Article

New Approach for Binormal Spherical Image in Terms of Inextensible Flow in E^3

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Abstract

In this work, we study the properties of the binormal spherical indicatrices (images) in terms of inextensible flows in E^3. Using the Frenet frame of the given curve, we present partial differential equations. We give some characterizations for curvatures of a curve in E^3.

Key Words: binormal images, fluid flow, E3, partial differential equation.

1 Introduction

For centuries, fluid flow researchers have been studying fluid flows in various ways, and today fluid flow is still an important field of research. The areas in which fluid flow plays a role are numerous. Gaseous flows are studied for the development of cars, aircraft and spacecrafts, and also for the design of machines such as turbines and combustion engines. Liquid flow research is necessary for naval applications, such as ship design, and is widely used in civil engineering projects such as harbour design and coastal protection. In chemistry, knowledge of fluid flow in reactor tanks is important; in medicine, the flow in blood vessels is studied. Numerous other examples could be mentioned. In all kinds of fluid flow research, visualization is an key issue, [1,2].

The local theory of space curves are mainly developed by the Frenet--Serret theorem which expresses the derivative of a geometrically chosen basis of E^3 by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidean space. It is a wellknown concept in the local differential geometry of curves. Such curves are obtained in terms of the Frenet--Serret vector fields (for details, see [2]).

This study is organised as follows: Firstly, we study inextensible flows of curves in Euclidean 3-space. Secondly, using the Frenet frame of the given curve, we present partial differential equations. Finally, we give some characterizations for curvatures of a curve in Euclidean 3-space.

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2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^3$ are briefly presented; a more complete elementary treatment can be found in [2].

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by

$$\|a\| = \sqrt{\langle a, a \rangle}.$$

α is called a unit speed curve if velocity vector v of α satisfies $\|v\| = 1$. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$. If the tangent vector of this curve forms a constant angle with a fixed constant vector U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space $\mathbb{E}^3$ is defined by

$$\mathbb{S}^2 = \left\{ p = (p_1, p_2, p_3) \in \mathbb{E}^3 : \langle p, p \rangle = r^2 \right\}.$$

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet-Serret frame along the curve α in the space $\mathbb{E}^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet-Serret formulae are given in [9] written under matrix form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where

$$\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1,$$
$$\langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0.$$

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be positive oriented natural basis of $\mathbb{E}^3$. Cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Mixed product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is defined by the determinant

$$[\mathbf{u},\mathbf{v},\mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Torsion of the curve α is given by the aid of the mixed product

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

Definition 2.1. *Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$. The unit binormal vectors along the curve α generate a curve ($\mathbf{B}$) on the sphere of radius 1 about the origin. The curve $\phi = (\mathbf{B})$ is called the spherical image of $\mathbf{B}$ or more commonly, ($\mathbf{B}$) is called binormal spherical image of the curve α, [2].*

3 Inextensible Flows of Binormal Spherical Images in $\mathbb{E}^3$

Let $\alpha(u,t)$ is a one parameter family of smooth curves in $\mathbb{E}^3$.

Any flow of α can be represented as

$$\frac{\partial \alpha}{\partial t} = \mathsf{H}_1\mathbf{T} + \mathsf{H}_2\mathbf{N} + \mathsf{H}_3\mathbf{B},$$

where $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3$ are smooth functions of time and arclength.

Theorem 3.1. *Let ϕ be binormal spherical image of α. Then,*

$$\nabla_t \mathbf{T}^{\phi} = -\nabla_t \mathbf{N} = (\mathsf{H}_1\kappa - \mathsf{H}_3\tau + \frac{\partial \mathsf{H}_2}{\partial s})\mathbf{T} - \mathsf{C}\mathbf{B},$$

$$\begin{aligned}
\nabla_t \mathbf{N}^{\phi} &= [\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] + (\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{T} \\
&+ [\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] + (\mathsf{H}_1\kappa - \mathsf{H}_3\tau + \frac{\partial \mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N} \\
&+ [(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] - \frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{B},
\end{aligned}$$

