Article

Physics as Infinite-dimensional Geometry I: Identification of the Configuration Space Kähler Function

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Abstract

There are two basic approaches to quantum TGD. The first approach, which is discussed in this article, is a generalization of Einstein's geometrization program of physics to an infinite-dimensional context. Second approach is based on the identification of physics as a generalized number theory. The first approach relies on the vision of quantum physics as infinite-dimensional Kähler geometry for the "world of classical worlds" (WCW) identified as the space of 3-surfaces in in certain 8-dimensional space. There are three separate approaches to the challenge of constructing WCW Kähler geometry and spinor structure. The first approach relies on direct guess of Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach relies on the construction of spinor structure based on the hypothesis that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of WCW spinor structure.

In this article the proposal for Kähler function based on the requirement of 4-dimensional General Coordinate Invariance implying that its definition must assign to a given 3-surface a unique space-time surface. Quantum classical correspondence requires that this surface is a preferred extremal of some some general coordinate invariant action, and so called Kähler action is a unique candidate in this respect. The preferred extremal has interpretation as an analog of Bohr orbit so that classical physics becomes and exact part of WCW geometry and therefore also quantum physics.

The basic challenge is the explicit calculation of WCW Kähler function K. Two assumptions lead to the identification of K as a sum of Chern-Simons type terms associated with the ends of the causal diamond and with the light-like wormhole throats at which the signature of the induced metric changes. The first assumption is the weak form of electric magnetic duality generalizing the standard electric-magnetic duality. Second assumption is that the Kähler current for the preferred extremals is proportional to instanton current so that the Coulomb interaction term in the Kähler action vanishes and it reduces to Chern-Simons term. This requires the condition $j_K \wedge dj_K = 0$ as integrability condition implying that the flow parameter of the flow lines of j_K defines a global space-time coordinate. This inspires a generalization of the earlier solution ansatz for the field equations to a condition that various conserved currents are Beltrami fields proportional to the instanton current. This would realize the vision about reduction to almost topological QFT.

Second challenge is the understanding of the space-time correlates of quantum criticality. The realization that the hierarchy of Planck constant realized in terms of coverings of the imbedding space follows from basic quantum TGD leads to a further understanding. The extreme non-linearity of canonical momentum densities as functions of time derivatives of the imbedding space coordinates implies that the correspondence between these two variables is not 1-1 so that it is natural to introduce coverings of $CD \times CP_2$. This leads also to a precise geometric characterization of the criticality of the preferred extremals.

Keywords: Kähler geometry, infinite-dimensional geometry, quantum criticality, electric-magnetic duality, Chern-Simons action, topological QFT.

1 Introduction

The motivation or the construction of configuration space geometry is the postulate that physics reduces to the geometry of classical spinor fields in the the "world of the classical worlds" (WCW) identified as the infinite-dimensional configuration space of 3-surfaces of some subspace of $M^4 \times CP_2$. The first candidates were $M^4_+ \times CP_2$ and $M^4 \times CP_2$, where M^4 and M^4_+ denote Minkowski space and its light cone respectively. The recent identification of WCW is as the union of sub-WCWs consisting of light-like 3-surface representing generalized Feynman diagrams in $CD \times CP_2$, where CD is intersection of future and past directed light-cones of M^4. The details of this identification will be discussed later.

Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses [28]. One of the basic features of the Kähler geometry is that it is solely determined by the so called Kähler function, which defines both the Kähler form J and the components of the Kähler metric g in complex coordinates via the formulas [29]:

$$\begin{aligned} J &= i\partial_k\partial_{\bar{l}}K dz^k \wedge d\bar{z}^l \ , \\ ds^2 &= 2\partial_k\partial_{\bar{l}}K dz^k d\bar{z}^l \ . \end{aligned} \tag{1.1}$$

Kähler form is covariantly constant two-form and can be regarded as a representation of imaginary unit in the tangent space of the configuration space

$$J_{mr}J^{rn} = -g_m^{\ n} \ . \tag{1.2}$$

As a consequence Kähler form defines also symplectic structure in configuration space [30].

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1.1 Configuration space Kähler metric from Kähler function

The task of finding Kähler geometry for the configuration space reduces to that of finding the Kähler function. The main constraints on the Kähler function result from the requirement of General Coordinate Invariance (GCI) -or more technically Diff^4 symmetry and Diff degeneracy. GCI requires that the definition of the Kähler function assigns to a given 3-surface X^3 a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with X^3. The natural guess inspired by quantum classical correspondence is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of CP_2 coordinates and that the space-time surface corresponds to a preferred extremal of Kähler action.

One can end up with the identification of the preferred extremal via several routes. Kähler action contains Kähler coupling strength as a temperature like parameter and this leads to the idea of quantum criticality fixing this parameter. One could go even even further, and require that space-time surfaces are critical in the sense that there exist an infinite number of vanishing second variations of Kähler action defining conserved Noether charges. The approach based on the modified Dirac action indeed leads naturally to this picture [10]. Kähler coupling strength should be however visible in the solutions of field equations somehow before one can say that these two criticalities have something to do with each other. Since Kähler coupling strength does not appear in the field equations it can make its way to thge field equations only via boundary conditions. This is achieved if one accepts the weak form of self-duality [10] generalizing the standard electric-magnetic duality [57]. The weak form of electric-magnetic duality roughly states that for the partonic 2-surfaces the induced Kähler electric field is proportional to the Kähler magnetic field strength. The proportionality constant turns out to be essentially the Kähler coupling strength. The simplest hypothesis is that Kähler coupling strength has single universal value for given value of Planck constant and the weak form of self-duality fixes it.

If Kähler action would define a strictly deterministic variational principle, Diff^4 degeneracy and invariance would be achieved by restricting the consideration to 3-surfaces Y^3 at the boundary of M^4_+ and by defining Kähler function for 3-surfaces X^3 at $X^4(Y^3)$ and diffeo-related to Y^3 as $K(X^3) = K(Y^3)$. This reduction might be called quantum gravitational holography. The classical non-determinism of the Kähler action introduces complications which might be overcome in zero energy ontology (ZEO). ZEO and strong from of GCI lead to the effective replacement of X^3 with partonic 2-surfaces at the ends of CD plus the 4-D tangent space distribution associated with them as basic geometric objects so that one can speak about effective 2-dimensionality and strong form of gravitational holography.

1.2 Configuration space metric from symmetries

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan Freed [46] has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that configuration space is a union symmetric spaces [26] labeled by zero modes not appearing in the line element as differentials and having interpretations as classical degrees providing a rigorous formulation of quantum measurement theory. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and imbedding space which is a product of four-dimensional Minkowski space or its future light cone with CP_2 [29].

1.3 Topics of the article

In the sequel I will first consider the basic properties of the configuration space, propose an identification of the Kähler function as Kähler action for a preferred extremal of Kähler action and discuss various physical and mathematical motivations behind the proposed definition. The key feature of the Kähler action is the failure of classical determinism in its standard form, and various implications of the failure are discussed. In the last section representing the progress that has taken place during last months (and induced by the birds's eye of view forced by the writing of this article series) the weak form of electric-magnetic duality and the argument reducing the hierarchy of Planck constants to the non-linearity of Kähler action are discussssed. The basic results besides the understanding of the hierarchy of Planck constants, are a concrete geometric understanding of the criticality of the preferred extremals and the reduction of quantum TGD to almost topological TGD via the reduction of Kähler action to Chern-Simons terms. This also leads to a generalization of the earlier solution ansatz for field equations [16].

2 Configuration space

The view about configuration space or world of classical worlds (WCW) has developed considerably during the last two decades. Here only the recent view is summarized in order to not load reader with unessential details.

2.1 Basic notions

The notions of imbedding space, 3-surface (and 4-surface), and configuration space or "world of classical worlds" (WCW), are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize GCI. During years these notions have however evolved considerably.

2.1.1 The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [17, 18, 19].

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields [39] (including their algebraic extensions) representing the pages of the book. As matter fact, this gluing idea generalizes to the level of WCW.

2. With the discovery of zero energy ontology [11, 9] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [14] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [15] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and possibly also factor spaces of CD and CP_2 to form a book like structure. There are good physical and mathematical arguments suggesting that only the singular coverings should be allowed [19, A6]. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

2.1.2 The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial and the recent view is an outcome of a long and tedious process involving many hastily done mis-interpretations.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence implied by GCI. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the GCI in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory (for super-conformal theories see [42]). Light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces. Therefore it seems that one must choose between light-like and space-like 3-surfaces or assume generalized GCI requiring that equivalently either space-like 3-surfaces or light-like 3-surfaces at the ends of CDs can be identified as the fundamental geometric objects. General GCI requires that the basic objects correspond to the partonic 2-surfaces identified as intersections of these 3-surfaces plus common 4-D tangent space distribution. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. Since the information about normal space of the 2-surface is needed one has only effective 2-dimensionality. Weak form of self-duality [7, A2] however implies that the normal data (flux Hamiltonians associated with Kähler electric field) reduces to magnetic flux Hamiltonians. This is essential for conformal symmetries and also simplifies the construction enormously.

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further but inessential complication relates to the hierarchy of Planck constants forcing to generalize the notion of imbedding space and also to the fact that for non-standard values of Planck constant there is symmetry breaking due to preferred plane M^2 preferred homologically trivial geodesic sphere of CP_2 having interpretation as geometric correlate for the selection of quantization axis. For given sector of CH this means union over choices of this kind.

The basic vision forced by the generalization of GCI has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

The study of the modified Dirac equation led to the realization that classical field equations for Kähler action can be seen as consistency conditions for the modified Dirac action and led to the identification of preferred extremals in terms of criticality. This identification follows naturally also from quantum criticality.

1. The conjecture was that generalized eigen modes of the modified Dirac operator D_{C-S} associated with Chern-Simons action [51] code for the information about preferred extremal of Kähler action and that vacuum functional identified as Dirac determinant defined as product of generalized eigenvalues equals to exponent of Kähler action for a preferred extremal [9, A3].

