Article

The Application of Zeta Regularization Method to the Calculation of Certain Divergent Series and Integrals

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Abstract

In this paper, we generalize the Zeta regularization method to the divergent integrals $\int_0^\infty x^s dx$ for positive 's.' Using the Euler-Maclaurin summation formula we express a divergent integral in terms of a linear combination of divergent series, which can be regularized using the Riemann Zeta function, $\zeta(s)$ s >0. For the case of the pole at s=1, we use a property of the functional determinant to obtain the regularization $\sum_{n=0}^{\infty}\frac{1}{(n+a)} = -\frac{\Gamma'}{\Gamma}(a)$. With the aid of the Laurent series, we extend the Zeta regularization to the case of integral $\int_0^\infty f(x)dx$. We believe that this method can be of interest in the regularization of the divergent UV integrals in quantum field theory since it does not have the problems of the analytic regularization or dimensional regularization.

Key Words: Riemann Zeta Function, functional determinant, Zeta regularization, divergent series.

1. Zeta Regularization for Divergent Integrals

In mathematics and physics, one sometimes must evaluate divergent series of the form $\sum_{n=1}^{\infty} n^k$ which is divergent unless Re (k) >1. These sums with k=1 or k=3 appears in several calculations of string theory and Casimir effect. See for example [3], the result $\sum_{n=1}^{\infty} n^3 = \frac{1}{120}$ appears to give the correct result for the Casimir force $\frac{F_c}{A} = -\frac{\hbar c \pi^2}{240 a^4}$ where A is the area and *a* is the separation between the 2 plates.

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The idea behind Zeta regularization method is to take for granted that for every 's' the identity $\sum_{n=1}^{\infty} n^s = \zeta(s)$ holds, although this formula is valid only for Re (s) > 1.

To extend the definition of the Riemann Zeta function to negative real numbers, one needs to use the functional equation for the Riemann function

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{1}$$

This gives the expressions $\sum_{n=i}^{\infty} n^0 = -\frac{1}{2}$, $\sum_{n=i}^{\infty} n = -\frac{1}{12}$ and $\sum_{n=i}^{\infty} n^2 = 0$.

Due to the pole at s=1, the Harmonic series $\sum_{n=1}^{\infty} n^{-1}$ is NOT zeta regularizable, although it can be given a finite value $\sum_{n=1}^{\infty} n^{-1} = \gamma = 0.577215..$ this value can be justified by using the theory of Zeta-regularized infinite products (determinants).

Zeta regularization for divergent integrals

Let $f(x) = x^{m-s}$ with Re (m-s) < -1, then the Euler-Maclaurin summation formula for this function reads

$$\begin{aligned}\int_a^{\infty} x^{m-s}dx &= \frac{m-s}{2}\int_a^{\infty} x^{m-1-s}dx + \zeta(s-m) - \sum_{i=1}^{a} i^{m-s} + a^{m-s} \\ &-\sum_{r=1}^{\infty}\frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}(m-2r+1-s)\int_a^{\infty} x^{m-2r-s}dx\end{aligned} \qquad a \in N \tag{2}$$

Here in formula (2) all the series and integrals are convergent. But formula (2) is usually worthless since it is easy to prove that $\int_a^\infty x^{-k}dx = \frac{a^{1-k}}{k-1}$ for Re(k) >1 and $\zeta(m-s) = \sum_{i=1}^\infty i^{m-s}$. So nothing new can be obtained from (2).

The idea here is to use the functional equation (1) for the Riemann and Zeta function to extend the definition of equation (2) to the whole complex plane except s=1.

In case (m-s) is positive, there will be no pole at x=0. S, one we can put a=0 and take the limit $s \to 0^+$

$$\int_0^\infty x^m dx = \frac{m}{2}\int_0^\infty x^{m-1}dx + \zeta(-m) - \sum_{r=1}^\infty \frac{B_{2r} m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\infty x^{m-2r}dx \qquad (3)$$

Formula (3) is the analytic continuation of formula (2) with a=0 and can be used to obtain a finite definition for otherwise divergent integrals. This equation has an infinite number of terms but the Gamma function has a pole at x= 0 and at x being some negative integer.

