer valued, satisfies this constraint. The quantization of E/T to integer values implies quantization of both T and "energy" spectrum and forces so called super conformal invariance [49, 51] in TGD applications, which is indeed a basic symmetry of the theory.

There are infinitely many ways to choose the elementary events and each choice corresponds to a decomposition of the infinite set of integers n labelling the powers of p to disjoint subsets. These subsets can be also infinite. One can assign to this kind of decomposition a resolution which is the poorer the larger the subsets involved are. p-Adic thermodynamics would represent the situation in which the resolution is maximal since each set contains only single pinary digit. Note the analogy with the basis of completely localized wave functions in a lattice.

3. How to map p-adic transition probabilities to real ones?

p-Adic variants of TGD, if they exist, give rise to S-matrices and transition probabilities P_{ij}, which are p-adic numbers.

1. The p-adic probabilities defined by rows of S-matrix mapped to real numbers using canonical identification respecting the $q = r/s$ decomposition of rational number or its appropriate generalization should define real probabilities.

2. The simplest example would simple renormalization for the real counterparts of the p-adic probabilities $(P_{ij})_R$ obtained by canonical identification (or more probably its appropriate modification).

$$\begin{aligned} P_{ij} &= \sum_{k\geq 0} P_{ij}^k p^k \ , \\ P_{ij} &\rightarrow \sum_{k\geq 0} P_{ij}^k p^{-k} \equiv (P_{ij})_R \ , \\ (P_{ij})_R &\rightarrow \frac{(P_{ij})_R}{\sum_j (P_{ij})_R} \equiv P_{ij}^R \ . \end{aligned} \tag{5.2}$$

The procedure converges rapidly in powers of p and resembles renormalization procedure of quantum field theories. The procedure automatically divides away one four-momentum delta function from the square of S-matrix element containing the square of delta function with no well defined mathematical meaning. Usually one gets rid of the delta function interpreting it as the inverse of the four-dimensional measurement volume so that transition rate instead of transition probability is obtained. Of course, also now same procedure should work either as a discrete or a continuous version.

3. Probability interpretation would suggest that the real counterparts of p-adic probabilities sum up to unity. This condition is rather strong since it would hold separately for each row and column of the S-matrix.

4. A further condition would be that the real counterparts of the p-adic probabilities for a given prime p are identical with the transition probabilities defined by the real S-matrix for real space-time sheets with effective p-adic topology characterized by p. This condition might allow to deduce all relevant phase information about real and corresponding p-adic S-matrices using as an input only the observable transition probabilities.

4. What it means that p-adically independent events are not independent in real sense?

A further condition would be that p-adic quantum transitions represent also in the real sense independent elementary events so that the real counterpart for a sum of the p-adic probabilities for a finite number of transitions equals to the sum of corresponding real probabilities. This condition is definitely too strong in the generl case since only a single transition could correspond to a given p-adic norm of transition probability P_{ij} with i fixed. In p-adic thermodynamics it can be satisfied if the degeneracy for an energy eigenstate for a given eigen value $L_0 = n$ is not larger than p. This condition fails for large values of n for super Virasoro representations since the degeneracy grows exponentially. This has not practical implications for the large values of p considered.

The crucial question concerns the physical difference between the real counterpart for the sum of the p-adic transition probabilities and for the sum of the real counterparts of these probabilities, which are in general different:

$$(\sum_j P_{ij})_R \neq \sum_j (P_{ij})_R \ . \tag{5.3}$$

The suggestion is that p-adic sum of the transition probabilities corresponds to the experimental situation, when one does not monitor individual transitions but using some common experimental signature only looks whether the transition leads to this set of the final states or not. When one looks each transition separately or effectively performs different experiment by considering only one transition channel in each experiment one must use the sum of the real probabilities. More precisely, the choice of the experimental signatures divides the set U of the final states to a disjoint union $U = \cup_i U_i$ and one must define the real counterparts for the transition probabilities P_{iU_k} as

$$\begin{aligned} P_{iU_k} &= \sum_{j\in U_k} P_{ij} \ , \\ P_{iU_k} &\rightarrow (P_{iU_k})_R \ , \\ (P_{iU_k})_R &\rightarrow \frac{(P_{iU_k})_R}{\sum_l (P_{iU_l})_R} \equiv P^R_{iU_k} \ . \end{aligned} \tag{5.4}$$

The assumption means deep a departure from the ordinary probability theory. If p-adic physics is the physics of cognitive systems, there need not be anything mysterious in the dependence of the behavior of system on how it is monitored. At least half-jokingly one might argue that the behavior of an intelligent system indeed depends strongly on whether the boss is nearby or not. The precise definition for the monitoring could be based on the decomposition of the density matrix representing the entangled subsystem into a direct sum over the subspaces associated with the degenerate eigenvalues of the density matrix. This decomposition provides a natural definition for the notions of the monitoring and resolution.

The renormalization procedure is in fact familiar from standard physics. Assume that the labels j correspond to momenta. The division of momentum space to cells of a given size so that the individual momenta inside cells are not monitored separately means that momentum resolution is finite. Therefore one must perform p-adic summation over the cells and define the real probabilities in the proposed manner. p-Adic effects resulting from the difference between p-adic and real summations could be the counterpart of the renormalization effects in QFT. It should be added that similar resolution can be defined also for the initial states by decomposing them into a union of disjoint subsets.

5.1.3 p-Adic thermodynamics

The p-adic field theory limit as such is not expected to give a realistic theory at elementary particle physics level. The point is that particles are expected to be either massless or possess mass of order 10^{-4} Planck mass. The p-adic description of particle massivation described in [5] shows that p-adic thermodynamics provides the proper formulation of the problem. What is thermalized is Virasoro generator L_0 (mass squared contribution is not included to L_0 so that states do not have a fixed conformal weight). Temperature is quantized purely number theoretically in low temperature limit ($exp(H/kT) \rightarrow p^{L_0/T}$, $T = 1/n$): in fact, the partition function does not even exist in high temperature phase. The extremely small mixing of massless states with Planck mass states implies massivation and predictions of the p-adic thermodynamics for the fermionic masses are in excellent agreement with experimental masses. Thermodynamic approach also explains the emergence of the length scale L_p for a given p-adic condensation level and one can develop arguments explaining why primes near prime powers of two are favored.

It should be noticed that rational p-adic temperatures $1/T = k/n$ are possible, if one poses the restriction that thermal probabilities are non-vanishing only for some subalgebra of the Super Virasoro algebra isomorphic to the Super Virasoro algebra itself. The generators L_{kn},G_{kn}, where k is a positive integer, indeed span this kind of a subalgebra by the fractality of the Super Virasoro algebra and $p^{L_0/T}$ is integer valued with this restriction.

One might apply thermodynamics approach should also in the calculation of S-matrix. What is is needed is thermodynamical expectation value for the transition amplitudes squared over incoming and outgoing states. In this expectation value 3-momenta are fixed and only mass squared varies.

