Article

On Lorentz-invariant Theory of Gravitation Part 3: Optical-mechanical Analogy and the Particle-wave Duality In the Theory of Gravity

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Abstract

The two features of the behavior of bodies and fields in mechanics, electromagnetic theory, quantum theory and theory of gravity are compared in the article: the optical-mechanical analogy and the wave-particle dualism. It is shown that in terms of mathematical description the two features are similar in all cases. On this basis, it can hypothesized that the gravitational equation of general relativity is a generalization of the wave equation of the gravitational field. Convincing reasons are given to justify this hypothesis. It is assumed that the wave equation of the nonlinear theory of elementary particles (NTEP) can be considered as a contender for the role of gravitational equations.

Keywords: gravitation theory, Lorentz-invariant gravitation theory, non-linear quantum theory, space-time background, physical vacuum.

Abbreviations: NTEP - nonlinear theory of elementary particles, EM - electromagnetic; HJE - Hamilton-Jacobi equation, HEE - Hilbert-Einstein equations of general relativity.

1.0. Introduction: Statement of subject

The general problem of our research is the construction of a theory of gravity based on the achievements of the Hilbert-Einstein gravitation theory that would save all the achievements of the last, but would lack theirs drawbacks (see (Logunov, 2006; Kyriakos, 2012a)).

Because of the extremely low values of gravity force for the masses of elementary particles, gravity applies to macrophenomena and must be described by classical physics. On the other hand, a body mass is the sum of the masses of the elementary particles that make up this body. In the modern theory and the nonlinear theory of elementary particles (NTEP) (see articles in the "Prespacetime Journal") the production of mass of elementary particles is described by a special mechanism. However, the mechanics of elementary particles is the wave mechanics. At the same time, the mechanics of the motion of bodies in a gravitational field is the mechanics of macroscopic bodies and thus the classical mechanics. In this case what is the connection between them, from the physical point of view and from the point of view of mathematical description?

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From a physical point of view, the relationship between the wave theory and mechanics of particles, in the form of optical-mechanical analogy of equations (recall that optics is the science of electromagnetic waves of the light range) was first discovered by R. Hamilton

Based on the study of the general laws of motion of classical mechanics, Hamilton came to the conclusion that motion along a trajectory of a massive particle in a field corresponds to the motion of a massless particle - photon - along the ray of light in a medium with a variable index of refraction. In the absence of the field or at constant refractive index of medium, the one and the other particles move in a straight line. In another words, at free motion a particle moves in a Euclidean space, but at non-free motion it moves in a curved space, which, in general is a Riemannian space.

Mathematically the equation of motion of a material body is the Hamilton-Jacobi equation (HJE). In this way, as we mentioned in the review (Kyriakos, 2012a), it is used in GRT (Landau and Lifshitz, 1980). Hamilton noted that this equation also describes the motion of light rays. The dual use of this equation is namely the mathematical expression of the optical-mechanical analogy.

But this analogy was not complete. Conventional optics is divided into wave and geometrical (ray) optics. The geometrical optics is the limited case of wave optics at very high frequencies of waves. Namely in this limit, the analogy between the equations of motion of the particle and the equation of the light beam is detected . But this analogy is not related to the connection of the equations of motion of a particle with the wave equation of light; so it has long been considered nothing more than an interesting mathematical conclusion.

De Broglie was the first who pointed out the possibility of an extended analogy, at least, for elementary particles. Based on Lorentz transformations he has shown that the elementary particles (such as electron) must have wave properties. In other words, de Broglie showed that the analogy between light and a material particle is complete: light and particle have both wave and particle properties (which is called "wave-particle dualism"). De Broglie showed mathematically that the motion of an electron in a hydrogen atom can be regarded as a movement of the waves according to Fermat's principle for light waves. The existence of the wave properties of electrons has been confirmed by experiment and raised questions about the mathematical description of the motion of the electron as a wave.

As is known, the equation of motion - the wave equation of motion of the electron (for non-relativistic speeds) was found by Schrödinger (Schroedinger, 1982).. The most important for us are the two pointsont which Schrödinger relied (see Schrodinger. First and second posts):

1) Schrödinger derived his equation from HJE, postulating the proportionality of the HJE main function ("action") with the phase of the electron wave.