2. The next step of progress was the realization that the requirement that the conservation of the Noether currents associated with the modified Dirac equation requires that the second variation of the Kähler action vanishes. In strongest form this condition would be satisfied for all variations and in weak sense only for those defining dynamical symmetries. The interpretation is as a space-time correlate for quantum criticality and the vacuum degeneracy of Kähler action makes the criticality plausible. The weak form of electric-magnetic duality gives a precise formulation for how Kähler coupling strength is visible in the properties of the preferred extremals. A generalization of the ideas of the catastrophe theory to infinite-dimensional context results [8]. These conditions make sense also in p-adic context and have a number theoretical universal form.

The notion of number theoretical compactication led to important progress in the understanding of the preferred extremals and the conjectures were consistent with what is known about the known extremals.

1. The conclusion was that one can assign to the 4-D tangent space $T(X^4(X_l^3)) \subset M^8$ a subspace $M^2(x) \subset M^4$ having interpretation as the plane of non-physical polarizations. This in the case that the induced metric has Minkowskian signature. If not, and if co-hyper-quaternionic surface is in question, similar assigned should be possible in normal space. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in M^2 degrees of freedom.

2. In number theoretical framework $M^2(x)$ has interpretation as a preferred hyper-complex sub-space of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8 (for classical numbers fields see [36, 37, 38]). The condition $M^2(x) \subset T(X^4(X_l^3)))$ in principle fixes the tangent space at X_l^3, and one has good hopes that the boundary value problem is well-defined and could fix $X^4(X^3)$ at least partially as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2(x) \subset M^4$ plays also other important roles.

3. At the level of H the counterpart for the choice of $M^2(x)$ seems to be following. Suppose that $X^4(X_l^3)$ has Minkowskian signature. One can assign to each point of the M^4 projection $P_{M^4}(X^4(X_l^3))$ a sub-space $M^2(x) \subset M^4$ and its complement $E^2(x)$, and the distributions of these planes are integrable and define what I have called Hamilton-Jacobi coordinates which can be assigned to the known extremals of Kähler with Minkowskian signature. This decomposition allows to slice space-time surfaces by string world sheets and their 2-D partonic duals. Also a slicing to 1-D light-like surfaces and their 3-D light-like duals Y_l^3 parallel to X_l^3 follows under certain conditions on the induced metric of $X^4(X_l^3)$. This decomposition exists for known extremals and has played key role in the recent developments. Physically it means that 4-surface (3-surface) reduces effectively to 3-D (2-D) surface and thus holography at space-time level.

4. The weakest form of number theoretic compactification [18, A5] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

If one takes M^-H duality seriously, one must conclude that one can choose any partonic 2-surface in the slicing of X^4 as a representative. This means gauge invariance reflect in the definition of Kähler function as $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$ having no effect on Kähler metric and Kähler form.

Although the details of this vision might change it can be defended by its ability to fuse together all great visions about quantum TGD. In the sequel the considerations are restricted to 3-surfaces in $M^4_\pm \times CP_2$. The basic outcome is that Kähler metric is expressible using the data at partonic 2-surfaces $X^2 \subset \delta M^4_+ \times CP_2$. The generalization to the actual physical situation requires the replacement of $X^2 \subset \delta M^4_+ \times CP_2$ with unions of partonic 2-surfaces located at light-like boundaries of CDs and sub-CDs.

2.1.3 The notion of configuration space

From the beginning there was a problem related to the precise definition of the configuration space ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the basis question is "M^4_+ or M^4?" and that this question had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of the configuration space or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that configuration space (WCW) is a union of configuration spaces associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. It must be however emphasized that Kähler function depends on partonic 2-surfaces at both ends of space-time surface so that WCW is topologically Cartesian product of corresponding symmetric spaces. WCW metric must therefore have parts corresponding to the partonic 2-surfaces (free part) and also an interaction term depending on the partonic 2-surface at the opposite ends of the light-like 3-surface. The conclusion is that geometrization reduces to that for single like of generalized Feynman diagram containing partonic 2-surfaces at its ends. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case corresponding to a line of generalized Feynman diagram. One can also deduce the free part of the metric by restricting the consideration to partonic 2-surfaces at single end of generalized Feynman diagram.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the configuration space metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at configuration space level and TGD is a genuine theory of gravitation at this level.

2. Configuration space can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub-configuration space associated with given CD in terms of the generalization of coset construction [45] for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). Recall that super Kac-Moody algebras [43] and super-Virasoro algebras [42] are central also for string models. Configuration space in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Formally, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about configuration space which I gave up as too naive!

4. Generalized coset construction [45] and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level: the identical actions of Super Virasoro generators for super-symplectic and super Kac-Moody algebras implies that inertial and gravitational four-momenta are identical.

2.2 Constraints on the configuration space geometry

The constraints on the WCW result both from the infinite dimension of the configuration space and from physically motivated symmetry requirements. There are three basic physical requirements on the configuration space geometry: namely four-dimensional GCI in strong form, Kähler property and the decomposition of configuration space into a union $\cup_i G/H_i$ of symmetric spaces G/H_i, each coset space allowing G-invariant metric such that G is subgroup of some 'universal group' having natural action on 3-surfaces. Together with the infinite dimensionality of the configuration space these requirements pose extremely strong constraints on the configuration space geometry. In the following we shall consider these requirements in more detail.

2.2.1 Diff^4 invariance and Diff^4 degeneracy

Diff^4 plays fundamental role as the gauge group of General Relativity. In string models $Diff^2$ invariance ($Diff^2$ acts on the orbit of the string) plays central role in making possible the elimination of the time like and longitudinal vibrational degrees of freedom of string. Also in the present case the elimination of the tachyons (time like oscillatory modes of 3-surface) is a physical necessity and Diff^4 invariance provides an obvious manner to do the job.

In the standard path l integral formulation the realization of Diff^4 invariance is an easy task at the formal level. The problem is however that path integral over four-surfaces is plagued by divergences and doesn't make sense.

In the present case the configuration space consists of 3-surfaces and only $Diff^3$ emerges automatically as the group of re-parameterizations of 3-surface. Obviously one should somehow define the action of Diff^4 in the space of 3-surfaces. Whatever the action of Diff^4 is it must leave the configuration space metric invariant. Furthermore, the elimination of tachyons is expected to be possible only provided the time like deformations of the 3-surface correspond to zero norm vector fields of the configuration space so that 3-surface and its Diff^4 image have zero distance. The conclusion is that configuration space metric should be both Diff^4 invariant and Diff^4 degenerate.

The problem is how to define the action of Diff^4 in $C(H)$. Obviously the only manner to achieve Diff^4 invariance is to require that the very definition of the configuration space metric somehow associates a unique space time surface to a given 3-surface for Diff^4 to act on. The obvious physical interpretation of this space time surface is as "classical space time" so that "Classical Physics" would be contained in configuration space geometry. In fact, this space-time surface is analogous to Bohr orbit so that semiclassical quantization rules become an exact part of the quantum theory. It is this requirement, which has turned out to be decisive concerning the understanding of the WCW geometry.

2.2.2 Decomposition of the configuration space into a union of symmetric spaces G/H

The extremely beautiful theory of finite-dimensional symmetric spaces constructed by Elie Cartan suggests that configuration space should possess decomposition into a union of coset spaces $CH = \cup_i G/H_i$ such that the metric inside each coset space G/H_i is left invariant under the infinite dimensional isometry group G. The metric equivalence of surfaces inside each coset space G/H_i does not mean that 3-surfaces inside G/H_i are physically equivalent. The reason is that the vacuum functional is exponent of Kähler action which is not isometry invariant so that the 3-surfaces, which correspond to maxima of Kähler function for a given orbit, are in a preferred position physically. For instance, one can imagine of calculating functional integral around this maximum perturbatively. Symmetric space property [26] actually allows also much more powerful non-perturbative approach based on harmonic analysis [27] in symmetric spaces [10]. The sum of over i means actually integration over the zero modes of the metric (zero modes correspond to coordinates not appearing as coordinate differentials in the metric tensor).

The coset space G/H is a symmetric space only under very special Lie-algebraic conditions. Denoting the decomposition of the Lie-algebra g of G to the direct sum of H Lie-algebra h and its complement t by $g = h \oplus t$, one has

$$[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ .$$

This decomposition turn out to play crucial role in guaranteing that G indeed acts as isometries and that the metric is Ricci flat.

The four-dimensional $Diff$ invariance indeed suggests to a beautiful solution of the problem of identifying G. The point is that any 3-surface X^3 is $Diff^4$ equivalent to the intersection of $X^4(X^3)$ with the light cone boundary. This in turn implies that 3-surfaces in the space $\delta H = \delta M^4_+ \times CP_2$ should be all what is needed to construct configuration space geometry. The group G can be identified as some subgroup of diffeomorphisms of δH and H_i contains that subgroup of G, which acts as diffeomorphisms of the 3-surface X^3. Since G preserves topology, configuration space must decompose into union $\cup_i G/H_i$, where i labels 3-topologies and various zero modes of the metric. For instance, the elements of the Lie-algebra of G invariant under configuration space complexification correspond to zero modes.

The reduction to the light cone boundary, identifiable as the moment of big bang, looks perhaps odd at first. In fact, it turns out that the classical non-determinism of Kähler action does not allow the complete reduction to the light cone boundary: physically this is a highly desirable implication but means a considerable mathematical challenge.

2.2.3 Kähler property

Kähler property implies that the tangent space of the configuration space allows complexification and that there exists a covariantly constant two-form J_{kl}, which can be regarded as a representation of the imaginary unit in the tangent space of the configuration space:

$$J_k{}^r J_{rl} = -G_{kl} \ . \tag{2.1}$$

There are several physical and mathematical reasons suggesting that configuration space metric should possess Kähler property in some generalized sense.