5.1.4 Generalization of the notion of information

TGD inspired theory of consciousness, in particular the formulation of Negentropy Maximization Principle (NMP) in p-adic context, has forced to rethink the notion of the information concept. In TGD state preparation process is realized as a sequence of self measurements. Each self measurement means a decomposition of the sub-system involved to two unentangled parts. The decomposition is fixed highly uniquely from the requirement that the reduction of the entanglement entropy is maximal.

The additional assumption is that bound state entanglement is stable against self measurement. This assumption is somewhat ad hoc and it would be nice to get rid of it. The only manner to achieve this seems to be a generalized definition of entanglement entropy allowing to assign a negative value of entanglement entropy to the bound state entanglement, so that bound state entanglement would actually carry information, in fact conscious information (experience of understanding). This would be very natural since macro-temporal quantum coherence corresponds to a generation of bound state entanglement, and is indeed crucial for ability to have long lasting non-entropic mental images.

The generalization of the notion of number concept leads immediately to the basic problem. How to generalize the notion of entanglement entropy that it makes sense for a genuinely p-adic entanglement? What about the number-theoretically universal entanglement with entanglement probabilities, which correspond to finite extension of rational numbers? One can also ask whether the generalized notion of information could make sense at the level of the space-time as suggested by quantum-classical correspondence.

In the real context Shannon entropy is defined for an ensemble with probabilities p_n as

$$S = -\sum_n p_n log(p_n) \ . \tag{5.5}$$

As far as theory of consciousness is considered, the basic problem is that Shannon entropy is always non-negative so that as such it does not define a genuine information measure. One could define information as a change of Shannon entropy and this definition is indeed attractive in the sense that quantum jump is the basic element of conscious experience and involves a change. One can however argue that the mere ability to transfer entropy to environment (say by aggressive behavior) is not all that is involved with conscious information, and even less so with the experience of understanding or moment of heureka. One should somehow generalize the Shannon entropy without losing the fundamental additivity property.

1. p-Adic entropies

The key observation is that in the p-adic context the logarithm function $log(x)$ appearing in the Shannon entropy is not defined if the argument of logarithm has p-adic norm different from 1. Situation changes if one uses an extension of p-adic numbers containing $log(p)$: the conjecture is that this extension is finite-dimensional. One might however argue that Shannon entropy should be well defined even without the extension.

p-Adic thermodynamics inspires a manner to achieve this. One can replace $log(x)$ with the logarithm $log_p(|x|_p)$ of the p-adic norm of x, where log_p denotes p-based logarithm. This logarithm is integer valued ($log_p(p^n) = n$), and is interpreted as a p-adic integer. The resulting p-adic entropy

$$\begin{aligned} S_p &= \sum_n p_n k(p_n) \ , \\ k(p_n) &= -log_p(|p_n|) \ . \end{aligned} \tag{5.6}$$

is additive: that is the entropy for two non-interacting systems is the sum of the entropies of composites. Note that this definition differs from Shannon's entropy by the factor $log(p)$. This entropy vanishes identically in the case that the p-adic norms of the probabilities are equal to one. This means that it is possible to have non-entropic entanglement for this entropy.

One can consider a modification of S_p using p-adic logarithm if the extension of the p-adic numbers contains $log(p)$. In this case the entropy is formally identical with the Shannon entropy:

$$S_p = -\sum_n p_n log(p_n) = -\sum_n p_n \left[-k(p_n) log(p) + p^{k_n} log(p_n/p^{k_n}\right] \ . \tag{5.7}$$

It seems that this entropy cannot vanish.

One must map the p-adic value entropy to a real number and here canonical identification can be used:

$$\begin{aligned} S_{p,R} &= (S_p)_R \times log(p)) \ , \\ (\sum_n x_n p^n)_R &= \sum_n x_n p^{-n} \ . \end{aligned} \tag{5.8}$$

The real counterpart of the p-adic entropy is non-negative.

2. Number theoretic entropies and metabolic energy

In the case that the probabilities are rational or belong to a finite-dimensional extension of rationals, it is possible to regard them as real numbers or p-adic numbers in some extension of p-adic numbers for any p. The visions that rationals and their finite extensions correspond to islands of order in the seas of chaos of real and p-adic transcendentals suggests that states having entanglement coefficients in finite-dimensional extensions of rational numbers are somehow very special. This is indeed the case. The p-adic entropy entropy $S_p = -\sum_n p_n log_p(|p_n|) log(p)$ can be interpreted in this case as an ordinary rational number in an extension containing $log(p)$.

What makes this entropy so interesting is that it can have also negative values in which case the interpretation as an information measure is natural. In the real context one can fix the value of the value of the prime p by requiring that S_p is maximally negative, so that the information content of the ensemble could be defined as

$$I \equiv Max\{-S_p, \ p \text{ prime}\} \ . \tag{5.9}$$

This information measure is positive when the entanglement probabilities belong to a finite-dimensional extension of rational numbers. Thus kind of entanglement is stable against NMP [25], and has a natural interpretation as a negentropic entanglement.

There is no need to interpret negentropic entanglement as bound state entanglement as was the original proposal. This together with the vision about life as something in the intersection of the real and p-adic worlds inspires the idea about a connection between information and metabolism in living matter. Metabolic energy could be carried by negentropic entanglement and the feed of metabolic energy would be also feed of negentropy. In particular, the poorly understood high energy phosphate bond could be identified as a bond involving negentropic entanglement [20]. The prediction would be that the negentropic states of real systems form a number theoretical hierarchy according to the prime p and and dimension of algebraic extension characterizing the entanglement.

Number theoretically state function reduction and state preparation could be seen as information generating processes in the intersection of real and p-adic worlds. p-Adic $\leftrightarrow$ real transitions make sense in the intersection with interpretation asa realization of intentional action and build-up of cognitive representations. Later an argument that these processes have a purely number theoretical interpretation will be developed based on the generalized notion of unitarity allowing the U-matrix to have matrix elements between the sectors of the state space corresponding to different number fields.

5.2 How to define integration and p-adic Fourier analysis, integral calculus, and p-adic counterparts of geometric objects?

p-Adic differential calculus exists and obeys essentially the same rules as ordinary differental calculus. The only difference from real context is the existence of p-adic pseudoconstants: any function which depends on finite number of pinary digits has vanishing p-adic derivative. This implies non-determinism of p-adic differerential equations. One can defined p-adic integral functions using the fact that indefinite integral is the inverse of differentiation. The basis problem with the definite integrals is that p-adic numbers are not well-ordered so that the crucial ordering of the points of real axis in definite integral is not unique. Also p-adic Fourier analysis is problematic since direct counterparts of ep(ix) and trigonometric functions are not periodic. Also exp(-x) fails to converse exponentially since it has p-adic norm equal to 1. Note also that these functions exists only when the p-adic norm of x is smaller than 1.