2) Schrödinger noted the link of the spatial interval with the action, following from HJE .
Later, the link of HJE, as a ray equation, with the theory of gravity was investigated by V. Fock (Fock, 1964). The connection between the theory of gravity with the wave equation was not discussed, since it concerns macrobodies, which practically do not have wave properties. However,

the full optical-mechanical analogy raises the question of the relation of the body motion in the gravitational field with the wave equation and we come back to this question later.

In his book V. Fock showed that a Lorentz transformation should be considered as invariant transformations of HJE. Using this approach, he pointed out that the space-time interval in the pseudo-Euclidean space is determined by the HJE main function, i.e. by action.

Unfortunately, the relationship of HJE with the space-time interval in the case of a curved space-time (in particular, in the pseudo-Riemannian space) not considered by V.Fock. This problem has not been solved and is one of the subjects to study. To take some steps in this direction, let us consider briefly the results of the above studies in terms of our problem.

2.0. Action function and its physical meaning.

2.1. Classical mechanics

The notion of function as "action" (the name given by Leibniz) was the result of the work of many famous scientists for nearly 200 years (Polak, 1959). The task in which the need occur for such a function is called the problem of functional extremum. The problem can be briefly stated as follows:

1) It is needed to find the equation of motion of a massive particle under the action of any forces between two points *A* and *B*. Clearly, the number of possible paths of motion between *A* and *B* can be infinite. But it was verified experimentally that the particle moves according to the Newton's law (or one of its equivalents), only along one definite path. Therefore, the question arose:

2) is there a physical value $S(x, y, z, t)$ that determines the choice from a set of trajectories, only the trajectory that corresponds to the task.

For some tasks was found, the function $p \cdot \Delta s$, where p is momentum of the body, and Δs is the element of path, so that the function $S' = \sum_{\Delta s \to 0} p \cdot \Delta s = \int_A^B p \cdot ds$ must have the extremum value (maximum or minimum) for real trajectory. Function S was called "action." For other problems was found function $T\Delta t$, where T is the kinetic energy of the particle, Δt is time traffic, so that action $S'' = \sum_{\Delta t \to 0} T \cdot \Delta t = \int_{t1}^{t2} T \cdot dt$. Later, for a wide range of mechanics problems was found the function $L\Delta t = (T - V)\Delta t$, where V is the potential energy, and $L = T - V$ is called the Lagrange function, so that action $S''' = \int (T - V)dt$ (note the fact that the action includes products "momentum x distance" and "energy x time".

The real trajectory of motion is determined by the variation of the action is equal to zero: $\delta S = 0$. The consequence of this equation is the Euler-Lagrange equation. But Hamilton found that the equation of motion can be written in the form of Hamilton-Jacobi (HJE) by means of action function. Its physical meaning is found in quantum mechanics - from the wave theory of the electron (a question, which we will discuss later). Next, look at specific forms HJE for different physical problems.

2.1.1. *Relativistic and non-relativistic Hamilton-Jacobi equation* (Landau and Lifshitz, 1980)

We can say that at the free movement (i.e. in absence of field) the particles move in a pseudo-Euclidean space, and in the case of motion in the field - in a curved space (which generally is a Riemannian space).

In the absence of the field (in the case of a photon it is equivalent to the presence of constant refractive index of the medium), the massless (photon) and massive (electron) particles move in a straight line.

In the case of massless particles – the photon - the equation of the wave front of light is the homogenous Hamilton-Jacobi equation (HJE) and has the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = 0, \qquad (3.2.1)$$

which is an equation of first order and second degree. As is well known (Landau and Lifshitz, Field Theory), the action S is associated with the momentum $\vec{p}$ and Hamiltonian $\hat{H}$ (total energy) by the relations: $\vec{p} = \frac{\partial S}{\partial \vec{r}}$ (i.e. $\frac{\partial S}{\partial t} = \varepsilon$, $\frac{\partial S}{\partial x} = p_x$, $\frac{\partial S}{\partial y} = p_y$, $\frac{\partial S}{\partial z} = p_z$) and $\hat{H} = -\frac{\partial S}{\partial t}$. In this case $\hat{H} = \hat{H}(q_{1,}..., q_n; p_{1,}..., p_n; t)$, and $S = S(q_{1,}..., q_n, t)$. The relationship $\hat{H} = -\frac{\partial S}{\partial t}$, considering $\vec{p} = \frac{\partial S}{\partial \vec{r}}$, actually is the HJE.