1. The deepest motivation comes from the need to geometrize hermitian conjugation which is basic mathematical operation of quantum theory.

2. Kähler property turns out to be a necessary prerequisite for defining divergence free configuration space integration. We will leave the demonstration of this fact later although the argument as such is completely general.

3. Kähler property very probably implies an infinite-dimensional isometry group. The study of the loop groups $Map(S^1, G)$ [46] shows that loop group allows only single Kähler metric with well defined Riemann connection and this metric allows local G as its isometries!

   To see this consider the construction of Riemannian connection for $Map(X^3, H)$. The defining formula for the connection is given by the expression

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y,Z) + Y(Z,X) - Z(X,Y) \\ &+ ([X,Y],Z) + ([Z,X],Y) - ([Y,Z],X) \end{aligned} \tag{2.2}$$

X, Y, Z are smooth vector fields in $Map(X^3, G)$. This formula defines $\nabla_X Y$ uniquely provided the tangent space of Map is complete with respect to Riemann metric. In the finite-dimensional case completeness means that the inverse of the covariant metric tensor exists so that one can solve the components of connection from the conditions stating the covariant constancy of the metric. In the case of the loop spaces with Kähler metric this is however not the case.

Now the symmetry comes into the game: if X, Y, Z are left (local gauge) invariant vector fields defined by the Lie-algebra of local G then the first three terms drop away since the scalar products of left invariant vector fields are constants. The expression for the covariant derivative is given by

$$\nabla_X Y = (Ad_X Y - Ad_X^* Y - Ad_Y^* X)/2 \tag{2.3}$$

where Ad_X^* is the adjoint of Ad_X with respect to the metric of the loop space.

At this point it is important to realize that Freed's argument does not force the isometry group of the configuration space to be $Map(X^3, M^4 \times SU(3))$! Any symmetry group, whose Lie algebra is complete with respect to the configuration space metric (in the sense that any tangent space vector is expressible as superposition of isometry generators modulo a zero norm tangent vector) is an acceptable alternative.

The Kähler property of the metric is quite essential in one-dimensional case in that it leads to the requirement of left invariance as a mathematical consistency condition and we expect that dimension three makes no exception in this respect. In 3-dimensional case the degeneracy of the metric turns out to be even larger than in 1-dimensional case due to the four-dimensional Diff degeneracy. So we expect that the metric ought to possess some infinite-dimensional isometry group and that the above formula generalizes also to the 3-dimensional case and to the case of local coset space. Note that in M^4 degrees of freedom $Map(X^3, M^4)$ invariance would imply the flatness of the metric in M^4 degrees of freedom.

The physical implications of the above purely mathematical conjecture should not be underestimated. For example, one natural looking manner to construct physical theory would be based on the idea that configuration space geometry is dynamical and this approach is followed in the attempts to construct string theories [56]. Various physical considerations (in particular the need to obtain oscillator operator algebra) seem to imply that configuration space geometry is necessarily Kähler. The above result however states that configuration space Kähler geometry cannot be dynamical quantity and is dictated solely by the requirement of internal consistency. This result is extremely nice since it has been already found that the definition of the configuration space metric must somehow associate a unique classical space time and "classical physics" to a given 3-surface: uniqueness of the geometry implies the uniqueness of the "classical physics".

4. The choice of the imbedding space becomes highly unique. In fact, the requirement that configuration space is not only symmetric space but also (contact) Kähler manifold inheriting its (degenerate) Kähler structure from the imbedding space suggests that spaces, which are products of four-dimensional Minkowski space with complex projective spaces CP_n, are perhaps the only possible candidates for H. The reason for the unique position of the four-dimensional Minkowski space turns out to be that the boundary of the light cone of D-dimensional Minkowski space is metrically a sphere S^{D-2} despite its topological dimension $D-1$: for $D = 4$ one obtains two-sphere allowing Kähler structure and infinite parameter group of conformal symmetries!

5. It seems possible to understand the basic mathematical structures appearing in string model in terms of the Kähler geometry rather nicely.

   (a) The projective representations of the infinite-dimensional isometry group (not necessarily Map!) correspond to the ordinary representations of the corresponding centrally extended group [44]. The representations of Kac Moody group indeed play central role in string models [53, 54] and configuration space approach would explain their occurrence, not as a result of some quantization procedure, but as a consequence of symmetry of the underlying geometric structure.

   (b) The bosonic oscillator operators of string models would correspond to centrally extended Lie-algebra generators of the isometry group acting on spinor fields of the configuration space.

   (c) The "fermionic" fields (Ramond fields, [53, 54]) should correspond to gamma matrices of the configuration space. Fermionic oscillator operators would correspond simply to contractions of isometry generators j_A^k with complexified gamma matrices of configuration space

$$\begin{aligned} \Gamma_A^{\pm} &= j_A^k \Gamma_k^{\pm} \\ \Gamma_k^{\pm} &= (\Gamma^k \pm J^k_{\ l} \Gamma^l)/\sqrt{2} \end{aligned} \tag{2.4}$$

   ($J^k_{\ l}$ is the Kähler form of the configuration space) and would create various spin excitations of the configuration space spinor field. $\Gamma_k^{\pm}$ are the complexified gamma matrices, complexification made possible by the Kähler structure of the configuration space.

This suggests that some generalization of the so called Super Kac Moody algebra of string models [53, 54] should be regarded as a spectrum generating algebra for the solutions of field equations in configuration space.

Although the Kähler structure seems to be physically well motivated there is a rather heavy counter argument against the whole idea. Kähler structure necessitates complex structure in the tangent space of the configuration space. In CP_2 degrees of freedom no obvious problems of principle are expected: configuration space should inherit in some sense the complex structure of CP_2.

In Minkowski degrees of freedom the signature of the Minkowski metric seems to pose a serious obstacle for complexification: somehow one should get rid of two degrees of freedom so that only two Euclidian degrees of freedom remain. An analogous difficulty is encountered in quantum field theories: only two of the four possible polarizations of gauge boson correspond to physical degrees of freedom: mathematically the wrong polarizations correspond to zero norm states and transverse states span a complex Hilbert space with Euclidian metric. Also in string model analogous situation occurs: in case of D-dimensional Minkowski space only $D-2$ transversal degrees of freedom are physical. The solution to the problem seems therefore obvious: configuration space metric must be degenerate so that each vibrational mode spans effectively a 2-dimensional Euclidian plane allowing complexification.

We shall find that the definition of Kähler function to be proposed indeed provides a solution to this problem and also to the problems listed before.

1. The definition of the metric doesn't differentiate between 1- and N-particle sectors, avoids spin statistics difficulty and has the physically appealing property that one can associate to each 3-surface a unique classical space time: classical physics is described by the geometry of the configuration space and d the geometry of the configuration space is determined uniquely by the requirement of mathematical consistency.

2. Complexification is possible only provided the dimension of the Minkowski space equals to four and is due to the effective 3-dimensionality of light-cone boundary.

3. It is possible to identify a unique candidate for the necessary infinite-dimensional isometry group G. G is subgroup of the diffeomorphism group of $\delta M^4_+ \times CP_2$. Essential role is played by the fact that the boundary of the four-dimensional light cone, which, despite being topologically 3-dimensional, is metrically two-dimensional Euclidian sphere, and therefore allows infinite-parameter groups of isometries as well as conformal and symplectic symmetries and also Kähler structure unlike the higher-dimensional light cone boundaries. Therefore configuration space metric is Kähler only in the case of four-dimensional Minkowski space and allows symplectic $U(1)$ central extension without conflict with the no-go theorems about higher dimensional central extensions.

   The study of the vacuum degeneracy of Kähler function defined by Kähler action forces to conclude that the isometry group must consist of the symplectic transformations of $\delta H = \delta M^4_+ \times CP_2$. The corresponding Lie algebra can be regarded as a loop algebra associated with the symplectic group of $S^2 \times CP_2$, where S^2 is $r_M = constant$ sphere of light cone boundary. Thus the finite-dimensional group G defining loop group in case of string models extends to an infinite-dimensional group in TGD context. This group has a monstrous size. The radial Virasoro localized with respect to $S^2 \times CP_2$ defines naturally complexification for both G and H. The general form of the Kähler metric deduced on basis of this symmetry has same qualitative properties as that deduced from Kähler function identified as preferred extremal of Kähler action. Also the zero modes, among them isometry invariants, can be identified.

4. The construction of the configuration space spinor structure is based on the identification of the configuration space gamma matrices as linear superpositions of the oscillator operators associated with the second quantized induced spinor fields. The extension of the symplectic invariance to super symplectic invariance fixes the anti-commutation relations of the induced spinor fields, and configuration space gamma matrices correspond directly to the super generators. Physics as number theory vision suggests strongly that configuration space geometry exists for 8-dimensional imbedding space only and that the choice $M^4_+ \times CP_2$ for the imbedding space is the only possible one.

3 Identification of the Kähler function

There are three approaches to the construction of the WCW geometry: a direct physics based guess of the Kähler function, a group theoretic approach based on the hypothesis that CH can be regarded as a union of symmetric spaces, and the approach based on the construction of WCW spinor structure first by second quantization of induced spinor fields. Here the first approach is discussed.

3.1 Definition of Kähler function

3.1.1 Kähler metric in terms of Kähler function

Quite generally, Kähler function K defines Kähler metric in complex coordinates via the following formula

$$J_{k\bar{l}} = ig_{k\bar{l}} = i\partial_k\partial_{\bar{l}}K \ . \tag{3.1}$$

Kähler function is defined only modulo a real part of holomorphic function so that one has the gauge symmetry

$$K \rightarrow K + f + \overline{f} \ . \tag{3.2}$$

Let X^3 be a given 3-surface and let X^4 be any four-surface containing X^3 as a sub-manifold: $X^4 \supset X^3$. The 4-surface X^4 possesses in general boundary. If the 3-surface X^3 has nonempty boundary δX^3 then the boundary of X^3 belongs to the boundary of X^4: $\delta X^3 \subset \delta X^4$.