The following considerations support the view that the p-adic variant of a geometric objects, integration and p-adic Fourier analysis exists but only when one considers highly symmetric geometric objects such as symmetric spaces. This is wellcome news from the point of view of physics. At the level of space-time surfaces this is problematic. The field equations associated with Kähler action and modified Dirac equation make sense. Kähler action defined as integral over p-adic space-time surface fails to exist. If however the Kähler function identified as Kähler for a preferred extremal of Kähler action is rational or algebraic function of preferred complex coordinates of WCW with ratonal coefficients, its p-adic continuation is expected to exist.

5.2.1 Circle with rotational symmetries and its hyperbolic counterparts

Consider first circle with emphasis on symmetries and Fourier analysis.

1. In this case angle coordinate ϕ is the natural coordinate. It however does not make sense as such p-adically and one must consider either trigonometric functions or the phase $exp(i\phi)$ instead. If one wants to do Fourier analysis on circle one must introduce roots $U_{n,N} = exp(in2\pi/N)$ of unity. This means discretization of the circle. Introducing all roots $U_{n,p} = exp(i2\pi n/p)$, such that p divides N, one can represent all $U_{k,n}$ up to $n = N$. Integration is naturally replaced with sum by using discrete Fourier analysis on circle. Note that the roots of unity can be expressed as products of powers of roots of unity $exp(in2\pi/p^k)$, where p^k divides N.

2. There is a number theoretical delicacy involved. By Fermat's theorem $a^{p-1}\ mod\ p = 1$ for $a = 1, ...p-1$ for a given p-adic prime so that for any integer M divisible by a factor of $p-1$ the M:th roots of unity exist as ordinary p-adic numbers. The problem disappears if these values of M are excluded from the discretization for a given value of the p-adic prime. The manner to achieve this is to assume that N contains no divisors of $p-1$ and is consistent with the notion of finite measurement resolution. For instance, $N = p^n$ is an especially natural choice guaranteing this.

3. The p-adic integral defined as a Fourier sum does not reduce to a mere discretization of the real integral. In the real case the Fourier coefficients must approach to zero as the wave vector $k = n2\pi/N$ increases. In the p-adic case the condition consistent with the notion of finite measurement resolution for angles is that the p-adic valued Fourier coefficients approach to zero as n increases. This guarantees the p-adic convergence of the discrete approximation of the integral for large values of N as n increases. The map of p-adic Fourier coefficients to real ones by canonical identification could be used to relate p-adic and real variants of the function to each other.

This finding would suggests that p-adic geometries -in particular the p-adic counterpart of CP_2, are discrete. Variables which have the character of a radial coordinate are in natural manner p-adically continuous whereas phase angles are naturally discrete and described in terms of algebraic extensions. The conclusion is disappoing since one can quite well argue that the discrete structures can be regarded as real. Is there any manner to escape this conclusion?

1. Exponential function $exp(ix)$ exists p-adically for $|x|_p \leq 1/p$ but is not periodic. It provides representation of p-adic variant of circle as group $U(1)$. One obtains actually a hierarchy of groups $U(1)_{p,n}$ corresponding to $|x|_p \leq 1/p^n$. One could consider a generalization of phases as products $Exp_p(N, n2\pi/N + x) = exp(in2\pi n/N)exp(ix)$ of roots of unity and exponent functions with an imaginary exponent. This would assign to each root of unity p-adic continuum interpreted as the analog of the interval between two subsequent roots of unity at circle.

The hierarchies of measurement resolutions coming as $2\pi/p^n$ would be naturally accompanied by increasingly smaller p-adic groups $U(1)_{p,n}$.

2. p-Adic integration would involve summation plus possibly also an integration over each p-adic variant of discretization interval. The summation over the roots of unity implies that the integral of $\int exp(inx)dx$ would appear for $n = 0$. Whatever the value of this integral is, it is compensated by a normalization factor guaranteing orthonormality.

3. If one interprets the p-adic coordinate as p-adic integer without the identification of points differing by a multiple of n as different points the question whether one should require p-adic continuity arises. Continuity is obtained if $U_n(x + mp^m) = U_n(x)$ for large values of m. This is obtained if one has $n = p^k$. In the spherical geometry this condition is not needed and would mean quantization of angular momentum as $L = p^k$, which does not look natural. If representations of translation group are considered the condition is natural and conforms with the spirit of the p-adic length scale hypothesis.

The hyperbolic counterpart of circle corresponds to the orbit of point under Lorentz group in two 2-D Minkowski space. Plane waves are replaced with exponentially decaying functions of the coordinate η replacing phase angle. Ordinary exponent function $exp(x)$ has unit p-adic norm when it exists so that it is not a suitable choice. The powers p^n existing for p-adic integers however approach to zero for large values of $x = n$. This forces discretization of η or rather the hyperbolic phase as powers of p^x, $x = n$. Also now one could introduce products of $Exp_p(nlog(p) + z) = p^n exp(x)$ to achieve a p-adic continuum. Also now the integral over the discretization interval is compensated by orthonormalization and can be forgotten. The integral of exponential function would reduce to a sum $\int Exp_p dx = \sum_k p^k = 1/(1 - p)$. One can also introduce finite-dimensional but non-algebraic extensions of p-adic numbers allowing e and its roots $e^{1/n}$ since e^p exists p-adically.

5.2.2 Plane with translational and rotational symmetries

Consider first the situation by taking translational symmetries as a starting point. In this case Cartesian coordinates are natural and Fourier analysis based on plane waves is what one wants to define. As in the previous case, this can be done using roots of unity and one can also introduce p-adic continuum by using the p-adic variant of the exponent function. This would effectively reduce the plane to a box. As already noticed, in this case the quantization of wave vectors as multiples of $1/p^k$ is required by continuity.

One can take also rotational symmetries as a starting point. In this case cylindrical coordinates (ρ, ϕ) are natural.

1. Radial coordinate can have arbitrary values. If one wants to keep the connection $\rho = \sqrt{x^2 + y^2}$ with the Cartesian picture square root allowing extension is natural. Also the values of radial coordinate proportional to odd power of p are problematic since one should introduce $\sqrt{p}$: is this extension internally consistent? Does this mean that the points $\rho \propto p^{2n+1}$ are excluded so that the plane decomposes to annuli?

2. As already found, angular momentum eigen states can be described in terms of roots of unity and one could obtain continuum by allowing also phases defined by p-adic exponent functions.

3. In radial direction one should define the p-adic variants for the integrals of Bessel functions and they indeed might make sense by algebraic continuation if one consistently defines all functions as Fourier expansions. Delta-function renormalization causes technical problems for a continuum of radial wave vectors. One could avoid the problem by using expontentially decaying variants of Bessel function in the regions far from origin, and here the already proposed description of the hyperbolic counterparts of plane waves is suggestive.