$$\begin{aligned}
\nabla_t \mathbf{B}^{\phi} &= [\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] - (\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{T} \\
&+ [(\mathsf{H}_1\kappa - \mathsf{H}_3\tau + \frac{\partial \mathsf{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] - \mathsf{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N} \\
&+ [\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] + (\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{B},
\end{aligned}$$

where $\mathsf{H}_1,\mathsf{H}_2,\mathsf{H}_3$ *are smooth functions of time and arc length and*

$$\mathsf{C}=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Proof. Using definition of ϕ, we have

$$\nabla_t\mathbf{T}^{\phi}=-\nabla_t\mathbf{N}=(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial\mathsf{H}_2}{\partial s})\mathbf{T}-\mathsf{C}\mathbf{B}.$$

Using the Frenet-Serret formula, we have

$$\mathbf{N}^{\phi}=\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\mathbf{T}-\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathbf{B}.$$

Then,

$$\begin{aligned}\nabla_t\mathbf{N}^{\phi}&=\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\mathbf{T}-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{B}\\&+[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\nabla_t\mathbf{T}-[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\nabla_t\mathbf{B}.\end{aligned}$$

Hence above equation becomes

$$\begin{aligned}\nabla_t\mathbf{N}^{\phi}&=[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{T}\\&+[\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial\mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N}\\&+[(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{B}.\end{aligned}$$

Also,

$$\mathbf{B}^{\phi}=\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathbf{T}+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\mathbf{B}.$$

This implies

$$\begin{aligned}\nabla_t\mathbf{B}^{\phi}&=\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{T}+\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\mathbf{B}\\&+[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\nabla_t\mathbf{T}+[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\nabla_t\mathbf{B}.\end{aligned}$$

Since

$$\nabla_t\mathbf{B}^{\phi}=[\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{T}$$

$$+[(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-\mathrm{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N}$$
$$+[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{B}.$$

Then, we obtain the theorem. So, theorem is proved.

Corollary 3.2.

$$\frac{\partial \phi}{\partial t}=-(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})\mathbf{T}-\mathrm{C}\mathbf{N},$$

where $\mathrm{H}_2,\mathrm{H}_3$ *are smooth functions of time and arclength.*

Proof. It is obvious from Theorem (3.1). The proof is completed.

Now we can express this lemma:

Lemma 3.3. *Let* $\frac{\partial \phi}{\partial t}$ *be inextensible flow of* ϕ. *Then,*

$$\mathrm{C}\kappa=\frac{\partial}{\partial s}(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s}),$$

where $\mathrm{H}_2,\mathrm{H}_3$ *are smooth functions of time and arclength.*

Proof. Assume that $\frac{\partial \phi}{\partial t}$ be inextensible flow of ϕ. Then,

$$\nabla_s\frac{\partial \phi}{\partial t}=[\mathrm{C}\kappa-\frac{\partial}{\partial s}(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})]\mathbf{T}-[\frac{\partial \mathrm{C}}{\partial s}+\kappa(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})]\mathbf{N}-\mathrm{C}\,\tau\mathbf{B}.$$

By the straight-forward calculation, we have lemma. Hence the proof is completed.

Theorem 3.4. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible flow of* α. *If* ϕ *is binormal spherical image of* α, *then,*

$$[\frac{\partial}{\partial t}(\tau\kappa^{\phi})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}+(\tau\kappa^{\phi})[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]$$
$$=\frac{\partial}{\partial s}(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})-\mathrm{f}_1\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s}),$$

where $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3$ *are smooth functions of time and arclength*

$$\mathrm{C}=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Proof. Using the formula of the curvature, we write a relation

$$\nabla_t \nabla_s \mathbf{T}^\phi - \nabla_s \nabla_t \mathbf{T}^\phi = R(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t})\mathbf{T}^\phi.$$

Since, we immediately arrive at

$$R(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t})\mathbf{T}^\phi = -\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})R(\mathbf{N},\mathbf{T})\mathbf{N}.$$