Considering the action as a phase of the wave, we see that the wave vector plays in geometrical optics the role of the particle's momentum in mechanics, and frequency plays the role of the Hamiltonian, i.e. of the total energy of the particle.

This equation can be written in linearized form as:

$$\frac{\partial S}{\partial t} = c\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2}, \qquad (3.2.2)$$

One can assume that the motion of a photon in an external field with the energy-momentum ε_{ex}, $\vec{p}_{ex}$, is described by HJE of type:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+\varepsilon_{ex}\right)^2-\left(\frac{\partial S}{\partial x}-p_{x\,ex}\right)^2-\left(\frac{\partial S}{\partial y}-p_{y\,\mathrm{ex}}\right)^2-\left(\frac{\partial S}{\partial z}-p_{z\,\mathrm{ex}}\right)^2=0\,, \qquad (3.2.3)$$

(note that ε_{ex}, $\vec{p}_{ex}$ can represent any field: electromagnetic, gravitational, etc.). This corresponds to the motion of a photon in an inhomogeneous medium with a refractive index that depends on coordinates and time. Obviously, the trajectory will not be a straight line, but a curve.

In the case of the free motion of a massive relativistic particle with mass m (such as an electron without external field) it is easy to obtain the relativistic HJE:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial y}\right)^2-\left(\frac{\partial S}{\partial z}\right)^2=m^2c^2\,, \qquad (3.2.4)$$

which can be written more concisely in vector form $\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\vec{\nabla}S\right)^2=m^2c^2$ or in covariant form $g^{ik}\left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial y^k}\right)+m^2c^2=0$.

In the case of the motion of a massive particle in an external field with the energy-momentum ε_{ex}, $\vec{p}_{ex}$, HJE is:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+\varepsilon_{ex}\right)^2-\left(\frac{\partial S}{\partial x}-p_{x\,ex}\right)^2-\left(\frac{\partial S}{\partial y}-p_{y\,\mathrm{ex}}\right)^2-\left(\frac{\partial S}{\partial z}-p_{z\,\mathrm{ex}}\right)^2=m^2c^2\,, \qquad (3.2.5)$$

In the nonrelativistic case, this equation in Cartesian coordinate system is:

$$\frac{\partial S}{\partial t}+\sum\frac{1}{2m}\left\{\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right\}=-V\,, \qquad (3.2.6)$$

2.2. Quantum mechanics

Schrödinger as the starting point of his search of the electron wave equation took the HJE equation. The latter is not a wave equation, but, as it appears, it is closely connected with it. Consider this relationship according to Schrödinger.

2.2.1. *The geometric representation of the particle motion as wave and as particles*

In his Nobel lecture Schrödinger described the basic idea of wave mechanics as follows (Schrödinger, 1933): “I would define the present state of our knowledge as follows. The ray or the particle path corresponds to a *longitudinal* relationship of the propagation process (i.e. *in the direction* of propagation), the wave surface on the other hand to a *transversal* relationship (i.e.

norma1 to it). *Both* relationships are without doubt real; one is proved by photographed particle paths, the other by interference experiments. To combine both in a uniform system has proved impossible so far. Only in extreme cases does either the transversal, shell-shaped or the radial, longitudinal relationship predominate to such an extent that we *think* we can make do with the wave theory alone or with the particle theory alone".

"According to the wave theory of light, the light rays, strictly speaking, have only fictitious significance. They are not the physical paths of some particles of light, but are a mathematical device, the so-called orthogonal trajectories of wave surfaces, imaginary guide lines as it were, which point in the direction normal to the wave surface in which the latter advances (cf. Fig. 1 which shows the simplest case of concentric spherical wave surfaces and accordingly rectilinear rays, whereas Fig. 2 illustrates the case of curved rays)".