4. One could try to understand the situation also using Cartesian coordinates. In the case of sphere this is achieved by introducing two coordinate patches with Cartesian coordinates. Pythagorean phases are rational phases (orthogonal triangles for which all sides are integer valued) and form a dense set on circle. Complex rationals (orthogonal triangles with integer valued short sides) define a more general dense subset of circle. In both cases it is difficult to imagine a discretized version of integration over angles since discretization with constant angle increrement is not possible.

5.2.3 The case of sphere and more general symmetric space

In the case of sphere spherical coordinates are favored by symmetry considerations. For spherical coordinates $sin(\theta)$ is analogous to the radial coordinate of plane. Legedre polynomials expressible as polynomials of $sin(\theta)$ and $cos(\theta)$ are expressible in terms of phases and the integration measure $sin^2(\theta)d\theta d\phi$ reduces the integral of S^2 to summation. As before one can introduce also p-adic continuum. Algebraic cutoffs in both angular momentum l and m appear naturally. Similar cutoffs appear in the representations of quantum groups and there are good reasons to expect that these phenomena are correlated.

Exponent of Kähler function appears in the integration over configuration space. From the expression of Kähler gauge potential given by $A_\alpha = J_\alpha{}^\theta \partial_\theta K$ one obtains using $A_\alpha = cos(\theta)\delta_{\alpha,\phi}$ and $J_{\theta\phi} = sin(\theta)$ the expression $exp(K) = sin(\theta)$. Hence the exponent of Kähler function is expressible in terms of spherical harmonics.

The completion of the discretized sphere to a p-adic continuum- and in fact any symmetric space- could be performed purely group theoretically.

1. Exponential map maps the elements of the Lie-algebra to elements of Lie-group. This recipe generalizes to arbitrary symmetric space G/H by using the Cartan decomposition $g = t + h$, $[h, h] \subset h$,$[h, t] \subset t$,$[t, t] \subset h$. The exponentiation of t maps t to G/H in this case. The exponential map has a p-adic generalization obtained

by considering Lie algebra with coefficients with p-adic norm smaller than one so that the p-adic exponent function exists. As a matter fact, one obtains a hierarchy of Lie-algebras corresponding to the upper bounds of the p-adic norm coming as p^{-k} and this hierarchy naturally corresponds to the hierarchy of angle resolutions coming as $2\pi/p^k$. By introducing finite-dimensional transcendental extensions containing roots of e one obtains also a hierarchy of p-adic Lie-algebras associated with transcendental extensions.

2. In particular, one can exponentiate the complement of the $SO(2)$ sub-algebra of $SO(3)$ Lie-algebra in p-adic sense to obtain a p-adic completion of the discrete sphere. Each point of the discretized sphere would correspond to a p-adic continuous variant of sphere as a symmetric space. Similar construction applies in the case of CP_2. Quite generally, a kind of fractal or holographic symmetric space is obtained from a discrete variant of the symmetric space by replacing its points with the p-adic symmetric space.

3. In the N-fold discretization of the coordinates of M-dimensional space t one $(N-1)^M$ discretization volumes which is the number of points with non-vanishing t-coordinates. It would be nice if one could map the p-adic discretization volumes with non-vanishing t-coordinates to their positive valued real counterparts by applying canonical identification. By group invariance it is enough to show that this works for a discretization volume assignable to the origin. Since the p-adic numbers with norm smaller than one are mapped to the real unit interval, the p-adic Lie algebra is mapped to the unit cell of the discretization lattice of the real variant of t. Hence by a proper normalization this mapping is possible.

The above considerations suggests that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The above considerations suggest that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The following arguments however support this identification.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M unless one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. Otherwise the measurement resolution is just $\Delta\phi = 2\pi/p^n$. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer and assume that state function reduction selects one particular prime (no superposition of quantum states with different p-adic topologies). For $N = p^nM$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$. One could assign to a given measurement resolution all the p-adic primes appearing as factors in N so that the notion of multi-p p-adicity would make sense. One can also consider the identification of the measurement resolution as $\Delta\phi = |N/M|_p = 2\pi/p^k$. This interpretation is supported by the approach based on infinite primes [17, A6].

5.2.4 What about integrals over partonic 2-surfaces and space-time surface?

One can of course ask whether also the integrals over partonic 2-surfaces and space-time surface could be p-adicized by using the proposed method of discretization. Consider first the p-adic counterparts of the integrals over the partonic 2-surface X^2.

1. WCW Hamiltonians and Kähler form are expressible using flux Hamiltonians defined in terms of X^2 integrals of JH_A, where H_A is $\delta CD \times CP_2$ Hamiltonian, which is a rational function of the preferred coordinates defined by the exponentials of the coordinates of the sub-space t in the appropriate Cartan algebra decomposition. The flux factor $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ is scalar and does not actually depend on the induced metric.

2. The notion of finite measurement resolution would suggest that the discretization of X^2 is somehow induced by the discretization of $\delta CD \times CP_2$. The coordinates of X^2 could be taken to be the coordinates of the projection of X^2 to the sphere S^2 associated with $\delta M^4_\pm$ or to the homologically non-trivial geodesic sphere of CP_2 so that the discretization of the integral would reduce to that for S^2 and to a sum over points of S^2.

3. To obtain an algebraic number as an outcome of the summation, one must pose additional conditions guaranteing that both H_A and J are algebraic numbers at the points of discretization (recall that roots of unity are involved). Assume for definiteness that S^2 is $r_M = constant$ sphere. If the remaining preferred coordinates are functions of the preferred S^2 coordinates mapping phases to phases at discretization points, one obtains the desired outcome. These conditions are rather strong and mean that the various angles defining CP_2 coordinates -at least the two cyclic angle coordinates- are integer multiples of those assignable to S^2 at the points of discretization. This would be achieved if the preferred complex coordinates of CP_2 are powers of the preferred complex coordinate of S^2 at these points. One could say that X^2 is algebraically continued from a rational surface in the discretized variant of $\delta CD \times CP_2$. Furthermore, if the measurement resolutions come as $2\pi/p^n$ as p-adic continuity actually requires and if they correspond to the p-adic group $G_{p,n}$ for which group parameters satisfy $|t|_p \leq p^{-n}$, one can precisely characterize how a p-adic prime characterizes the real partonic 2-surface. This would be a fulfillment of one of the oldest dreams related to the p-adic vision.

A even more ambitious dream would be that even the integral of the Kähler action for preferred extremals could be defined using a similar procedure. The conjectured slicing of Minkowskian space-time sheets by string world sheets and partonic 2-surfaces encourages these hopes.

1. One could introduce local coordinates of H at both ends of CD by introducing a continuous slicing of $M^4 \times CP_2$ by the translates of $\delta M^4_\pm \times CP_2$ in the direction of the time-like vector connecting the tips of CD. As space-time coordinates one could select four of the eight coordinates defining this slicing. For instance, for the regions of the space-time sheet representable as maps $M^4 \rightarrow CP_2$ one could use the preferred M^4 time coordinate, the radial coordinate of δM^4_+, and the angle coordinates of $r_M = constant$ sphere.