3.1.2 Induced Kähler form and its physical interpretation

Induced Kähler form defines a Maxwell field and it is important to characterize precisely its relationship to the gauge fields as they are defined in gauge theories. Kähler form J is related to the corresponding Maxwell field F via the formula

$$J = xF \ , \ x = \frac{g_K}{\hbar} \ . \tag{3.3}$$

Similar relationship holds true also for the other induced gauge fields. The inverse proportionality of J to $\hbar$ does not matter in the ordinary gauge theory context where one routinely choses units by putting $\hbar = 1$ but becomes very important when one considers a hierachy of Planck constants [15].

Unless one has $J = (g_K/\hbar_0)$, where $\hbar_0$ corresponds to the ordinary value of Planck constant, $\alpha_K = g_K^2/4\pi\hbar$ together the large Planck constant means weaker interactions and convergence of the functional integral defined by the exponent of Kähler function and one can argue that the convergence of the functional integral is what forces the hierarchy of Planck constants. This is in accordance with the vision that Mother Nature likes theoreticians and takes care that the perturbation theory works by making a phase transition increasing the value of the Planck constant in the situation when perturbation theory fails. This leads to a replacement of the M^4 (or more precisely, causal diamond CD) and CP_2 factors of the imbedding space ($CD \times CP_2$) with its $r = \hbar/\hbar_0$-fold singular covering (one can consider also singular factor spaces). If the components of the space-time surfaces at the sheets of the covering are identical, one can interpret r-fold value of Kähler action as a sum of r identical contributions from the sheets of the covering with ordinary value of Planck constant and forget the presence of the covering. Physical states are however different even in the case that one assumes that sheets carry identical quantum states and anyonic phase could correspond to this kind of phase [22].

3.1.3 Kähler action

One can associate to Kähler form Maxwell action and also Chern-Simons anomaly term proportional to $\int_{X^4} J \wedge J$ in well known manner. Chern Simons term is purely topological term and well defined for orientable 4-manifolds, only. Since there is no deep reason for excluding non-orientable space-time surfaces it seems reasonable to drop Chern Simons term from consideration. Therefore Kähler action $S_K(X^4)$ can be defined as

$$S_K(X^4) = k_1 \int_{X^4;X^3 \subset X^4} J \wedge (*J) \ . \tag{3.4}$$

The sign of the square root of the metric determinant, appearing implicitly in the formula, is defined in such a manner that the action density is negative for the Euclidian signature of the induced metric and such that for a Minkowskian signature of the induced metric Kähler electric field gives a negative contribution to the action density.

The notational convention

$$k_1 \equiv \frac{1}{16\pi\alpha_K} \ , \tag{3.5}$$

where α_K will be referred as Kähler coupling strength will be used in the sequel. If the preferred extremals minimize/maximize [18, A5] the absolute value of the action in each region where action density has a definite sign, the value of α_K can depend on space-time sheet.

3.1.4 Kähler function

One can define the Kähler function in the following manner. Consider first the case $H = M^4_+ \times CP_2$ and neglect for a moment the non-determinism of Kähler action. Let X^3 be a 3-surface at the light-cone boundary $\delta M^4_+ \times CP_2$. Define the value $K(X^3)$ of Kähler function K as the value of the Kähler action for some preferred extremal in the set of four-surfaces containing X^3 as a sub-manifold:

$$K(X^3) = K(X^4_{pref}) \ , \ X^4_{pref} \subset \{X^4 | X^3 \subset X^4\} \ . \tag{3.6}$$

The most plausible identification of preferred extremals is in terms of quantum criticality in the sense that the preferred extremals allow an infinite number of deformations for which the second variation of Kähler action vanishes. Combined with the weak form of electric-magnetic duality forcing appearence of Kähler coupling strength in the boundary conditions at partonic 2-surfaces this condition might be enough to fix preferred extremals completely.

3.2 What are the values of the Kähler coupling strength?

Since the vacuum functional of the theory turns out to be essentially the exponent $exp(K)$ of the Kähler function, the dynamics depends on the normalization of the Kähler function. Since the Theory of Everything should be unique it would be highly desirable to find arguments fixing the normalization or equivalently the possible values of the Kähler coupling strength α_K. Also a discrete spectrum of values is acceptable.

The quantization of Kähler form could result in the following manner. It will be found that Abelian extension of the isometry group results by coupling spinors of the configuration space to a multiple of Kähler potential. This means that Kähler potential plays role of gauge connection so that Kähler form must be integer valued by Dirac quantization condition for magnetic charge. So, if Kähler form is co-homologically nontrivial it is quantized.

Unfortunately, the exact definition of renormalization group concept is not at all obvious. There is however a much more general but more or less equivalent manner to formulate the condition fixing the value of α_K. Vacuum functional $exp(K)$ is analogous to the exponent $exp(-H/T)$ appearing in the definition of the partition function of a statistical system and S-matrix elements and other interesting physical quantities are integrals of type $\langle O\rangle = \int exp(K)O\sqrt{G}dV$ and therefore analogous to the thermal averages of various observables. α_K is completely analogous to temperature. The critical points of a statistical system correspond to critical temperatures T_c for which the partition function is nonanalytic function of $T - T_c$ and according RGE hypothesis critical systems correspond to fixed points of renormalization group evolution. Therefore, a mathematically more precise manner to fix the value of α_K is to require that some integrals of type $\langle O\rangle$ (not necessary S-matrix elements) become nonanalytic at $1/\alpha_K - 1/\alpha_K^c$.

This analogy suggests also a physical motivation for the unique value or value spectrum of α_K. Below the critical temperature critical systems suffer something analogous to spontaneous magnetization. At the critical point critical systems are characterized by long range correlations and arbitrarily large volumes of magnetized and non-magnetized phases are present. Spontaneous magnetization might correspond to the generation of Kähler magnetic fields: the most probable 3-surfaces are Kähler magnetized for subcritical values of α_K. At the critical values of α_K the most probable 3-surfaces contain regions dominated by either Kähler electric and or Kähler magnetic fields: by the compactness of CP_2 these regions have in general outer boundaries.

This suggests that 3-space has hierarchical, fractal like structure: 3-surfaces with all sizes (and with outer boundaries) are possible and they have suffered topological condensation on each other. Therefore the critical value of α_K allows the richest possible topological structure for the most probable 3-space. In fact, this hierarchical structure is in accordance with the basic ideas about renormalization group invariance. This hypothesis has highly nontrivial consequences even at the level of ordinary condensed matter physics.

Renormalization group invariance is closely related with criticality. The self duality of the Kähler form and Weyl tensor of CP_2 indeed suggest RG invariance. The point is that in $N = 4$ super-symmetric field theories duality transformation relates the strong coupling limit for ordinary particles with the weak coupling limit for magnetic monopoles and vice versa. If the theory is self-dual these limits must be identical so that action and coupling strength must be RG invariant quantities. This form of self-duality cannot hold true in TGD. The weak form of self-duality discussed in [7, A2] roughly states that for the partonic 2-surface the induce Kähler electric field is proportional to the Kähler magnetic field strength. The proportionality constant is essentially Kähler coupling strength. The simplest hypothesis is that Kähler coupling strength has single universal valiue and the weak form of self-duality fixes it. The proportionality $\alpha_K = g_K^2/4\pi\hbar$ and the proposed quantization of Planck constant requiring a generalization of the imbedding space imply that Kähler coupling strength varies but is constant at a given page of the "Big Book" defined by the generalized imbedding space [15] .

3.3 What preferred extremal property means?

The requirement that the 4-surface having given 3-surface as its sub-manifold is absolute minimum of the Kähler action is the most obvious guess for the principle selecting the preferred extremals and has been taken as a working hypothesis for about one and half decades. Quantum criticality of Quantum TGD should have however led to the idea that preferred extremals are critical in the sense that space-time surface allows deformations for which second variation of Kähler action vanishes so that the corresponding Noether currents are conserved.

Further insights emerged through the realization that Noether currents assignable to the modified Dirac equation are conserved only if the first variation of the modified Dirac operator D_K defined by Kähler action vanishes. This is equivalent with the vanishing of the second variation of Kähler action -at least for the variations corresponding to dynamical symmetries having interpretation as dynamical degrees of freedom which are below measurement resolution and therefore effectively gauge symmetries.

The vanishing of the second variation in interior of $X^4(X_l^3)$ is what corresponds exactly to quantum criticality so that the basic vision about quantum dynamics of quantum TGD would lead directly to a precise identification of the preferred extremals.

The vanishing of second variations of preferred extremals -at least for deformations representing dynamical symmetries, suggests a generalization of catastrophe theory of Thom, where the rank of the matrix defined by the second derivatives of potential function defines a hierarchy of criticalities with the tip of bifurcation set of the catastrophe representing the complete vanishing of this matrix. In the recent case this theory would be generalized to infinite-dimensional context. There are three kind of variables now but quantum classical correspondence (holography) allows to reduce the types of variables to two.

1. The variations of $X^4(X_l^3)$ vanishing at the intersections of $X^4(X_l^3)$ with the light-like boundaries of causal diamonds CD would represent behavior variables. At least the vacuum extremals of Kähler action would represent extremals for which the second variation vanishes identically (the "tip" of the multi-furcation set).

2. The zero modes of Kähler function would define the control variables interpreted as classical degrees of freedom necessary in quantum measurement theory. By effective 2-dimensionality (or holography or quantum classical correspondence) meaning that the configuration space metric is determined by the data coming from partonic 2-surfaces X^2 at intersections of X_l^3 with boundaries of CD, the interiors of 3-surfaces X^3 at the boundaries of CDs in rough sense correspond to zero modes so that there is indeed huge number of them. Also the variables characterizing 2-surface, which cannot be complexified and thus cannot contribute to the Kähler metric of configuration space represent zero modes. Fixing the interior of the 3-surface would mean fixing of control variables. Extremum property would fix the 4-surface and behavior variables if boundary conditions are fixed to sufficient degree.