Some examples of formula (3) are :

$$I_0 = \zeta(0) + 1 = \int_0^\infty dx \quad I_1 = \frac{I_0}{2} + \zeta(-1) = \int_0^\infty x dx$$

$$I_2 = \left(\frac{I_0}{2} + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0 = \int_0^\infty x^2 dx \qquad (4)$$

$$I_3 = \frac{3}{2}\left(\frac{1}{2}\left(I_0 + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0\right) + \zeta(-3) - B_2 a_{31} I_0 = \int_0^\infty x^3 dx$$

So our method can provide finite 'regularization' to divergent integrals.

With the aid of the zeta regularization algorithm, one gets finite results for divergent integrals. IN any case, formulae (2), (3) and (4) are consistent with the definition of the sum of a series, when this series is a convergent one.

In fact if $\int_0^{\Lambda} x^m dx$ is finite, for finite Λ, one can use the properties of the Riemann and Hurwitz Zeta function [5] to get the sum of the k-th powers of n on the interval $[0, \Lambda]$

$$\sum_{i=0}^{\Lambda-1} i^m = \zeta(-m) - \zeta(-m,\Lambda)\,, \quad \zeta(s,\Lambda) = \sum_{n=0}^{\infty}(n+\Lambda)^{-s}\,, \ \mathrm{Re}(s) > 1$$

We also have the following formulae for any values of the regulator Λ

$$\int_0^{\Lambda} x^m dx = \frac{m}{2}\int_0^{\Lambda} x^{m-1}dx + \zeta(-m) - \zeta(-m,\Lambda) - \sum_{r=1}^{\infty}\frac{B_{2r}\,m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^{\Lambda} x^{m-2r}dx \qquad (5)$$

For integer 'm' $\zeta_H(-m,x) = -\frac{B_{m+1}(x)}{m+1}$ one finds the Bernoulli Polynomials. The powers of Λ would cancel the integral $\int_0^{\Lambda} x^m dx = \frac{\Lambda^{m+1}}{m+1}$. So in the end in formula (5) one gets the usual definition of Zeta regularization $\zeta_H(-m) = -\frac{B_{m+1}(0)}{m+1}$ integer.

Of course one could argue that a 'simpler' regularization of the divergent integrals should be $I(s) = \int_0^{\infty} dx(x+a)^s = -\frac{a^{s+1}}{s+1}$ and $I(-1) = \int_0^{\infty} dx(x+a)^{-1} = -\log a$. This is obtained by just dropping out the term proportional to $\log\infty$ or ∞^{s+1} inside the integral to make it finite.

However, if one plugs this result into the Euler-Maclaurin summation formulae (2), (3) or (5), the terms involving 'a' would cancel and one finally finds that $\zeta_H(-m) = 0$ for every 'm' which clearly is against the definition of zeta regularization of a series.

For the case of the logarithmic divergence, taking the finite part of the integral obtained from differentiation with respect to the external parameter 'a' apparently works.

For the case of the integrals $\int_a^\infty x^{m-s}\log^k(x)dx$, we can simply differentiate k-times with respect to regulator 's' in order to obtain finite values in terms of $\zeta(-s)$ and $\zeta'(-s)$.

The of negative values of 's' unless m=-1 (for other negative values of m one can make a change of variable $xq=1$) is treated in the next section.

Zeta-regularized determinants and the harmonic series

Given an operator A with an infinite set of nonzero Eigenvalues $\{\lambda_n\}_{n=0}^\infty$ one can define a Zeta function and a Zeta-regularized determinant [10]

$$Tr\left\{A^{-s}\right\}=\zeta_A(s)=\sum_{n=0}^\infty \lambda_n^{-s} \qquad \det(A)=\prod_{n=0}^\infty \lambda_n=\exp\left(-\frac{d\zeta_A(0)}{ds}\right) \tag{6}$$

The proof of the second formula in (6) is straightforward: The derivative of the generalized zeta function is $\zeta_A'(s)=-\sum_{n=0}^\infty \frac{\log\lambda_n}{\lambda_n^s}$, let s=0, use the property of the logarithm $\log(a.b)=\log a+\log b$ and take the exponential on both sides.