2. Kähler action density should have algebraic values and this would require the strengthening of the proposed conditions for X^2 to apply to the entire slicing meaning that the discretized space-time surface is a rational surface in the discretized $CD \times CP_2$. If this condition applies to the entire space-time surface it would effectively mean the discretization of the classical physics to the level of finite geometries. This seems quite strong implication but is consistent with the preferred extremal property implying the generalized Bohr rules.

5.2.5 Tentative conclusions

These findings suggest following conclusions.

1. Exponent functions play a key role in the proposed p-adicization. This is not an accident since exponent functions play a fundamental role in group theory and p-adic variants of real geometries exist only under symmetries- possibly maximal possible symmetries- since otherwise the notion of Fourier analysis making possible integration does not exist. The inner product defined in terms of integration reduce for functions representable in Fourier basis to sums and can be carried out by using orthogonality conditions. Convolution involving integration reduces to a product for Fourier components. In the case of imbedding space and WCW these conditions are satisfied but for space-time surfaces this is not possible.

2. There are several manners to choose the Cartan algebra already in the case of sphere. In the case of plane one can consider either translations or rotations and this leads to different p-adic variants of plane. Also the realization of the hierarchy of Planck constants leads to the conclusion that the extended imbedding space and therefore also WCW contains sectors corresponding to different choices of quantization axes meaning that quantum measurement has a direct geometric correlate. One an imagine also other discretizations and choices of preferred coordinates and the interpretation is that they correspond to different cognitive representations and to to different p-adic physics. This means a refinement of General Coordinate Invariance taking into account cognition.

3. The above described 2-D examples represent symplectic geometries for which one has natural decomposition of coordinates to canonical pairs of cyclic coordinate (phase angle) and corresponding canonical conjugate co-ordinate. p-Adicization depends on whether the conjugate corresponds to an angle or noncompact coordinate. In both cases it is however possible to define integration. For instance, in the case of CP_2 one would have two canonically conjugate pairs and one can define the p-adic counterparts of CP_2 partial waves by generalizing the procedure applied to spherical harmonics. Products of functions expressible using partial waves can be decomposed by tensor product decomposition to spherical harmonics and can be integrated. In particular inner products can be defined as integrals. The Hamiltonians generating isometries are rational functions of phases: this inspires the hope that also WCW Hamiltonians also rational functions of preferred WCW coordinates and thus allow p-adic variants.

4. Discretization by introducing algebraic extensions seems unavoidable in the p-adicization of geometrical objects but one can have p-adic continuum as the analog of the discretization interval and in the function basis expressible in terms of phase factors and p-adic counterparts of exponent functions. As already described, the exponential map for Lie group provide an elegant manner to realize this. This would give a precise meaning for the p-adic counterparts of the imbedding space and WCW if the latter is a symmetric space allowing coordinatization in terms of phase angles and conjugate coordinates. The intersection of p-adic and real worlds in a given measurement resolution would be unique and correspond to the points defining the discretization.

5.3 p-Adic imbedding space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

5.3.1 p-Adic Riemannian geometry depends on cognitive representation

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal purely algebraic generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. Canonical identification would make it possible to define p-adic variant of Riemann integral formally allowing to calculate arc lengths and similar quantities but looks like a trick. The realization that the p-adic variant of harmonic analysis makes it possible to define definite integrals in the case of symmetric space became possible only after a detailed vision about what quantum TGD is [10] had emerged.

Symmetry considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ to a high degree but not uniquely. This non-uniqueness might relate to the distinction between different cognitive representations. For instance, in the case of Euclidian plane one can introduce linear or cylindrical coordinates and the manifest symmetries dictating the preferred coordinates correspond to translational and rotational symmetries in these two cases and give rise to different p-adic variants of the plane. Both linear and cylindrical coordinates are fixed only modulo the action of group consisting of translations and rotations and the degeneracy of choices can be interpreted in terms of a choice of quantization axies of angular momentum and momenta.

The most natural looking manner to define the p-adic counterpart of M^4 is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of M^4 linear Minkowski coordinates are an obvious choice but also the counterparts of Robertson-Walker coordinates for M^4_+ defined as $[t,(z,x,y)] = a \times [cosh(\eta), sinh(\eta)(cos(\theta), sin(\theta)cos(\phi), sin(\theta)sin(\phi))]$ expressible in terms of phases and their hyperbolic counterparts and transforming nicely under the Cartan algebra of Lorentz group are possible. p-Adic variant is obtained by introducing finite measurement resolution for angle and replacing angle range by finite number of roots of unity. Same applies to hyperbolic angles.

Rational CP_2 could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary 3×3-matrices. CP_2 could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational 3×3-matrix representable in terms of Pythagorean phases [48] and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

This option is not very natural as far as integration is considered. CP_2 however allows the analog of spherical coordinates for S^2 expressible in terms of angle variables alone and this suggests the introduction of the variant of CP_2 for which the coordinate values correspond to roots of unity. Completion would be performed in the same manner as for rational CP_2. This non-uniqueness need not be a drawback but could reflect the fact that the p-adic cognitive representation of real geometry are geometrically non-equivalent. This means a refinement of the principle of General Coordinate Invariance taking into account the fact that the cognitive representation of the real world affects the world with cognition included in a delicate manner.

5.3.2 The identification of the fundamental p-adic length scale

The fundamental p-adic length scale correponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the 'radius' R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the 'cosmological constant' Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 \ = \ 2\Lambda \ . \tag{5.10}$$

The relationship between R and l is uniquely fixed:

$$R^2 \ = \ \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} \ . \tag{5.11}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π:s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod \ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

   (b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of 1/3 rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in

the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R ,$$

rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

5.3.3 p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counter part of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p\ mod\ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal planewaves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naively expected p, in the expression of the p-adic length scale can be undertood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{5.12}$$

where m_{kl} is the standard M^4 metric $(1, -1, -1, -1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k:th coordinate axis the expression

$$s = R\sqrt{p} m^k \ . \tag{5.13}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{5.14}$$

The p-adic distance along the k:th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p} l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R \frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{5.15}$$

What is remarkable is that the shortest distance in the range $m^k = 1, ..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

5.3.4 The two variants of CP_2

As noticed, CP_2 allows two variants based on rational discretization and on the discretization based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rtional variant of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/colums are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 \;=\; \{x = exp(\sum_k it_k X_k) \; ; \; |t_k|_p < 1\} = \{x = 1 + iy \; ; \; |y|_p < 1\} \; . \tag{5.16}$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h \;=\; diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2)\} \; . \tag{5.17}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1 + O(p)$. For $p \; mod \; 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \; ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \; ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the 'physical' $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \; mod \; 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic submanifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \; , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \; . \end{aligned} \tag{5.18}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \; .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \; . \tag{5.19}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2\partial_i\partial_{\bar{j}}K = R^2\frac{(\delta_{i\bar{j}}r^2 - \bar{\xi}_i\xi_j)}{(1+r^2)^2} \ . \tag{5.20}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