3. The complex variables characterizing X^2 would represent third kind of variables identified as quantum fluctuating degrees of freedom contributing to the configuration space metric. Quantum classical correspondence requires 1-1 correspondence between zero modes and these variables. This would be essentially holography stating that the 2-D "causal boundary" X^2 of $X^3(X^2)$ codes for the interior. Preferred extremal property identified as criticality condition would realize the holography by fixing the values of zero modes once X^2 is known and give rise to the holographic correspondence $X^2 \to X^3(X^2)$. The values of behavior variables determined by extremization would fix then the space-time surface $X^4(X_l^3)$ as a preferred extremal.

4. Clearly, the presence of zero modes would be absolutely essential element of the picture. Quantum criticality, quantum classical correspondence, holography, and preferred extremal property would all represent more or less the same thing. One must of course be very cautious since the boundary conditions at X_l^3 involve normal derivative and might bring in delicacies forcing to modify the simplest heuristic picture.

One must be very cautious with what one means with the preferred extremal property and criticality.

1. Does one assign criticality with the partonic 2-surfaces at the ends of CDs? Does one restrict it with the throats for which light-like 3-surface has also degenerate induced 4-metric? Or does one assume stronger form of holography requiring a slicing of space-time surface by partonic 2-surfaces and string world sheets and assign criticality to all partonic 2-surfaces. This kind of slicing is suggested by the study of the extremals [16], required by the number theoretic vision ($M^8 - H$ duality [19, A6]), and also by the purely physical condition that a stringy realization of GCI is possible.

2. What is the exact meaning of the preferred extremal property? The assumption that the variations of Kähler action leaving 3-surfaces at the ends of CDs invariant would not be consistent with the effective 2-dimensionality. The assumption that the critical deformations leave invariant only partonic 2-surfaces would imply genuine 2-dimensionality. Should one assume that critical deformations leave invariant partonic 2-surface and 3-D tangent space in the direction of space-like 3-surface or light-like 3-surface but not both. This would be consistent with effective 3-dimensionality and would explain why Kac-Moody symmetries associated with the light-like 3-surfaces act as gauge symmetries. This is also essential for the realization of Poincare invariance since the quantization of the light-cone proper time distance between CDs implies that infinitesimal Poincare transformations lead out of CD unless compensated by Kac-Moody type transformations acting like gauge transformations. In the similar manner it would explain why symplectic transformations of δCD act like gauge transformations.

3. Could one pose the criticality condition for all partonic 2-surfaces in the slicing or only for the throats of light-like 3-surfaces? This hypothesis looks natural but is not necessary. Light-like throats are very singular objects criticality might apply only to their variations only in the limiting sense and it might be necesary to assume criticality for all partonic 2-surfaces.

3.4 Why non-local Kähler function?

Kähler function is nonlocal functional of 3-surface. Non-locality of the Kähler function seems to be at odds with basic assumptions of local quantum field theories. Why this rather radical departure from the basic assumptions of local quantum field theory? The answer is shortly given: configuration space integration appears in the definition of the inner product for WCW spinor fields and this inner product must be free from perturbative divergences. Consider now the argument more closely.

In the case of finite-dimensional symmetric space with Kähler structure the representations of the isometry group necessitate the modification of the integration measure defining the inner product so that the integration measure becomes proportional to the exponent $exp(K)$ of the Kähler function [47]. The generalization to infinite-dimensional case is obvious. Also the requirement of Kac-Moody symmetry leads to the presence of this kind of vacuum functional as will be found later. The exponent is in fact uniquely fixed by finiteness requirement. Configuration space integral is of the following form

$$\int \bar{S}_1 exp(K) S_1 \sqrt{g} dX \ . \tag{3.7}$$

One can develop perturbation theory using local complex coordinates around a given 3-surface in the following manner. The $(1,1)$-part of the second variation of the Kähler function defines the metric and therefore propagagator as contravariant metric and the remaining $(2,0)-$ and $(0,2)$-parts of the second variation are treated perturbatively. The most natural choice for the 3-surface are obviously the 3-surfaces, which correspond to extrema of the Kähler function.

When perturbation theory is developed around the 3-surface one obtains two ill-defined determinants.

1. The Gaussian determinant coming from the exponent, which is just the inverse square root for the matrix defined by the metric defining $(1, 1)$-part of the second variation of the Kähler function in local coordinates.

2. The metric determinant. The matrix representing covariant metric is however same as the matrix appearing in Gaussian determinant by the defining property of the Kähler metric: in local complex coordinates the matrix defined by second derivatives is of type $(1, 1)$. Therefore these two ill defined determinants (recall the presence of Diff degeneracy) cancel each other exactly for a unique choice of the vacuum functional!

Of course, the cancellation of the determinants is not enough. For an arbitrary local action one encounters the standard perturbative divergences. Since most local actions (Chern-Simons term is perhaps an exception [52]) for induced geometric quantities are extremely nonlinear there is no hope of obtaining a finite theory. For nonlocal action the situation is however completely different. There are no local interaction vertices and therefore no products of delta functions in perturbation theory.

A further nice feature of the perturbation theory is that the propagator for small deformations is nothing but the contravariant metric. Also the various vertices of the theory are closely related to the metric of the configuration space since they are determined by the Kähler function so that perturbation theory would have a beautiful geometric interpretation. Furthermore, since four-dimensional Diff degeneracy implies that the propagator doesn't couple to un-physical modes.

It should be noticed that divergence cancellation arguments do not necessarily exclude Chern Simons term from vacuum functional defined as imaginary exponent of $exp(ik_2 \int_{X^4} J \wedge J)$. The term is not well defined for non-orientable space-time surfaces and one must assume that k_2 vanishes for these surfaces. The presence of this term might provide first principle explanation for CP breaking. If k_2 is integer multiple of $1/(8\pi)$ Chern Simons term gives trivial contribution for closed space-time surfaces since instanton number is in question. By adding a suitable boundary term of form $exp(ik_3 \int_{\delta X^3} J \wedge A)$ it is possible to guarantee that the exponent is integer valued for 4-surfaces with boundary, too.

There are two arguments suggesting that local Chern Simons term would not introduce divergences. First, 3-dimensional Chern Simons term for ordinary Abelian gauge field is known to define a divergence free field theory [52]. The term doesn't depend at all on the induced metric and therefore contains no dimensional parameters (CP_2 radius) and its expansion in terms of CP_2 coordinate variables is of the form allowed by renormalizable field theory in the sense that only quartic terms appear. This is seen by noticing that there always exist symplectic coordinates, where the expression of the Kähler potential is of the form

$$A = \sum_k P_k dQ^k \ . \tag{3.8}$$

The expression for Chern-Simons term in these coordinates is given by

$$k_2 \int_{X^3} \sum_{k,l} P_l dP_k \wedge dQ^k \wedge dQ^l \ , \tag{3.9}$$

and clearly quartic CP_2 coordinates. A further nice property of the Chern Simons term is that this term is invariant under symplectic transformations of CP_2, which are realized as $U(1)$ gauge transformation for the Kähler potential.

4 Some properties of Kähler action

In this section some properties of Kähler action and Kähler function are discussed in light of experienced gained during about 15 years after the introduction of the notion.

4.1 Vacuum degeneracy and some of its implications

The vacuum degeneracy is perhaps the most characteristic feature of the Kähler action. Although it is not associated with the preferred extremals of Kähler action, there are good reasons to expect that it has deep consequences concerning the structure of the theory.

4.1.1 Vacuum degeneracy of the Kähler action

The basic reason for choosing Kähler action is its enormous vacuum degeneracy, which makes long range interactions possible (the well known problem of the membrane theories is the absence of massless particles [55]). The Kähler form of CP_2 defines symplectic structure and any 4-surface for which CP_2 projection is so called Lagrangian sub-manifold [31] (at most two dimensional manifold with vanishing induced Kähler form), is vacuum extremal due to the vanishing of the induced Kähler form. More explicitly, in the local coordinates, where the vector potential A associated with the Kähler form reads as $A = \sum_k P_k dQ^k$. Lagrangian manifolds are expressible locally in the following form

$$P_k = \partial_k f(Q^i) \ . \tag{4.1}$$

where the function f is arbitrary. Notice that for the general YM action surfaces with one-dimensional CP_2 projection are vacuum extremals but for Kähler action one obtains additional degeneracy.

There is also a second kind of vacuum degeneracy, which is relevant to the elementary particle physics. The so called CP_2 type vacuum extremals are warped imbeddings X^4 of CP_2 to H such that Minkowski coordinates are functions of a single CP_2 coordinate, and the one-dimensional projection of X^4 is random light like curve. These extremals have a non-vanishing action but vanishing Poincare charges. Their small deformations are identified as space-time counterparts of fermions and their super partners. Wormhole throats identified as pieces of these extremals are identified as bosons and their super partners.

The conditions stating light likeness are equivalent with the Virasoro conditions of string models and this actually led to the eventualo realization that conformal invariance [40] is a basic symmetry of TGD and that WCW can be regarded as a union of symmetric spaces with isometry groups having identification as symplectic and Kac-Moody type groups assignable to the partonic 2-surfaces.

4.1.2 Approximate symplectic invariance

Vacuum extremals have diffeomorphisms of M^4_+ and M^4_+ local symplectic transformations as symmetries. For non-vacuum extremals these symmetries leave induced Kähler form invariant and only induced metric breaks these symmetries. Symplectic transformations of CP_2 act on the Maxwell field defined by the induced Kähler form in the same manner as ordinary $U(1)$ gauge symmetries. They are however not gauge symmetries since gauge invariance is still present. In fact, the construction of the configuration space geometry relies on the assumption that symplectic transformations of $\delta M^4_+ \times CP_2$ which infinitesimally correspond to combinations of M^4_+ local CP_2 symplectic and CP_2-local M^4_+ symplectic transformations act as isometries of the configuration space. In zero energy ontology these transformations act simultanoeusly on all partonic 2-surfaces characterizing the space-time sheet representing a generalized Feynman diagram inside CD.