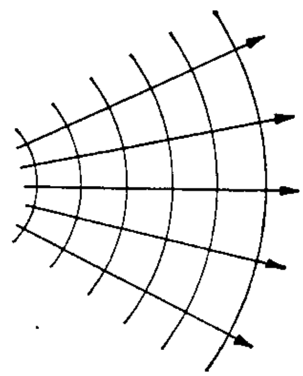

Fig. 1

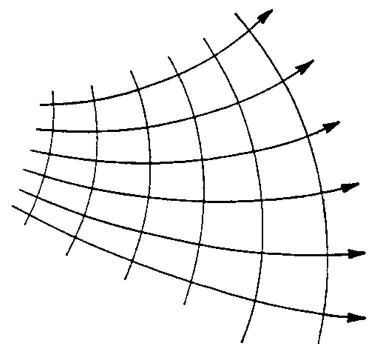

Fig. 2

Thus, on the one hand we have the wave pattern of motion (Schrödinger, Nobel lecture), which results in the following conclusion (Schrödinger, 1933):

"We identify the area of interference, the diffraction halo, with the atom; we assert that the atom in reality is merely the diffraction phenomenon of an electron wave captured us it were by the nucleus of the atom. It is no longer a matter of chance that the size of the atom and the wavelength are of the same order of magnitude: it is a matter of course".

On the other hand, we have the motion of the electron as a particle. But the differentiation of these movements is a difficult task (Schroedinger, 1982, second part):

It is clear that then the " system path " in the sense of classical mechanics, *i.e.* the path of the point of exact phase agreement, will completely lose its prerogative, because there exists a whole continuum of points before, behind, and near the particular point, in which there is almost as complete phase agreement, and which describe totally different " paths ". In other words, the wave group not only fills the whole path domain all at once but also stretches far beyond it in all directions.

In *this* sense do I interpret the "phase waves" which, according to de Broglie, accompany the path of the electron; in the sense, therefore, that no special meaning is to be attached to the electronic path itself (at any rate, in the interior of the atom), and still less to the position of the electron on its path. And in this sense I explain the conviction, increasingly evident to-day, *firstly*, that real meaning has to be denied to the *phase* of electronic motions in the atom ; *secondly*, that we can never assert that the electron at a definite instant is to be found on *any definite one* of the quantum paths, specialised by the quantum conditions; and *thirdly*, that the true laws of quantum mechanics do not consist of definite rules for the *single path*, but that in these laws the elements of the whole manifold of paths of a system are bound together by equations, so that apparently a certain reciprocal action exists between the different paths.

It is not incomprehensible that a careful analysis of the experimentally known quantities should lead to assertions of this kind, if the experimentally known facts are the outcome of such a structure of the real process as is here represented. All these assertions systematically contribute to the relinquishing of the ideas of "place of the electron" and "path of the electron". If these are not given up, contradictions remain. This contradiction has been so strongly felt that it has even been doubted whether what goes on in the atom could ever be described within the scheme of space and time. From the philosophical standpoint…
I would consider a conclusive decision, in this sense as equivalent to a complete surrender. For we cannot really alter our manner of thinking in space and time, and what we cannot comprehend within it we cannot understand at all. There *are* such things — but I do not believe that atomic structure is one of them. From our standpoint, however, there is no reason for such doubt, although or rather *because* its appearance is extraordinarily comprehensible. So might a person versed in geometrical optics, after many attempts to explain diffraction phenomena by means of the idea of the ray (trustworthy for his macroscopic optics), which always came to nothing, at last think that the *Laws of Geometry* are not applicable to diffraction, since he continually finds that light rays, which he imagines as *rectilinear* and *independent* of each other, now suddenly show, even in homogeneous media, the most remarkable *curvatures*, and obviously *mutually influence* one another».

From this analysis follows that the particle motion as a wave and the particle motion as a material body determine one another. It can be assumed that a particle in physical vacuum at each infinitesimal step of motion selects its direction, by "touching" with waves the state of physical vacuum around himself.

2.2.2. *Relationship between the differential equation of particle motion as mechanical body and as corresponding wave*

The Hamiltonian analogy between mechanics and optics in quantum mechanics, Schrödinger considered in detail (Schroedinger, 1982, first and second parts):

“Let us throw more light on the *general* correspondence which exists between the Hamilton-Jacobi differential equation of a mechanical problem and the "allied" *wave equation*.