6 Quantum physics in the intersection of p-adic and real worlds

The p-adicization of quantum TGD means several challenges. One should define the notions of Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be understood. Also the construction of Kähler geometry of "world of classical worlds" (WCW) in p-adic context should be carried out and the notion of WCW spinor fields should be defined in the p-adic context. The crucial technical problems relate to the notion of integral and Fourier analysis, which are the central elements of any physical theory. The basic challenge is to overcome the fact that although the field equations assignable to a given variational principle make sense p-adically, the action defined as an integral over arbitrary space-time surface has no natural p-adic counterpart as such in the generic case. What raises hopes that these challenges could be overcome is the symmetric space property of WCW and the idea of algebraic continuation. If WCW geometry is expressible in terms of rational functions with rational coefficients it allows a generalization to the p-adic context. Also integration can be reduced to Fourier analysis in the case of symmetric spaces. I have discussed the p-adicization and fusion of real and p-adic physics in earlier article [29] and will not go to it here anymore. Suffice it to say that the notion of symmetric space allowing to algebraize the integration is central element of the approach.

The intersection of real and p-adic worlds is especially interesting as far as the physics of living system is considered in TGD framework and is discussed in this section.

6.1 What it means to be in the intersection of real and p-adic worlds?

The first question is what one really means when one speaks about a partonic 2-surface in the intersection of real and p-adic worlds or in the intersection of two p-adic worlds.

1. Many algebraic numbers can be regarded also as ordinary p-adic numbers: square roots of roughly one half of integers provide a simple example about this. Should one assume that all algebraic numbers representable as ordinary p-adic numbers belong to the intersection of the real and p-adic variants of partonic 2-surface (or to the intersection of two different p-adic number fields)? Is there any hope that the listing of the points in the intersection is possible without a complete knowledge of the number theoretic anatomy of p-adic number fields in this kind of situation? And is the set of common algebraic points for real and p-adic variants of the partonic 2-surface X^2 quite too large- say a dense sub-set of X^2?

   This hopeless looking complexity is simplified considerably if one reduces the considerations to algebraic extensions of rationals since these induce the algebraic extensions of p-adic numbers. For instance, if the p-adic number field contains some n:th roots of integers in the range $(1, p-1)$ as ordinary p-adic numbers they are identified with their real counterparts. In principle one should be able to characterize the -probably infinite-dimensional- algebraic extension of rationals which is representable by a given p-adic number field as p-adic numbers of unit norm. This does not look very practical.

2. At the level WCW one must direct the attention to the function spaces used to define partonic 2-surfaces. That is the spaces of rational functions or even algebraic functions with coefficients of polynomials in algebraic extensions of rational numbers making sense with arguments in all number fields so that algebraic extensions of rationals provide a neat hierarchy defining also the points of partonic 2-surfaces to be considered. If one considers only the algebraic points of X^2 belonging to the extension appearing in the definition the function space as common to various number fields one has good hopes that the number of common points is finite.

3. Already the ratios of polynomials with rational coefficients lead to algebraic extensions of rationals via their roots. One can replace the coefficients of polynomials with numbers in algebraic extensions of rationals. Also algebraic functions involving roots of rational functions can be considered and force to introduce the algebraic extensions of p-adic numbers. For instance, an n:th root of a polynomial with rational coefficients is well defined if n:th roots of p-adic integers in the range $(1, p-1)$ are well well-defined. One clearly obtains an infinite hierarchy of function spaces. This would give rise to a natural hierarchy in which one introduces n:th roots for a minimum number of p-adic integers in the range $(1, p-1)$ in the range $1 \leq n \leq N$. Note that also the roots of unity would be introduced in a natural manner.

   The situation is made more complex because the partonic 2-surface is in general defined by the vanishing of six rational functions so that algebraic extensions are needed. An exception occurs when six preferred imbedding space coordinates are expressible as rational functions of the remaining two preferred coordinates. In this case the number of common rational points consists of all rational points associated with the remaining two coordinates. This situation is clearly non-generic. Usually the number of common points is much smaller (the set of rational points satisfying $x^n + y^n = z^n$ for $n > 2$ is a good example). This however suggests that these surfaces are of special importance since the naive expectation is that the amplitude for transformation of intention to action or its reversal is especially large in this case. This might also explain why these surfaces are easy to understand mathematically.

4. These considerations suggest that the numbers common to reals and p-adics must be defined as rationals and algebraic numbers appearing explicitly in the algebraic extension or rationals associated with the function spaces used to define partonic 2-surfaces. This would make the deduction of the common points of partonic 2-surface a task possible at least in principle. Algebraic extensions of rationals rather than those of p-adic numbers would be in the fundamental role and induce the extensions of p-adic numbers.

6.2 Braids and number theoretic braids

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [25].

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of CDs and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining X^2 make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [25]. The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ...$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions antifermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW, the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of M-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [25] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic points and and then perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or antifermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

6.3 Number theoretical Quantum Mechanics

The vision about life as something in the intersection of the p-adic and real worlds requires a generalization of quantum theory to describe the U-process properly. One must answer several questions. What it means mathematically to be in this intersection? What the leakage between different sectors does mean? Is it really possible to formally extend quantum theory so that direct sums of Hilbert spaces in different number fields make sense? Or should one consider the possibility of using only complex, algebraic, or rational Hilbert spaces also in p-adic sectors so that p-adicization would take place only at the level of geometry?

6.3.1 What it means to be in the intersection of real and p-adic worlds?

The first question is what one really means when one speaks about a partonic 2-surface in the intersection of real and p-adic worlds or in the intersection of two p-adic worlds.

1. Many algebraic numbers can be regarded also as ordinary p-adic numbers: square roots of roughly one half of integers provide a simple example about this. Should one assume that all algebraic numbers representable as ordinary p-adic numbers belong to the intersection of the real and p-adic variants of partonic 2-surface (or to the intersection of two different p-adic number fields)? Is there any hope that the listing of the points in the intersection is possible without a complete knowledge of the number theoretic anatomy of p-adic number fields in this kind of situation? And is the set of common algebraic points for real and p-adic variants of the partonic 2-surface X^2 quite too large- say a dense sub-set of X^2?

   This hopeless looking complexity is simplified considerably if one reduces the considerations to algebraic extensions of rationals since these induce the algebraic extensions of p-adic numbers. For instance, if the p-adic number field contains some n:th roots of integers in the range $(1, p-1)$ as ordinary p-adic numbers they are identified with their real counterparts. In principle one should be able to characterize the -probably infinite-dimensional- algebraic extension of rationals which is representable by a given p-adic number field as p-adic numbers of unit norm. This does not look very practical.