For the case of the eigenvalues of a simple quantum harmonic oscillator in one dimension [10] $\lambda_n=n+a$, the Zeta function is just the Hurwitz Zeta function. So one can define a zeta-regularized infinite product in the form

$$\prod_{n=0}^\infty (n+a)=\exp\left(-\frac{d\zeta_H(0,a)}{ds}\right) \qquad \frac{d\zeta_H(0,a)}{ds}=\log\Gamma(a)-\log\left(\sqrt{2\pi}\right) \tag{7}$$

In the case a=1, one finds the zeta-regularized product of all the natural numbers $\prod_{n=0}^\infty (n+1)=\sqrt{2\pi}$ [see 5]. If one take the derivative with respect to 'a', one finds the same regularized value as Ramanujan did [2], i.e., $\sum_{n=0}^\infty \frac{1}{(n+a)}=-\frac{\Gamma'}{\Gamma}(a)$ a > 0.

Harmonic series appear due to a logarithmic divergence of the integral $\int_0^\infty \frac{dx}{(n+a)}$. Let m= -1 in formula (2) and use a regulator's' , $s \to 0^+$, one obtains the Euler-Maclaurin summation formula

$$\int_0^\infty \frac{dx}{(n+a)^{s+1}} = -\frac{1}{2a} + \sum_{n=0}^\infty \frac{1}{(n+a)^{1+s}} + \sum_{r=1}^\infty \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial u^{2r-1}} \left(\frac{1}{(x+a)^{s+1}} \right)_{x=0} \quad (8)$$

Since s >0 the integral and the series in (8) will converge. Now one can integrate over 'a' in (8) and use the definition of the logarithm $\lim_{s\to 0^+} \frac{x^s - 1}{s} = \log x$ to regularize the integral $\int_0^\infty \frac{dx}{(n+a)^{s+1}}$ as $s \to 0^+$ in terms of the function $-\frac{\Gamma'}{\Gamma}(a)$ plus some finite corrections due to the Euler-Maclaurin summation formula.

A faster method is simply to differentiate with respect to 'a' inside the integral $\int_0^\infty \frac{dx}{(n+a)^2} = -\frac{dI}{da}$ which is convergent for every 'a' and equal to $-\frac{1}{a}$. Integration over 'a' again gives the value $-\log a + c$ plus a constant 'c' which does not depend on the value of 'a' in the integral in question. The proof that 'c' is unique no matter what 'a' is comes from the fact that the difference $\int_0^\infty dx \left(\frac{1}{x+a} - \frac{1}{x+b} \right) = \log\left(\frac{b}{a}\right)$.

For the case a=0, the derivative of the Hurwitz Zeta is $\frac{d\zeta_H(0,0)}{ds} = -\log\left(\sqrt{2\pi}\right)$. So if one approximates the divergent integral by a series, one can get the regularized result $\int_0^\infty \frac{dx}{x} \approx \sum_{n=0}^\infty \frac{1}{n} = 0$.

Regularization of divergent integrals $\int_0^\infty dx f(x)$

In general, the divergent integrals that appear in quantum field theory, e.g., $\int \frac{d^4 p}{\left(p^2+m^2\right)^2}$ or $\int \frac{d^4 p}{\left((p-q)^2+m^2\right)} \frac{1}{p^2}$ are invariant under rotations.

If one uses 4-dimesional polar coordinates, one can reduce these integrals to the case $\frac{2\pi^{d/2}}{\Gamma\left(d/2\right)} \int_0^\infty dr f(r) r^{d-1}$ and the UV divergences appear when $r \to \infty$.

Depending on the value of 'd' one can have several types of divergences $\int_0^\Lambda dr f(r) r^{d-1} \approx a\Lambda^{m+1} + b\log\Lambda$. If b =0 and m =2, the UV divergences are quadratic; if m =0 the divergences are linear ; and in case a = 0 and b =1 the divergences are of logarithmic type, e.g., $\int \frac{d^4 p}{\left(p^2+m^2\right)^2}$ has only a logarithmic divergence in dimension 4.

To study the rate of divergence, one can expand the function into a Laurent series valid for $z \to \infty$, $f(x) = \sum_{n=-\infty}^{n=k} c_n (x+a)^n$ where 'k' is a finite number which means that the function $f(x)$ has a power law divergence for big 'x'.