The fact that CP_2 symplectic transformations do not act as genuine gauge transformations means that $U(1)$ gauge invariance is effectively broken. This has non-trivial implications. The field equations allow purely geometric vacuum 4-currents not possible in Maxwell's electrodynamics [16]. For the known extremals (massless extremals) they are light-like and a possible interpretation is in terms of Bose-Einstein condensates of collinear massless bosons.

4.1.3 Spin glass degeneracy

Vacuum degeneracy means that all surfaces belonging to $M^4_+ \times Y^2$, Y^2 any Lagrangian sub-manifold of CP_2 are vacua irrespective of the topology and that symplectic transformations of CP_2 generate new surfaces Y^2. If preferred extremals are obtained as small deformations of vacuum extremals (for which the criticality is maximal), one expects therefore enormous ground state degeneracy, which could be seen as 4-dimensional counterpart of the spin glass degeneracy. This degeneracy corresponds to the hypothesis that configuration space is a union of symmetric spaces labeled by zero modes which do not appear at the line-element of the configuration space metric.

Zero modes define what might be called the counterpart of spin glass energy landscape and the maxima Kähler function as a function of zero modes define a discrete set which might be called reduced configuration space. Spin glass degeneracy turns out to be crucial element for understanding how macro-temporal quantum coherence emerges in TGD framework. One of the basic ideas about p-adicization is that the maxima of Kähler function define the TGD counterpart of spin glass energy landscape [17, 20]. The hierarchy of discretizations of the symmetric spaces corresponding to a hierarchy of measurement resolutions [10] could allow an identification in terms of a hierarchy spin glass energy landscapes so that the algebraic points of the WCW would correspond to the maxima of Kähler function. The hierarchical structure would be due to the failure of strict non-determinism of Kähler action allowing in zero energy ontology to add endlessly details to the space-time sheets representing zero energy states in shorter scale.

4.1.4 Generalized quantum gravitational holography

The original naive belief was that the construction of the configuration space geometry reduces to $\delta H = \delta M^4_+ \times CP_2$. An analogous idea in string model context became later known as quantum gravitational holography. The basic implication of the vacuum degeneracy is classical non-determinism, which is expected to reflect itself as the properties of the Kähler function and configuration space geometry. Obviously classical non-determinism challenges the notion of quantum gravitational holography.

The hope was that a generalization of the notion of 3-surface is enough to get rid of the degeneracy and save quantum gravitational holography in its simplest form. This would mean that one just replaces space-like 3-surfaces with "association sequences" consisting of sequences of space-like 3-surfaces with time like separations as causal determinants. This would mean that the absolute minima of Kähler function would become degenerate: same space-like 3-surface at δH would correspond to several association sequences with the same value of Kähler function.

The life turned out to be more complex than this. CP_2 type extremals have Euclidian signature of the induced metric and therefore CP_2 type extremals glued to space-time sheet with Minkowskian signature of the induced metric are surrounded by light like surfaces X^3_l, which might be called elementary particle horizons. The non-determinism of the CP_2 type extremals suggests strongly that also elementary particle horizons behave non-deterministically and must be regarded as causal determinants having time like projection in M^4_+. Pieces of CP_2 type extremals are good candidates for the wormhole contacts connecting a space-time sheet to a larger space-time sheet and are also surrounded by an elementary particle horizons and non-determinism is also now present. That this non-determinism would allow the proposed simple description seems highly implausible.

Zero energy ontology realized in terms of a hierarchy of CDs seems to provide the most plausible treatment of the non-determinism and has indeed led to a breakthrough in the construction and understanding of quantum TGD. At the level of generalized Feynman diagrams sub-CDs containing zero energy states represent a hierarchy of radiative corrections so that the classical determinism is direct correlate for the quantum non-determinism. Determinism makes sense only when one has specified the length scale of measurement resolution. One can always add a CD containing a vacuum extremal to get a new zero energy state and a preferred extremal containing more details.

4.1.5 Classical non-determinism saves the notion of time

Although classical non-determinism represents a formidable mathematical challenge it is a must for several reasons. Quantum classical correspondence, which has become a basic guide line in the development of TGD, states that all quantum phenomena have classical space-time correlates. This is not new as far as properties of quantum states are considered. What is new that also quantum jumps and quantum jump sequences which define conscious existence in TGD Universe, should have classical space-time correlates: somewhat like written language is correlate for the contents of consciousness of the writer. Classical non-determinism indeed makes this possible. Classical non-determinism makes also possible the realization of statistical ensembles as ensembles formed by strictly deterministic pieces of the space-time sheet so that even thermodynamics has space-time representations. Space-time surface can thus be seen as symbolic representations for the quantum existence.

In canonically quantized general relativity the loss of time is fundamental problem. If quantum gravitational holography would work in the most strict sense, time would be lost also in TGD since all relevant information about quantum states would be determined by the moment of big bang. More precisely, geometro-temporal localization for the contents of conscious experience would not be possible. Classical non-determinism together with quantum-classical correspondence however suggests that it is possible to have quantum jumps in which non-determinism is concentrated in space-time region so that also conscious experience contains information about this region only.

4.2 Four-dimensional General Coordinate Invariance

The proposed definition of the Kähler function is consistent with GCI and implies also 4-dimensional Diff degeneracy of the Kähler metric. Zero energy ontology inspires strengthening of the GCI in the sense that space-like 3-surfaces at the boundaries of CD are physically equivalent with the light-like 3-surfaces connecting the ends. This imples that basic geometric objects are partonic 2-surfaces at the boundaries of CDs identified as the intersections of these two kinds of surfaces. Besides this the distribution of 4-D tangent planes at partonic 2-surfaces would code for physics so that one would have only effective 2-dimensionality. The failure of the non-determinism of Kähler action in the standard sense of the word affects the situation also and one must allow a fractal hierarchy of CDs inside CDs having interpretation in terms of radiative corrections.

4.2.1 Resolution of tachyon difficulty and absence of Diff anomalies

In TGD as in string models the tachyon difficulty is potentially present: unless the time like vibrational excitations possess zero norm they contribute tachyonic term to the mass squared operator of Super Kac Moody algebra. This difficulty is familiar already from string models [53, 54].

The degeneracy of the metric with respect to the time like vibrational excitations guarantees that time like excitations do not contribute to the mass squared operator so that mass spectrum is tachyon free. It also implies the decoupling of the tachyons from physical states: the propagator of the theory corresponds essentially to the inverse of the Kähler metric and therefore decouples from time like vibrational excitations. The experience with string model suggests that if metric is degenerate with respect to diffeomorphisms of $X^4(X^3)$ there are indeed good hopes that time like excitations possess vanishing norm with respect to configuration space metric.

The four-dimensional Diff invariance of the Kähler function implies that Diff invariance is guaranteed in the strong sense since the scalar product of two Diff vector fields given by the matrix associated with $(1,1)$ part of the second variation of the Kähler action vanishes identically. This property gives hopes of obtaining theory, which is free from Diff anomalies: in fact loop space metric is not Diff degenerate and this might be the underlying reason to the problems encountered in string models [53, 54].

4.2.2 Complexification of the configuration space

Strong form of GCI plays a fundamental role in the complexification of the configuration space. GCI in strong form reduces the basic building brick of WCW to the pairs of partonic 2-surfaces and their 4-D tangent space data associated with ends of light-like 3-surface at light-like boundaries of CD. At boths end the imbedding space is effectively reduces to $\delta M^4_+ \times CP_2$ (forgetting the complications due to non-determinism of Kähler action). Light cone boundary in turn is metrically 2-dimensional Euclidian sphere allowing infinite-dimensional group of conformal symmetries and Kähler structure. Therefore one can say that in certain sense configuration space metric inherits the Kähler structure of $S^2 \times CP_2$. This mechanism works in case of four-dimensional Minkowski space only: higher-dimensional spheres do not possess even Kähler structure. In fact, it turns out that the quantum fluctuating degrees of freedom can be regarded in well-defined sense as a local variant of $S^2 \times CP_2$ and thus as an infinite-dimensional analog of symmetric space as the considerations of [7, A2] demonstrate.

The details of the complexification were understood only after the construction of configuration space geometry and spinor structure in terms of second quantized induced spinor fields [9, A3]. This also allows to make detailed statements about complexification [7, A2].

4.2.3 Contravariant metric and Diff^4 degeneracy

Diff degeneracy implies that the definition of the contravariant metric, which corresponds to the propagator associated to small deformations of minimizing surface is not quite straightforward. We believe that this problem is only technical. Certainly this problem is not new, being encountered in both GRT and gauge theories [48]. In TGD a solution of the problem is provided by the existence of infinite-dimensional isometry group. If the generators of this group form a complete set in the sense that any vector of the tangent space is expressible as as sum of these generators plus some zero norm vector fields then one can restrict the consideration to this subspace and in this subspace the matrix $g(X,Y)$ defined by the components of the metric tensor indeed indeed possesses well defined inverse $g^{-1}(X,Y)$. This procedure is analogous to gauge fixing conditions in gauge theories and coordinate fixing conditions in General Relativity.

It has turned that the representability of WCW as a union of symmetric spaces makes possible an approach to WCW integration based on harmonic analysis replacing the perturbative approach based on perturbative functional integral. This approach allows also a p-adic variant and leads an effective discretization in terms of discrete variants of WCW for which the points of symmetric space consist of algebraic points. There is an infinite number of these discretizations [17] and the interpretation is in terms of finite measurement resolution. This gives a connection with the p-adicization program, infinite primes, inclusions of hyper-finite factors as representation of the finite measurement resolution, and the hierarchy of Planck constants [19, A6] so that various approaches to quantum TGD converge nicely.