2. At the level WCW one must direct the attention to the function spaces used to define partonic 2-surfaces. That is the spaces of rational functions or even algebraic functions with coefficients of polynomials in algebraic extensions of rational numbers making sense with arguments in all number fields so that algebraic extensions of rationals provide a neat hierarchy defining also the points of partonic 2-surfaces to be considered. If one considers only the algebraic points of X^2 belonging to the extension appearing in the definition the function space as common to various number fields one has good hopes that the number of common points is finite.

3. Already the ratios of polynomials with rational coefficients lead to algebraic extensions of rationals via their roots. One can replace the coefficients of polynomials with numbers in algebraic extensions of rationals. Also algebraic functions involving roots of rational functions can be considered and force to introduce the algebraic extensions of p-adic numbers. For instance, an n:th root of a polynomial with rational coefficients is well defined if n:th roots of p-adic integers in the range $(1, p-1)$ are well well-defined. One clearly obtains an infinite hierarchy of function spaces. This would give rise to a natural hierarchy in which one introduces n:th roots for a minimum number of p-adic integers in the range $(1, p-1)$ in the range $1 \leq n \leq N$. Note that also the roots of unity would be introduced in a natural manner.

   The situation is made more complex because the partonic 2-surface is in general defined by the vanishing of six rational functions so that algebraic extensions are needed. An exception occurs when six preferred imbedding space coordinates are expressible as rational functions of the remaining two preferred coordinates. In this case the number of common rational points consists of all rational points associated with the remaining two coordinates. This situation is clearly non-generic. Usually the number of common points is much smaller (the set of rational points satisfying $x^n + y^n = z^n$ for $n > 2$ is a good example). This however suggests that these surfaces are of special importance since the naive expectation is that the amplitude for transformation of intention to action or its reversal is especially large in this case. This might also explain why these surfaces are easy to understand mathematically.

4. These considerations suggest that the numbers common to reals and p-adics must be defined as rationals and algebraic numbers appearing explicitly in the algebraic extension or rationals associated with the function spaces used to define partonic 2-surfaces. This would make the deduction of the common points of partonic 2-surface a task possible at least in principle. Algebraic extensions of rationals rather than those of p-adic numbers would be in the fundamental role and induce the extensions of p-adic numbers.

Let us next try to summarize the geometrical picture at the level of WCW and WCW spinor fields.

1. WCW decomposes into WCWs associated with CDs and there unions. For the unions one has Cartesian product of WCWs associated with CDs. At the level of WCW spinor fields one has tensor product.
2. The WCW for a given CD decomposes into a union of sectors corresponding to various number fields and their algebraic extensions. The sub-WCW corresponding to the intersection consists of partonic 2-surfaces X^2 (plus distribution of 4-D tangent spaces $T(X^4)$ at X^2 - a complication which will not be considered in the sequel), whose mathematical representation makes sense in real number field and in some algebraic extensions of p-adic number fields. The extension of p-adic number fields needed for algebraic extension of rationals depends on p and is in general sub-extension of the extension of rationals. This sub-WCW is a sub-manifold of WCW itself. It has also a filtering by sub-manifolds of QCW. For instance, partonic 2-surfaces representable using ratios of polynomials with degree below fixed number N defines an inclusion hierarchy with levels labelled by N.
3. The spaces of WCW spinors associated with these sectors are dictated by the second quantization of induced spinor fields with dynamics dictated by the modified Dirac action in more or less one-one correspondence. The dimension for the modes of induced spinor field (solutions of the modified Dirac equation at the space-time surface holographically assigned with X^2 plus the 4-D tangent space-space distribution) in general depends on the partonic 2-surface and the classical criticality of space-time surface suggests an inclusion hierarchy of super-conformal algebras corresponding to a hierarchy of criticalities. For instance, the partonic 2-surfaces X^2 having polynomial representations in referred coordinates could correspond to simplest possible surfaces nearest to the vacuum extremals and having in a well define sense smallest (but possibly infinite) dimension for the space of spinor modes.
4. For each CD one can decompose the Hilbert space to a formal direct sum of orthogonal state spaces associated with various number fields

$$H = \oplus_F H_F \ . \tag{6.1}$$

Here F serves as a label for number fields. For the sake of simplicity and to get idea about what is involved, all complications due to algebraic extensions are neglected in the sequel so that only rational surfaces are regarded as being common to various sectors of WCW.

5. The states in the direct sum make sense only formally since the formal inner product of these states would be a sum of numbers in different number fields unless one assigns complex Hilbert space with each sector or restricts the coefficients to be rational which is of course also possible. This problem is avoided if the state function reduction process induces inside each CD a choice of the number field. One could say that state function is a number theoretical necessity at least in this sense.
   (a) Should the state function reduction in this sense involve a reduction of entanglement between distinct CDs is not clear. One could indeed consider the possibility of a purely number theoretical reduction not induced by NMP and taking place in the absence of entanglement with reduction probabilities determined by the probabilities assignable to various number fields which should be rational or at most algebraic. Hard experience however suggests that one should not make exceptions from principles.
   (b) The alternative is to allow the Hilbert spaces in question to have rational or at most algebraic coefficients in the intersection of real and various p-adic worlds. This means that the entanglement is algebraic and NMP need not lead to a pure state: the superposition of pairs of entangled states is however mathematically well defined since inner products give algebraic numbers. Cognitive entanglement stable under NMP would become possible. The experience of understanding could be a correlate for it. The pairs in the sum defining the entangled state defined the instances of a concept as a mapping of real world state to its symbol structurally analogous to a Boolean rule. The entangled states between different p-adic number fields would define maps between symbolic representations.
6. Assume that each H_F allows a decomposition to a direct sum of two orthogonal parts corresponding to WCW spinor fields localized to the intersection of number fields and to the complements of the intersection:

$$\begin{aligned} H &= H_{nm} \oplus H_m \ , \\ H_{nm} &= \oplus_F H_{nm,F} \ , \ \ H_m = \oplus_F H_{m,F} \ . \end{aligned} \tag{6.2}$$

Here nm stands for 'no mixing' (no mixing between different number fields and localization to the complement of the intersection) and m for 'mixing' (mixing between different number fields in the intersection). F labels the number fields. Orthogonal direct sum might be mathematically rather singular and un-necessarily strong assumption but the notion of number theoretical criticality favors it.

6.3.2 The general structure of U-matrix neglecting the complexities due to algebraic extensions

M-matrix is diagonal with respect to the number field for obvious reasons. U-matrix can however induce a leakage between different number fields as well as entanglement between different number fields when unions of CDs are considered. The simplest assumption is that this entanglement is induced by the leakage between different number fields for single CD but not directly. For instance, the members of entangled pair of real states associated with two CDs leak to various p-adic sectors and induce in this manner entanglement beween different number fields. One must however notice that the part of U-matrix acting in the tensor product of Hilbert spaces assignable to separate CDs must be considered separately: it seems that the entanglement inducing part of U is diagonal with respect to number field except in the intersection.

To simplify the rather complex situation consider first the U matrix for a given CD by neglecting the possibility of algebraic extensions of the p-adic number fields. Restrict also the consideration to single CD.