To compute a divergent integral, one adds and substracts a Polynomial plus a term proportional to $\frac{1}{x+a}$ in order to split the integral into a finite part and another divergent integrals

$$\int_0^\infty dx \left(f(x) - \sum_{n=0}^{k} b_n x^n - \frac{b_{-1}}{x+a} \right) + \sum_{n=0}^{k} b_n \int_0^\infty x^n dx + b_{-1} \int_0^\infty \frac{dx}{x+a} = \int_0^\infty f(x) dx \qquad (9)$$

where 'k' is chosen so that the first integral is finite. The first integral in (9) can be computed by numerical or exact methods yielding a finite value and the remaining integrals are just the logarithmic and power-law divergences. They can be regularized with the aid of formulae (2) (3), (4), (6) and (8) to get a finite value involving a linear combination of $\zeta(-m)$ m=0,1,2,....,k and

$$\int_a^\infty \frac{x^2dx}{x+1} = \int_a^\infty dx\left(\frac{x^2}{x+1} - 1 + x + \frac{1}{x}\right) + \frac{\zeta(0)}{2} - a + \frac{a^2}{2} + \frac{1}{2a} + \frac{\Gamma'}{\Gamma}(a) - \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial u^{2r-1}}\left(\frac{1}{x+a}\right)_{x=0} - \zeta(-1) + \frac{1}{2} \qquad (10)$$

The first integral in (10) is convergent and has exact value of $\log\left(\frac{a+1}{a}\right)$. In order to regularize the logarithmic integral, one has to use the result $\sum_{n=0}^{\infty}\frac{1}{(n+a)} = -\frac{\Gamma'}{\Gamma}(a)$ plus the Euler-Maclaurin summation formula.

2. Regularization of Multiple Integrals

Until now, I have only considered integrals in one variable (after change to polar coordinates). The question then is if one can apply my method of zeta regularization to more complicate integrals such as

$$I(s) = \int d^4q_1 \int d^4q_2 \ldots\ldots\ldots \int d^4q_n \prod_{i=1}^{\infty}\frac{1}{\left(1+q_i^2\right)} F(q_1, q_2, \ldots.., q_n)\left(R\left(q_1, q_2, \ldots.., q_n\right)\right)^{-s} \qquad (11)$$

where I have introduced a regulator depending on an external parameter 's' in order for the integral to converge for big 's' and use the analytic regularization to take the limit $s \to 0^+$.

This regulator must be chosen with care in order not to spoil any symmetries of the physical system. This regulator may be of the form

$$R\left(q_1, q_2, \ldots.., q_n\right) = 1 + \sum_{i=1}^{n} q_i^2 \qquad R\left(q_1, q_2, \ldots.., q_n\right) = \prod_{i=1}^{n}\left(1+q_i\right) \qquad (12)$$

The first ansatz is to define n-dimensional polar coordinates in order to rewrite (11) as a multiple integral depending on 'r' $\sqrt{\sum_{i=1}^{n} q_i^2} = r$ and several angles θ_i i= 1,2,3,4,..., n-1 in the form

$$I(s) = \int_{\Omega} d\Omega \int_0^{\infty} dr G(r,\theta_i) r^{n-1} \left(R(r,\theta_i)\right)^{-s} \qquad d\Omega = \prod_{i=1}^{n-1} d\theta_i \sin^{n-i-1}\left(\theta_i\right) \tag{13}$$

One may choose the first regulator in (13) so that it does not depend on the angular coordinates. The idea is that in case of (13) one has an ultraviolet divergence which appears whenever $r \to \infty$. One performs the integral over the angular variables $d\Omega = \prod_{i=1}^{n-1} d\theta_i \sin^{n-i-1}\left(\theta_i\right)$ being left with an integral $I(s) = \int_0^{\infty} dr U(r) r^{n-1} \left(1+r\right)^{-s}$. In order to regularize this, one defines a convergent integral (by substraction) plus some divergent terms

$$I(s) = \int_0^{\infty} dr \left(1+r\right)^{-s} \left(U(r) r^{n-1} - \sum_{i=-1}^{k} a_i (1+r)^i \right) + \sum_{i=-1}^{k} a_i \int_0^{\infty} \left(1+r\right)^{i-s} dr \tag{14}$$

where U (r) is the function obtained after integration over the angles and 'k' is a finite number for performing the minimal substraction of terms in order the first integral to converge even for s = 0. If the integral over the angles is too complicated to have an exact form, one can replace this integral over the angles by an approximate finite sum $d\Omega \to \sum_i$ (sum over all the angular variables) in order to make the integral easier to calculate by using Montercarlo methods of integration.