4.2.4 General Coordinate Invariance and WCW spinor fields

GCI applies also at the level of quantum states. WCW spinor fields are Diff^4 invariant. This in fact fixes not only classical but also quantum dynamics completely. The point is that the values of the configuration space spinor fields must be essentially same for all Diff^4 related 3-surfaces at the orbit X^4 associated with a given 3-surface. This would mean that the time development of Diff^4 invariant configuration spinor field is completely determined by its initial value at the moment of the big bang!

This is of course a naive over statement. The non-determinism of Kähler action and zero energy ontology force to take the causal diamond (CD) defined by the intersection of future and past directed light-cones as the basic structural unit of configuration space, and there is fractal hierarchy of CDs within CDs so that the above statement makes sense only for giving CD in measurement resolution neglecting the presence of smaller CDs. Strong form of GCI also implies factorization of WCW spinor fields into a sum of products associated with various partonic 2-surfaces. In particular, one obtains time-like entanglement between positive and negative energy parts of zero energy states and entanglement coefficients define what can be identified as M-matrix expressible as a "complex square root" of density matrix and reducing to a product of positive definite diagonal square root of density matrix and unitary S-matrix. The collection of orthonormal M-matrices in turn define unitary U-matrix between zero energy states. M-matrix is the basic object measured in particle physics laboratory.

4.3 Holomorphic factorization of Kähler function

One can guess the general form of the core part of the Kähler function as function of complex coordinates assignable to the partonic surfaces at positive and negative energy ends of CD. It its convenient to restrict the consideration to the simplest possible non-trivial case which is represented by single propagator line connecting the ends of CD.

1. The propagator line corresponds to a symmetric space defined as a coset space G/H of the symplectic group [30] and Kac-Moody group [43]. This coset space is as a manifold Cartesian product $(G/H)\times(G/H)$ of symmetric spaces G/H associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i=1,2 \ . \end{aligned} \tag{4.2}$$

Here $K_{kin,i}$ define "kinetic" terms and K_{int} defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories. K_{kin} would correspond to the Chern-Simons term assignable to the ends of the line and K_{int} to the Chern-Simons terms assignable to the wormhole throats.

4.4 Could the dynamics of Kähler action predict the hierarchy of Planck constants?

The original justification for the hierarchy of Planck constants came from the indications that Planck constant could have large values in both astrophysical systems involving dark matter and also in biology. The realization of the hierarchy in terms of the singular coverings and possibly also factor spaces of CD and CP_2 emerged from consistency conditions. The formula for the Planck constant involves heuristic guess work and physical plausibility arguments. There are good arguments in favor of the hypothesis that only coverings are possible. Only a finite number of pages of the Big Book correspond to a given value of Planck constant, biological evolution corresponds to a gradual dispersion to the pages of the Big Book with larger Planck constant, and a connection with the hierarchy of infinite primes and p-adicization program based on the mathematical realization of finite measurement resolution emerges.

One can however ask whether this hierarchy could emerge directly from the basic quantum TGD rather than as a separate hypothesis. The following arguments suggest that this might be possible. One finds also a precise geometric interpretation of preferred extremal property interpreted as criticality in zero energy ontology.

4.4.1 1-1 correspondence between canonical momentum densities and time derivatives fails for Kähler action

The basic motivation for the geometrization program was the observation that canonical quantization for TGD fails. To see what is involved let us try to perform a canonical quantization in zero energy ontology at the 3-D surfaces located at the light-like boundaries of $CD \times CP_2$.

1. In canonical quantization canonical momentum densities $\pi_k^0 \equiv \pi_k = \partial L_K/\partial(\partial_0 h^k)$, where $\partial_0 h^k$ denotes the time derivative of imbedding space coordinate, are the physically natural quantities in terms of which to fix the initial values: once their value distribution is fixed also conserved charges are fixed. Also the weak form of electric-magnetic duality given by $J^{03}\sqrt{g_4} = 4\pi\alpha_K J_{12}$ and a mild generalization of this condition to be discussed below can be interpreted as a manner to fix the values of conserved gauge charges (not Noether charges) to their quantized values since Kähler magnetic flux equals to the integer giving the homology class of the (wormhole) throat. This condition alone need not characterize criticality, which requires an infinite number of deformations of X^4 for which the second variation of the Kähler action vanishes and implies infinite number conserved charges. This in fact gives hopes of replacing π_k with these conserved Noether charges.
2. Canonical quantization requires that $\partial_0 h^k$ in the energy is expressed in terms of π_k. The equation defining π_k in terms of $\partial_0 h^k$ is however highly non-linear although algebraic. By taking squares the equations reduces to equations for rational functions of $\partial_0 h^k$. $\partial_0 h^k$ appears in contravariant and covariant metric at most quadratically and in the induced Kähler electric field linearly and by multplying the equations by $det(g_4)^3$ one can transform the equations to a polynomial form so that in principle $\partial_0 h^k$ can obtained as a solution of polynomial equations.
3. One can always eliminate one half of the coordinates by choosing 4 imbedding space coordinates as the coordinates of the spacetime surface so that the initial value conditions reduce to those for the canonical momentum densities associated with the remaining four coordinates. For instance, for space-time surfaces representable as map $M^4 \to CP_2$ M^4 coordinates are natural and the time derivatives $\partial_0 s^k$ of CP_2 coordinates are multivalued. One would obtain four polynomial equations with $\partial_0 s^k$ as unknowns. In regions where CP_2 projection is 4-dimensional -in particular for the deformations of CP_2 vacuum extremals the natural coordinates are CP_2 coordinates and one can regard $\partial_0 m^k$ as unknows. For the deformations of cosmic strings, which are of form $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, one can use coordinates of $M^2 \times S^2$, where S^2 is geodesic sphere as natural coordinates and regard as unknowns E^2 coordinates and remaining CP_2 coordinates.
4. One can imagine solving one of the four polynomials equations for time derivaties in terms of other obtaining N roots. Then one would substitute these roots to the remaining 3 conditions to obtain algebraic equations from which one solves then second variable. Obviously situation is very complex without additional symmetries. The criticality of the preferred extremals might however give additional conditions allowing simplifications. The reasons for giving up the canonical quantization program was following. For the vacuum extremals of Kähler action π_k are however identically vanishing and this means that there is an infinite number of value distributions for $\partial_0 h^k$. For small deformations of vacuum extremals one might however hope a finite number of solutions to the conditions and thus finite number of space-time surfaces carrying same conserved charges.

If one assumes that physics is characterized by the values of the conserved charges one must treat the the many-valuedness of $\partial_0 h^k$. The most obvious guess is that one should replace the space of space-like 4-surfaces corresponding to different roots $\partial_0 h^k = F^k(\pi_l)$ with four-surfaces in the covering space of $CD \times CP_2$ corresponding to different branches of the many-valued function $\partial_0 h^k = F(\pi_l)$ co-inciding at the ends of CD.

4.4.2 Do the coverings forces by the many-valuedness of $\partial_0 h^k$ correspond to the coverings associated with the hierarchy of Planck constants?

The obvious question is whether this covering space actually corresponds to the covering spaces associated with the hierarchy of Planck constants. This would conform with quantum classical correspondence. The hierarchy of Planck constants and hierarchy of covering spaces was introduced to cure the failure of the perturbation theory at quantum level. At classical level the multivaluedness of $\partial_0 h^k$ means a failure of perturbative canonical quantization and forces the introduction of the covering spaces. The interpretation would be that when the density of matter becomes critical the space-time surface splits to several branches so that the density at each branches is sub-critical. It is of course not at all obvious whether the proposed structure of the Big Book is really consistent with this hypothesis and one also consider modifications of this structure if necessary. The manner to proceed is by making questions.

1. The proposed picture would give only single integer characterizing the covering. Two integers assignable to CD and CP_2 degrees of freedom are however needed. How these two coverings could emerge?

(a) One should fix also the values of $\pi^n_k = \partial L_K/\partial h^k_n$, where n refers to space-like normal coordinate at the wormhole throats. If one requires that charges do not flow between regions with different signatures of the metric the natural condition is $\pi^n_k = 0$ and allows also multi-valued solution. Since wormhole throats carry magnetic charge and since weak form of electric-magnetic duality is assumed, one can assume that CP_2 projection is four-dimensional so that one can use CP_2 coordinates and regard $\partial_0 m^k$ as un-knows. The basic idea about topological condensation in turn suggests that M^4 projection can be assumed to be 4-D inside space-like 3-surfaces so that here $\partial_0 s^k$ are the unknowns. At partonic 2-surfaces one would have conditions for both π^0_k and π^n_k. One might hope that the numbers of solutions are finite for preferred extremals because of their symmetries and given by n_a for $\partial_0 m^k$ and by n_b for $\partial_0 s^k$. The optimistic guess is that n_a and n_b corresponds to the numbers of sheets for singular coverings of CD and CP_2. The covering could be visualized as replacement of space-time surfaces with space-time surfaces which have $n_a n_b$ branches. n_b branches would degenerate to single branch at the ends of diagrams of the generaled Feynman graph and n_a branches would degenerate to single one at wormhole throats.

(b) This picture is not quite correct yet. The fixing of π^0_k and π^n_k should relate closely to the effective 2-dimensionality as an additional condition perhaps crucial for criticality. One could argue that both π^0_k and π^n_k must be fixed at X^3 and X^3_l in order to effectively bring in dynamics in two directions so that X^3 could be interpreted as a an orbit of partonic 2-surface in space-like direction and X^3_l as its orbit in light-like direction. The additional conditions could be seen as gauge conditions made possible by symplectic and Kac-Moody type conformal symmetries. The conditions for π^k_0 would give n_b branches in CP_2 degrees of freedom and the conditions for π^n_k would split each of these branches to n_a branches.

(c) The existence of these two kinds of conserved charges (possibly vanishing for π^n_k) could relate also very closely to the slicing of the space-time sheets by string world sheets and partonic 2-surfaces.