Putting

$$R(\mathbf{N},\mathbf{T})\mathbf{N} = \mathrm{f}_1\mathbf{T} + \mathrm{f}_2\mathbf{N} + \mathrm{f}_3\mathbf{B},$$

where $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3$ are smooth functions of time and arclength.
Thus, we easily obtain that

$$R(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t})\mathbf{T}^\phi = -\mathrm{f}_1\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})\mathbf{T} - \mathrm{f}_2\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})\mathbf{N}$$
$$-\mathrm{f}_3\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})\mathbf{B}.$$

So, we conclude

$$\nabla_s \nabla_t \mathbf{T}^\phi = \frac{\partial}{\partial s}(\mathrm{H}_1\kappa - \mathrm{H}_3\tau + \frac{\partial \mathrm{H}_2}{\partial s})\mathbf{T} - \frac{\partial \mathrm{C}}{\partial s}\mathbf{B}$$
$$+[\kappa(\mathrm{H}_1\kappa - \mathrm{H}_3\tau + \frac{\partial \mathrm{H}_2}{\partial s}) + \mathrm{C}\tau]\mathbf{N}.$$

Also, we have the following

$$\nabla_t \nabla_s \mathbf{T}^\phi = [\frac{\partial}{\partial t}(\tau\kappa^\phi)\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} + (\tau\kappa^\phi)[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] + (\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{T}$$
$$+(\tau\kappa^\phi)[\mathrm{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] + (\mathrm{H}_1\kappa - \mathrm{H}_3\tau + \frac{\partial \mathrm{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N}$$
$$+[(\tau\kappa^\phi)[(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] - \frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]] - \frac{\partial}{\partial t}(\tau\kappa^\phi)\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{B}.$$

So, from these equalities, we obtain the theorem.

As a consequence of this theorem, we get the following corollaries.

Corollary 3.5.

$$(\tau\kappa^\phi)[\mathrm{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] + (\mathrm{H}_1\kappa - \mathrm{H}_3\tau + \frac{\partial \mathrm{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]$$
$$= [\kappa(\mathrm{H}_1\kappa - \mathrm{H}_3\tau + \frac{\partial \mathrm{H}_2}{\partial s}) + \mathrm{C}\tau] - \mathrm{f}_2\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s}),$$

where $\mathsf{f}_2, \mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3$ *are smooth functions of time and arclength*

$$\varpi = \langle \nabla_t \mathbf{N}, \mathbf{B} \rangle.$$

Corollary 3.6.

$$(\tau\kappa^{\phi})[(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] - \frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]] - \frac{\partial}{\partial t}(\tau\kappa^{\phi})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]$$
$$= -\frac{\partial \mathsf{C}}{\partial s} - \mathsf{f}_3\tau(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s}),$$

where $\mathsf{f}_3, \mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3$ *are smooth functions of time and arclength.*

Theorem 3.7. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible. If* ϕ *is binormal spherical image of* α, *then,*

$$[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\tau}{\sqrt{\kappa^2+\tau^2}} - (\tau\kappa^{\phi})(\mathsf{H}_1\kappa - \mathsf{H}_3\tau + \frac{\partial \mathsf{H}_2}{\partial s})$$
$$+ (\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] - (\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]$$
$$= [\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] + (\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]$$
$$- \kappa\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] + (\mathsf{H}_1\kappa - \mathsf{H}_3\tau + \frac{\partial \mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]$$
$$+ [\mathsf{f}_4\tau(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} - \mathsf{f}_7\tau(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}],$$

where $\mathsf{f}_4, \mathsf{f}_7, \mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3$ *are smooth functions of time and arclength*

$$\mathsf{C} = \langle \nabla_t \mathbf{N}, \mathbf{B} \rangle.$$

Proof. Using the formula of the curvature, we write a relation

$$\nabla_t \nabla_s \mathbf{N}^{\phi} - \nabla_s \nabla_t \mathbf{N}^{\phi} = R(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t})\mathbf{N}^{\phi}.$$