Substraction method

Once one has made the change of variable to spherical coordinates inside integral $I(q_1, q_2, ..., q_n)$ one could substract some terms to render the integral finite

$$I(s) = \int_{\Omega} d\Omega \int_0^{\infty} dr \left(G(r,\theta_i) r^{n-1} - \sum_{j=-1}^{k} f_j(\theta_i) \left(1+r\right)^{j-s} \right) + \int_{\Omega} \sum_{j=-1}^{k} f_j(\theta_i) d\Omega \int_0^{\infty} dr \left(1+r\right)^{j-s} \tag{15}$$

One may choose number 'k' and the functions $f_j(\theta_i)$ in such way that the first integral in (15) converge. For the second integral, one can perform integration over the angular variables and then use formulae (2) and (3) to regularize $\int_0^\infty (1+r)^m dr$.

Iterated integration on several variables

Another method is to consider the multiple integral as an interate integral and then make the substraction for every variable, e.g.,

$$\int \partial q_n \left(F(q_1, q_2, \ldots\ldots, q_{n-1}) - \sum_{i=-1}^{k} a_i(q_1, \ldots.., q_{n-1})(1+q_n)^i \right) + \int_0^\infty \partial q_n \sum_{i=-1}^{k} a_i(q_1, \ldots.., q_{n-1})(1+q_n)^i \qquad (16)$$

where ∂q_n means that the integral is made over the variable q_n while keeping the other variables constant , the number 'k' is chosen so the first integral is finite. This integral will depend on $I(q_1, \ldots\ldots.., q_n)$, the divergent integrals (even for the logarithmic case i=-1) can be regularized.

Now one can repeat the iterative process for the functions

$$\int \partial q_{n-1} \left(a_i(q_1, q_2 \ldots\ldots, q_{n-1}) - \sum_{j=-1}^{k} b_j(q_2, \ldots.., q_{n-2})(1+q_{n-1})^i \right) + \int_0^\infty \partial q_{n-1} \sum_{j=-1}^{k} b_i(q_1, \ldots.., q_{n-2})(1+q_{n-1})^j \qquad (17)$$

Using (16) and (17) for every step, one can reduce the dimension of the integral until one reach the one dimensional case which is easier to handle, e.g.,

$$\int_0^\infty dx \int_0^\infty dy \frac{xy}{x+y+1} = \int_0^\infty dx \int_0^\infty dy \left(\frac{xy}{x+y+1} - x + \frac{x+x^2}{y+1} \right) + \int_0^\infty x dx \int_0^\infty dy - \int_0^\infty \left(x+x^2 \right) dx \int_0^\infty \frac{dy}{y+1} \qquad (18)$$

$$\int_0^\infty dx \left(f(x) - bx^2 + (a-b)x \right) \qquad f(x) = \int_0^\infty \frac{dy}{(y+1)} \frac{x^3+x^2}{\left(x+y+1\right)} \qquad (19)$$

where $a = \left(\int_0^\infty dx \right)_{reg}$ and . For an initial given integral with an overlapping divergence as $x \to \infty$ $y \to \infty$, a substraction is made to get a finite integral over 'y'.

Repeating the same process one can regularize the integral over 'x'. In order to integrate the finite part of the integral, one can use several numerical methods.

For example, the integral $f(x)=\int_0^\infty \frac{dy}{(y+1)}\frac{x^3+x^2}{(x+y+1)}$ can be calculated numerically to give $f(x)\approx\sum_j \frac{1}{(y_j+1)}\frac{x^3+x^2}{(x+y_j+1)}$ in order to avoid terms with $\log(x)$, $ar\tan(x)$ or similar ones in (18) and (19).

Another method to calculate (19) would be to introduce a regulator in the form $\left(1+\sqrt{x^2+y^2}\right)^{-s}=R(s,x,y)$ which will make (19) to converge for certain values of 's'. If one uses polar coordinates in (x,y), one has

$$\int_0^\infty dx\int_0^\infty dy(xy)\frac{R(x,y,s)}{x+y+1}=\int_0^\infty dr\int_0^{\pi/2} du\,\frac{r^2\sin(2u)(1+r)^{-s}}{2\left(\cos(u)+\sin(u)+r^{-1}\right)} \qquad (20)$$

Integration over the angular variable 'u' can be carried by numerical methods to produce some new divergent integrals that will only depend on the value of 'r' $\int_0^\infty drg_i(r)(1+r)^{-s}$. This integrals can be regularized with the same methods I used to give a finite meaning to 1-D integrals. But unlike the dimensional regularitzation, the 'regulator' is not the dimension of space-time, so we have no problem to change to spherical coordinates in d=4 to overcome the UV divergencies.