1. The unitarity conditions do not make sense in a completely general sense since one cannot add numbers belonging to different number fields. The problem can be circumvented if the U-matrix decomposes into a product of U-matrices, which both are such that unitarity conditions make sense for them. Here an essential assumption is that unit matrix and projection operators are number theoretically universal. In this spirit assume that for a given CD U decomposes to a product of two U-matrices U_{nm} inducing no mixing between different number fields and U_m inducing the mixing in the intersection:

$$U = U_{nm}U_m . \tag{6.3}$$

   Here the subscript 'nm' (no mixing) having nothing to do with the induces of U as a matrix means that the action is restricted to a dispersion in a sector of WCW characterized by particular number field. The subscript 'm' (mixing) in turn means that the action corresponds to a leakage between different number fields possible in the intersection of worlds corresponding to different number fields and that U_m acts non-trivially in this intersection.

2. Assume that U_{nm} decomposes into a formal direct sum of U-matrices associated with various number fields F:

$$U_{nm} = \oplus_F U_{nm,F} . \tag{6.4}$$

   $U_{nm,F}$ acts inside H_F in both WCW and spin degrees of freedom, does not mix states belonging to different number fields, and creates a state which is always mathematically completely well defined in particular number field although the direct sum over number fields is only formally defined. Unitarity condition gives a direct sum of projection operators to Hilbert spaces associated with various number fields. One can assume that this object is number theoretically universal.

3. U_m acts in the intersection of the real and p-adic worlds identified in the simplied picture in terms of surfaces representable using ratios of polynomials with rational coefficients. The resulting superposition of configuration space spinor fields in different number fields is as such not mathematical sensible although the expression of U_m is mathematically well-defined. If the leakage takes place with same probability amplitude irrespective of the quantum state, U_m is a unitary operator, not affecting at all the spinor indices of WCW spinor fields characterizing quantum numbers of the state and whose action is analogous to unitary mixing of the identical copies of the state in various number fields.

   The probability with which the intention is realized as action would not therefore depend at all on the quantum number fields, but only on the data at points common to the variants of the partonic 2-surface in various number fields. Intention would reduce completely to the algebraic geometry of partonic 2-surfaces. This assumption allows to write U in the form

$$U = U_{nm}U_m , \tag{6.5}$$

   where U_m acts as an identity operator in H_{nm}.

6.3.3 The general structure of U-matrix when algebraic extensions of rationals are allowed

Consider now the generalization of the previous argument allowing also algebraic extensions.

1. For each algebraic extension of rationals one can express WCW as a union of two parts. The first one corresponds to to 2-surfaces, which belong to the intersection of real and p-adic worlds. The second one corresponds to 2-surfaces in the algebraic extension of genuine p-adic numbers and having necessarily infinite size in real sense. Thefore the decomposition of U to a product $U = U_{nm}U_m$ makes sense also now.

2. It is natural to assume that U_m decomposes to a product of two operators: $U_m = U_H U_Q$. The strictly horizontal operator U_H connects only same algebraic extensions of rationals assigned to different number fields. Here one must think that p-adic number fields represent a large number of algebraic extensions of rationals without need for an algebraic extension in the p-adic sense. The second unitary operator U_Q describes the leakage between different algebraic extensions of rationals. Number theoretical universality encourages the assumption that this unitary operator reduces to an operator U_Q acting on algebraic extensions of rationals regarded effectively as quantum states so that it would be same for all number fields. One can even consider the possibility that U_Q depends on the extensions of rationals only and not at all on partonic 2-surfaces. One cannot assume that U_Q corresponds just to an inclusion to a larger state space since this would give an infinite number of identical copies of same state and imply a non-normalizable state. Physically U_Q would define dispersion in the space of algebraic extension of rationals defining the rational function space giving rise to the sub-WCW. The simplest possibility is that U_Q between different algebraic extensions is just the projection operator to their intersection multiplied by a numerical constant determined number theoretical in terms of ratios of dimensions of the algebraic extensions so that the diffusion between extensions products unit norm states.

One must take into account the consistency conditions from the web of inclusions for the algebraic extensions of rationals inducing extensions of p-adic numbers.

1. There is an infinite inverted pyramide-like web of natural inclusions of WCWs associated with algebraic extensions of ratonal numbers and one can assign a copy of this web to all number fields if a given p-adic number field is characterized by a web defined by algebraic extensions of rationals numbers, which it is able to represent without explicit introduction of the algebraic extension, so that the pyramide is same for all number fields. For instance, the WCW corresponding to p-adic numbers proper is included to the WCWs associated with any of its genuine algebraic extensions and defines the lower tip of the inverted pyramide. From this tip an arrow emerges connecting it to every algebraic extension defining a node of this web. Besides these arrows there are arrows from a given extension to all extensions containing it.

2. These geometric inclusions induce inclusions of the corresponding Hilbert spaces defined by rational functions and possibly by algebraic functions in which case sub-web must be considered (all n:th roots of integers in the range $(1, p-1)$ must be introduced simultaneously). Leakage can occur between different extensions only through WCW spinor fields located in the common intersection of these spaces containing always the rational surfaces. The intersections of WCWs associated with various extensions of p-adic number fields correspond to WCWs assignable to rational functions with coeffficients in various algebraic extensions of rationals using preferred coordinates of CD and CP_2.

Together with unitarity conditions this web poses strong constraints on the unitary matrices U_m and U_Q expressible conveniently in terms of commuting diagrams. There are two kinds of webs. The vertical webs are defined by the algebraic extensions of rationals. These form a larger web in which lines connect the nodes of identical webs associated with various p-adic number fields and represent algebraic extensions of rationals.

1. One has the general product decomposition $U = U_{nm} U_Q U_m$, where U_{nm} does not induce mixing between number fields, and U_m does it purely horizontally but without affecting quantum states in WCW spin degreees of freedom, and $P(H_{nm})$ projects to the complement of the intersection of number fields holds true also now.

2. Each algebraic extension of rationals gives unitary conditions for the corresponding $U_{nm,F}$ for each p-adic number field with extensions included. These conditions are relatively simple and no commuting diagrams are needed.

3. In the horizontal web U_m mixes the states in the intersections of two number fields but connects only same algebraic extensions so that the lines are strictly horizontal. U_Q acts strictly vertically in the web formed by algebraic extension of rationals and its action is unitary. One has infinite number of commuting diagrams involving U_m and U_Q since the actions along all routes connecting given points between p_1 and p_2 must be identical.

4. If algebraic universality holds in the sense that U_m is expressible using only the data about the common points of 2-surfaces in the intersection defined by particular extensions using some universal functions, and U_Q is purely number theoretical unitary matrix having no dependence on partonic 2-surfaces, one can hope that the constraints due to commuting diagrams in the web of horizontal inclusions can be satisfied automatically and only the unitarity constraints remain. This web of inclusions brings strongly in mind the web of inclusions of hyper-finite factors.

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