Then we obtain

$$R(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t})\mathbf{N}^{\phi} = \tau(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} R(\mathbf{N}, \mathbf{T})\mathbf{T}$$
$$- \tau(\mathsf{H}_2\tau + \frac{\partial \mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}} R(\mathbf{N}, \mathbf{T})\mathbf{B}.$$

Putting

$$R(\mathbf{N}, \mathbf{B})\mathbf{T} = \mathsf{f}_4\mathbf{T} + \mathsf{f}_5\mathbf{N} + \mathsf{f}_6\mathbf{B},$$
$$R(\mathbf{N}, \mathbf{B})\mathbf{B} = \mathsf{f}_7\mathbf{T} + \mathsf{f}_8\mathbf{N} + \mathsf{f}_9\mathbf{B},$$

where f_i are smooth functions of time and arclength.

Thus it is easy to obtain that

$$R(\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t})\mathbf{N}^{\phi}=[\mathsf{f}_4\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}-\mathsf{f}_7\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{T}$$
$$+[\mathsf{f}_5\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}-\mathsf{f}_8\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{N}$$
$$+[\mathsf{f}_6\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}-\mathsf{f}_9\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\mathbf{B}.$$

By using Serret--Frenet formulas, we have

$$\nabla_s\nabla_t\mathbf{N}^{\phi}=[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]$$
$$-\kappa\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial \mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{T}$$
$$+[\kappa\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]$$
$$+\frac{\partial}{\partial s}[\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial \mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]$$
$$-\tau(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{N}$$
$$+[\frac{\partial}{\partial s}[(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]$$
$$+\tau\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial \mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{B}.$$

By a direct computation, we have

$$\nabla_t\nabla_s\mathbf{N}^{\phi}=[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-(\tau\kappa^{\phi})(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial \mathsf{H}_2}{\partial s})$$
$$+(\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{T}$$
$$+[\frac{\partial}{\partial t}(\tau\kappa^{\phi})+(\tau\tau^{\phi})[(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial \mathsf{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]$$
$$-\mathsf{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{N}+[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}+\mathsf{C}(\tau\kappa^{\phi})$$
$$+(\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_2\tau+\frac{\partial \mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{B}.$$

Combining above equations, we have theorem. Hence the proof is completed.

In the light of Theorem 3.7, we express the following corollaries without proofs:

Corollary 3.8.

$$\begin{aligned}&[\frac{\partial}{\partial t}(\tau\kappa^{\phi})+(\tau\tau^{\phi})[(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-\mathrm{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\\&=[\kappa\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]+\frac{\partial}{\partial s}[\mathrm{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\\&+(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]-\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\\&-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]+[\mathrm{f}_5\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}-\mathrm{f}_8\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}],\end{aligned}$$

where $\mathrm{f}_5,\mathrm{f}_8,\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3$ *are smooth functions of time and arclength*

$$\mathrm{C}=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Corollary 3.9.

$$\begin{aligned}&[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}+\mathrm{C}(\tau\kappa^{\phi})+(\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\\&=[\frac{\partial}{\partial s}[(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]+\tau\mathrm{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]\end{aligned}$$

$$+(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]+[\mathrm{f}_6\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}-\mathrm{f}_9\tau(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}],$$

where $\mathrm{f}_6,\mathrm{f}_9,\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3$ *are smooth functions of time and arclength*

$$\mathrm{C}=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Theorem 3.10. *Let* $\frac{\partial\alpha}{\partial t}$ *be inextensible. If* ϕ *is binormal spherical image of* α, *then,*

$$\begin{aligned}&-[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}+(\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\\&=[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(\mathrm{H}_2\tau+\frac{\partial \mathrm{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\\&-\kappa(\mathrm{H}_1\kappa-\mathrm{H}_3\tau+\frac{\partial \mathrm{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-\mathrm{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\end{aligned}$$

$$+[f_4\tau(H_2\tau+\frac{\partial H_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}+f_7\tau(H_2\tau+\frac{\partial H_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}],$$