If the integrand $F(q_1,q_2,.......,q_n)$ had no singularities for $q_j>0$, one may expand this integrand into a multiple Laurent series of several variables, and then perform the substraction $\sum_{m1,m2,.....,mn=-1}^{s1,s2,...,sn} C_{m1,m2,....,mn}(q_1+b_1)^{m1}(q_2+b_2)^{m2}......(q_n+b_n)^{mn}$

$$\int d^4q_1\int d^4q_2.........\int d^4q_n\left(F-\sum_{m1,m2,.....,mn=-1}^{s1,s2,...,sn} C_{m1,m2,....,mn}(q_1+b_1)^{m1}(q_2+b_2)^{m2}......(q_n+b_n)^{mn}\right) \qquad (21)$$

plus some corrections due to divergent integrals $\int_0^\infty (q_i+b_i)^m dq_i$ m=-1,0,1,..... .

In many cases, although the integrals given in (17) and (18) are finite, they will have no exact expression or the exact expression will be too complicated. In that case, one can use the Gauss-Laguerre Quadrature formula (in case the interval is $[0,\infty)$) to approximate the integral by a sum over the zeros of Laguerre Polynomials $\sum_{i=0}^{n} w_i f(q_1, q_2, ..., q_{n-1}, x_i)$ with the weight expressed in terms of Laguerre Polynomials and their roots

$$w_i = \frac{x_i}{(n+1)^2 \left(L_{n+1}(x_i)\right)^2}, \; L_n(x_i) = 0 \tag{22}$$

3. Conclusions

In this paper, I have extended the definition of the Zeta regularization of a series in order to apply the extension to the Zeta regularization of a divergent integral $\int_0^\infty x^m dx$ m >0 by using the Zeta regularization technique combined with the Euler-Maclaurin summation formula.

For introduction to the Zeta regularization techniques, the readers are directed to the book by Elizalde [4] or the book Brendt [2]. These techniques are based on the mathematical discoveries of Ramanujan and its method of summation equivalent to the Zeta regularization algorithm [2]. Another good reference (but a bit more advanced) is the book by Zeidler [12]. Further, for the case of Zeta-regularized determinants [7] is a good online reference to study infinite product, e.g., $\prod_{n=0}^{\infty}(n+1) = \log\sqrt{2\pi}$.

Since the Riemann Zeta funciton has a pole at s=1, there is an apparent contradiction that harmonic series could not be regularized. However, using the definition of a functional determinant $\prod_{n=0}^{\infty} \frac{E_n}{\mu}$ $E_n = n + a$, one gets the finite result for the generalized harmonic series $\sum_{n=0}^{\infty} \frac{1}{n+a} = -\frac{\Gamma'(a)}{\Gamma(a)}$ with the aid of the Euler-Maclaurin summation formula. This result for the harmonic series can be used to obtain an approximate regularized value for the logarithmic integral $\int_0^\infty dx \frac{1}{x+a}$.

For the case of other types of divergent integrals $\int_0^\infty dx(x+a)^m$, one can again use Euler-Maclaurin summation formula to express this divergent integrals in terms of the negative values of the Hurwitz or Riemann Zeta function $\zeta_H(s,1)=\zeta(s)$, $\zeta_H(-m,1)$ (UV).

I also believe that a similar procedure can be applied to extend the Zeta regularization algorithm to multiple (multi-loop) integrals $\int d^4q_1 \int d^4q_2 .... \int d^4q_n F(q_1,q_2,.......,q_n)$.

One of the main advantages of this algorithm is that the dimension of the space does not appear explicitly. So method developed here does not have the same problems as dimensional regularization and can be used when the Dirac matrices $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ appear.

The imposition in formula (2) that 'a' must be a natural number is in order to avoid oddities in the process of Zeta regularization with the Zeta and Hurwitz Zeta function, since unless 'a' is a positive integer the equality $\zeta(-1,a)=\sum_{n=0}^{\infty}(n+a) \neq \sum_{n=0}^{\infty} n + \sum_{n=0}^{\infty} a = \frac{-1}{12} - \frac{a}{2}$ does not hold.

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