2. Should one then treat these branches as separate space-time surfaces or as a single space-time surface? The treatment as a single surface seems to be the correct thing to do. Classically the conserved changes would be $n_a n_b$ times larger than for single branch. Kähler action need not (but could!) be same for different branches but the total action is $n_a n_b$ times the average action and this effectively corresponds to the replacement of the $\hbar_0/g^2_K$ factor of the action with $\hbar/g^2_K$, $r \equiv \hbar/\hbar_0 = n_a n_b$. Since the conserved quantum charges are proportional to $\hbar$ one could argue that $r = n_a n_b$ tells only that the charge conserved charge is $n_a n_b$ times larger than without multi-valuedness. $\hbar$ would be only effectively $n_a n_b$ fold. This is of course poor man's argument but might catch something essential about the situation.

3. How could one interpret the condition $J^{03}\sqrt{g_4} = 4\pi\alpha_K J_{12}$ and its generalization to be discussed below in this framework? The first observation is that the total Kähler electric charge is by $\alpha_K \propto 1/(n_a n_b)$ same always. The interpretation would be in terms of charge fractionization meaning that each branch would carry Kähler electric charge $Q_K = ng_K/n_a n_b$. I have indeed suggested explanation of charge fractionization and quantum Hall effect based on this picture [15].

4. The vision about the hierarchy of Planck constants involves also assumptions about imbedding space metric. The assumption that the M^4 covariant metric is proportional to $\hbar^2$ follows from the physical idea about $\hbar$ scaling of quantum lengths as what Compton length is. One can always introduce scaled M^4 coordinates bringing M^4 metric into the standard form by scaling up the M^4 size of CD. It is not clear whether the scaling up of CD size follows automatically from the proposed scenario. The basic question is why the M^4 size scale of the critical extremals must scale like $n_a n_b$? This should somehow relate to the weak self-duality conditions implying that Kähler field at each branch is reduced by a factor $1/r$ at each branch. Field equations should posses a dynamical symmetry involving the scaling of CD by integer k and $J^{0\beta}\sqrt{g_4}$ and $J^{n\beta}\sqrt{g_4}$ by $1/k$. The scaling of CD should be due to the scaling up of the M^4 time interval during which the branched light-like 3-surface returns back to a non-branched one.

5. The proposed view about hierarchy of Planck constants is that the singular coverings reduce to single-sheeted coverings at $M^2 \subset M^4$ for CD and to $S^2 \subset CP_2$ for CP_2. Here S^2 is any homologically trivial geodesic sphere of CP_2 and has vanishing Kähler form. Weak self-duality condition is indeed consistent with any value of $\hbar$ and impies that the vacuum property for the partonic 2-surface implies vacuum property for the entire space-time sheet as holography indeed requires. This condition however generalizes. In weak self-duality conditions the value of $\hbar$ is free for any 2-D Lagrangian sub-manifold of CP_2.

   The branching along M^2 would mean that the branches of preferred extremals always collapse to single branch when their M^4 projection belongs to M^2. Magnetically charged light-light-like throats cannot have M^4 projection in M^2 so that self-duality conditions for different values of $\hbar$ do not lead to inconsistencies. For spacelike 3-surfaces at the boundaries of CD the condition would mean that the M^4 projection becomes light-like geodesic. Straight cosmic strings would have M^2 as M^4 projection. Also CP_2 type vacuum extremals for which the random light-like projection in M^4 belongs to M^2 would represent this of situation. One can ask whether the degeneration of branches actually takes place along any string like object $X^2 \times Y^2$, where X^2 defines a minimal surface in M^4. For these the weak self-duality condition would imply $\hbar = \infty$ at the ends of the string. It is very plausible that string like objects feed their magnetic fluxes to larger space-times sheets through wormhole contacts so that these conditions are not encountered.

4.4.3 Connection with the criticality of preferred extremals

Also a connection with quantum criticality and the criticality of the preferred extremals suggests itself. Criticality for the preferred extremals must be a property of space-like 3-surfaces and light-like 3-surfaces with degenerate

4-metric and the degeneration of the $n_a n_b$ branches of the space-time surface at the its ends and at wormhole throats is exactly what happens at criticality. For instance, in catastrophe theory roots of the polynomial equation giving extrema of a potential as function of control parameters co-incide at criticality. If this picture is correct the hierarchy of Planck constants would be an outcome of criticality and of preferred extremal property and preferred extremals would be just those multi-branched space-time surfaces for which branches co-incide at the boundaries of $CD \times CP_2$ and at the throats.

5 Weak form of electric-magnetic duality and explicit calculation of Kähler function

The basic technical problem of quantum TGD has been the explicit calculation of the Kähler function [10]. Here only the overall view is discussed. The identification as a Kähler action for a preferred extremal of Kähler action does not look a very practical approach since even the question what "preferred" means has been far from obvious. The notion which I have christened as a weak form of electric-magnetic duality however led to a dramatic progress in this problem [10]. One ends up to an expression of Kähler function as a Chern-Simons action for its extremal defined by the ends of the space-time sheet and wormhole throats and also to an expression as a Dirac determinant [A4, 10] and there are excellent hopes that a longstanding technical problem has been finally solved. Also the notion of preferred extremal finds a precise definition in terms of general solution ansatz for field equations forced by the reduction of TGD to almost topological quantum field theory and the theory can be solved also in the fermionic sector.

The notion of electric-magnetic duality [57] was proposed first by Olive and Montonen and is central in $\mathcal{N} = 4$ supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for CP_2 geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [7]. What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

1. The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

2. This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be $(2, -1, -1)$ and could be proportional to color hyper charge.

3. The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects in electro-weak scale: this should become manifest at LHC energies.

The weak form of electric-magnetic duality has equally dramatic implications concerning the mathematical understanding of the basic theory and its calculability.

1. The weak form electric-magnetic duality together with Beltrami flow property [58] of Kähler current leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable [A4, 10]. This has an enormous impact concerning the calculability of the theory. The basic observation is that if Kähler current is proportional to instanton current then the Coulomb interaction term in the decomposition of the Kähler action to interior and boundary terms vanishes and Kähler action reduces to a mere boundary term, which by the weak form of electric magnetic duality reduces to a Chern-Simons term. The proportionality of the Kähler current to instanton current implies also the vanishing of the 4-D Lorentz force if the CP_2 projection of the space-time surface has dimension less than four and has been conjectured to be a general property of solutions of field equations [16] so that the reduction to almost topological QFT has been implicitly prediction of TGD for almost decade but has remained unrecognized.

2. The requirement that WCW Kähler metric is non-trivial in M^4 degrees of freedom forces to replace CP_2 Kähler form with the sum of CP_2 and S^2 Kähler forms. The latter defines a magnetic monopole field of a monopole residing at the time-like line connecting the tips of CD. The non-vacuum extremals remain extremals and the vacuum extremals representable as graphs $M^4 \rightarrow CP_2$ are replaced with vacuum extremals for which the induced Kähler forms of CP_2 sum up to zero. The most general extremals of this kind have 3-D CP_2 projection which is a good news from the point of view of TGD based description of the classical gravitation.

3. One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT [A4, 10, 16]. The solution ansatz makes more detailed the older solution ansatz and is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. Kähler current in turn must be proportional to instanton current to achieve the vanishing of Coulomb term in Kähler action implying a reduction to almost topological QFT. The proposed solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow [58]) assignable to the instanton current. A general ansatz satisfying the integrability conditions is found.

   The solution ansatz applies also to the extremals of Chern-Simons action and to the conserved currents associated with the modified Dirac equation defined as contractions of the modified gamma matrices between the solutions of the modified Dirac equation. The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current (Kähler magnetic field associated with Chern-Simons action). Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics but Kähler current must still be proportional to instanton current. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the radients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural. This means dual interpretations in terms of hydrodynamics and field theory.

4. The general solution ansatz works for induced Kähler Dirac equation and Chern-Simons Dirac equation and reduces them to ordinary differential equations along flow lines. The induced spinor fields are simply constant along flow lines of indued spinor field for Dirac equation in suitable gauge. Also the generalized eigen modes of the modified Chern-Simons Dirac operator can be deduced explicitly if the throats and the ends of space-time surface at the boundaries of CD are extremals of Chern-Simons action. Chern-Simons Dirac equation reduces to ordinary differential equations along flow lines. One can deduce the general form of the spectrum and the explicit representation of the Dirac determinant in terms of geometric quantities characterizing the 3-surface (eigenvalues are inversely proportional to the lengths of strands of the flow lines in the effective metric defined by the modified gamma matrices). The resulting general form of Kähler function is consistent with the expression of the Kähler action of CP_2 type vacuum extremals conjectured from the argument leading to a general formula for gravitational constant.

5. Connections with various conjectures emerge. Infinite-primes [19, A7] provide a highly suggestive characterization for the spectrum of the eigenvalues expressible in terms of M^2 pseudo-momenta identifiable as hyper-complex primes of the projections of hyper-octonionic primes to hyper-complex plane M^2. This would also mean a number theoretical characterization of the geometry of 3-surfaces defining the lines of the generalized Feynman diagram. An arithmetic quantum field theory defined by infinite primes would correlate via the conservation of number theoretic momentum $\sum n_i log(p_i)$ the geometries for the lines of the generalized Feynman diagram arriving at a given vertex realizing therefore quantum classical correspondence. A precise connection with the p-adic length scale hypothesis and hierarchy of Planck constants emerges. Even the notion of number theoretic braid emerges also unavoidably so that it is fair to say that a large bundle of "must-be-trues" reduces to consequences of the weak form of electric-magnetic duality.

To sum up, the weak form of electric-magnetic duality gives excellent hopes that quantum TGD is exactly solvable theory. Of course, it must be made clear that the proportionalty of all conserved currents to instanton current defining Beltrami flow is very strong and the mathematical proof that this reduction almost obvious in the hydrodynamical picture is really possible is lacking.

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Books about TGD

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