where f_4, f_7, H_1, H_2, H_3 *are smooth functions of time and arclength*

$$C=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Proof. Using Theorem 3.1, we have

$$\begin{aligned}
\nabla_s\nabla_t\mathbf{B}^{\phi} &= [\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]] \\
&-\kappa(H_1\kappa-H_3\tau+\frac{\partial H_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-C[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{T} \\
&+[\kappa\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]] \\
&+\frac{\partial}{\partial s}[(H_1\kappa-H_3\tau+\frac{\partial H_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-C[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]] \\
&-\tau\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{N} \\
&+[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]+(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]] \\
&+\tau(H_1\kappa-H_3\tau+\frac{\partial H_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-C[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{B},
\end{aligned}$$

or, equivalently

$$\begin{aligned}
\nabla_t\nabla_s\mathbf{B}^{\phi} &= -[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}+(\tau\tau^{\phi})[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] \\
&+(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{T}-(\tau\tau^{\phi})[C[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] \\
&+(H_1\kappa-H_3\tau+\frac{\partial H_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]\mathbf{N}+[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \\
&-(\tau\tau^{\phi})[(H_2\tau+\frac{\partial H_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]\mathbf{B}.
\end{aligned}$$

Using the formula of the curvature, we write a relation

$$\nabla_t\nabla_s\mathbf{B}^{\phi}-\nabla_s\nabla_t\mathbf{B}^{\phi}=R(\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t})\mathbf{B}^{\phi}.$$

Since, we immediately arrive at

$$R(\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t})\mathbf{B}^{\phi} = \tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}R(\mathbf{N},\mathbf{T})\mathbf{T}$$
$$+\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}R(\mathbf{N},\mathbf{T})\mathbf{B}.$$

By means of obtained equations, we express

$$R(\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t})\mathbf{B}^{\phi} = [\mathsf{f}_4\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}+\mathsf{f}_7\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\mathbf{T}$$
$$+[\mathsf{f}_5\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}+\mathsf{f}_8\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\mathbf{N}$$
$$+[\mathsf{f}_6\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}+\mathsf{f}_9\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]\mathbf{B}.$$

Thus, we obtain the theorem.

In the light of Theorem 3.10, we express the following corollaries without proofs:

Corollary 3.11.

$$-(\tau\tau^{\phi})[\mathsf{C}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]+(\mathsf{H}_1\kappa-\mathsf{H}_3\tau+\frac{\partial\mathsf{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]$$
$$=+[\kappa\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]+\frac{\partial}{\partial s}[(\mathsf{H}_1\kappa$$
$$-\mathsf{H}_3\tau+\frac{\partial\mathsf{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]-\mathsf{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]-\tau\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]$$
$$+(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]+[\mathsf{f}_5\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}$$
$$+\mathsf{f}_8\tau(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}],$$

where $\mathsf{f}_5,\mathsf{f}_8,\mathsf{H}_1,\mathsf{H}_2,\mathsf{H}_3$ *are smooth functions of time and arclength*

$$\mathsf{C}=\langle\nabla_t\mathbf{N},\mathbf{B}\rangle.$$

Corollary 3.12.

$$[\frac{\partial}{\partial t}(\tau\tau^{\phi})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-(\tau\tau^{\phi})[(\mathsf{H}_2\tau+\frac{\partial\mathsf{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]-\frac{\partial}{\partial t}[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]]]$$

$$= [\frac{\partial}{\partial s}[\frac{\partial}{\partial t}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}] + (\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}]] + \tau(\mathrm{H}_1\kappa - \mathrm{H}_3\tau$$
$$+ \frac{\partial \mathrm{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2+\tau^2}}] - \mathrm{C}[\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}]]] + [\mathrm{f}_6\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})\frac{\tau}{\sqrt{\kappa^2+\tau^2}}$$
$$+ \mathrm{f}_9\tau(\mathrm{H}_2\tau + \frac{\partial \mathrm{H}_3}{\partial s})\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}],$$

where $\mathrm{f}_6, \mathrm{f}_9, \mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ *are smooth functions of time and arclength*

$$\mathrm{C} = \langle \nabla_t \mathbf{N}, \mathbf{B} \rangle.$$

Example 3.13. The time-helix is parametrized by

$$\gamma(u,t) = (A(t)\cos(u), A(t)\sin(u), B(t)u),$$

where A, B are functions only of time. The arc-length derivative is

$$\left(A^2 + B^2\right)^{\frac{1}{2}} \frac{\partial}{\partial s} = \frac{\partial}{\partial u}$$

and the evolution of γ is explicity given by

$$(\frac{\partial A}{\partial t}\cos(u), \frac{\partial A}{\partial t}\sin(u), \frac{\partial B}{\partial t}u) = \frac{1}{A^2+B^2}(-A\cos(u), -A\sin(u), 0).$$

Hence

$$\frac{\partial A}{\partial t} = -A\left(A^2+B^2\right)^{-1}, \frac{\partial B}{\partial t} = 0$$

and solitions are given by

$$\frac{A(t)^2}{2} + B^2 \log(A(t)) = -t + \frac{A(0)^2}{2} + B^2 \log(A(0)).$$

Note that, for positive B, $A(t)$ converges to, but never reaches, zero.

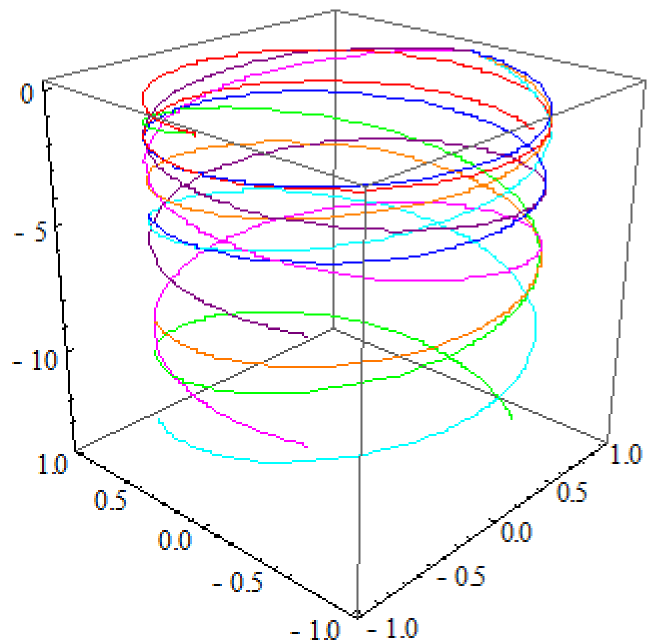

Figure 1: *A time-helix is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Green at the time* $t=1$, $t=1.2$, $t=1.4$, $t=1.6$, $t=1.8$, $t=2$, $t=2.2$, *respectively*.

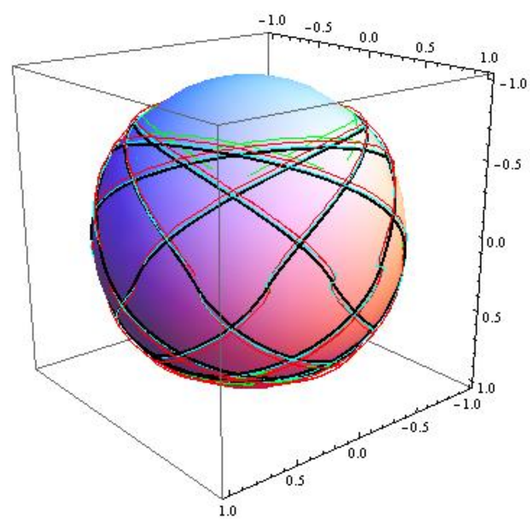

Figure 2: *Binormal spherical image is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Green at the time* $t=1$, $t=1.2$ $t=1.4$, $t=1.6$, $t=1.8$, $t=2$, $t=2.2$, *respectively.*

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