The *inner* connection between Hamilton's theory and the process of wave propagation is anything but a new idea. It was not only well known to Hamilton, but it also served him as the starting-point for his theory of mechanics, which grew out of his *Optics of Non-homogeneous Media*. Hamilton's variation principle can be shown to correspond to Format's *Principle* for a wave propagation in configuration space (*q*-space), and the Hamilton-Jacobi equation expresses Huygens' *Principle* for this wave propagation. Unfortunately this powerful and momentous conception of Hamilton is deprived, in most modem reproductions, of its beautiful raiment as a superfluous accessory, in favour of a more colourless representation of the analytical correspondence".

On this basis, Schroedinger derived his famous wave equation of the electron.

Let us consider the derivation of the Schrödinger wave equation as the steady state equation of atom (Stanyukovich, Kolesnikov et al., 1968).

For definiteness we consider a hydrogen atom, which consists of one proton (assuming it is immobile) and the electron moving around it (assuming it is a point with coordinates x, y, z). Let the total energy of the system be equal to $\varepsilon = const$. The task is to identify the values ε, which make the system stable.

We assume that the instantaneous (momentary) value of the total energy of the system is given by:

$$\varepsilon_m = T + V = \frac{\vec{p}^2}{2m} + V(x, y, z) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{e^2}{r}, \tag{3.2.7}$$

Substituting the values of the projections of momentum in the action, we find

$$\varepsilon_m = \frac{1}{2m}\left\{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right\} - \frac{e^2}{r}, \tag{3.2.7'}$$

We will now define the difference at evry point x, y, z:

$$\Delta\varepsilon(x, y, z) = \varepsilon_m - \varepsilon = \frac{1}{2m}\left\{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right\} - \frac{e^2}{r} - \varepsilon, \tag{3.2.8}$$

Let us integrate this difference over all possible values of the coordinates:

$$\Delta\tilde{\varepsilon} = \int \Delta\varepsilon(x, y, z)dxdydz = \int\left\{\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] - \frac{e^2}{r} - \varepsilon\right\}dxdydz, \tag{3.2.9}$$

It is natural now to assume that the stable movements of our system meet the minimum value of the integral $\Delta\tilde{\varepsilon}$ as a functional of the function $S(x, y, z)$. Varying $\Delta\varepsilon$ with respect to S and equating the variation to zero, we obtain:

$$\delta(\Delta\tilde{\varepsilon}) = \int\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right)\delta S\, dxdydz = 0, \tag{3.2.10}$$

From this follows the condition for the stability of motion of our system (condition of Lyapunov-Chetaev)

$$\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}=\Delta S=0\,, \tag{3.2.11}$$

This equation, however, does not contains the value of the constant ε, which makes the motion of the system stable. Therefore, Schrödinger considered transformation of the desired function

$$S(x,y,z)=A\ln\psi(x,y,z)\,, \tag{3.2.12}$$

As result of the transformation we obtain:

$$\Delta\varepsilon(x,y,z)=\frac{A^2}{2m\psi^2}\left\{\left(\frac{\partial\,\psi}{\partial\,x}\right)^2+\left(\frac{\partial\,\psi}{\partial\,y}\right)^2+\left(\frac{\partial\,\psi}{\partial\,z}\right)^2\right\}-\frac{e^2}{r}-\varepsilon\,, \tag{3.2.13}$$

Next Schrödinger considered another integral

$$\langle\Delta\varepsilon\rangle=\int\Delta\varepsilon(x,y,z)\psi^2(x,y,z)\,dxdydz\,, \tag{3.2.4}$$

Varying it on ψ and setting the result to zero, Schrödinger received the wave equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2m}{A^2}\left(\varepsilon+\frac{e^2}{r}\right)\psi=0\,, \tag{3.2.15}$$

where was found that $A=-i\hbar$

Note that equation (3.2.11) is equivalent to (3.2.15) if we use (3.2.12) in the form $S(x,y,z)=-i\hbar\ln\psi(x,y,z)$. It is easy to show this by direct substitution.

2.2.3. *The geometric representation of the Schrödinger (non-relativistic) equation*

(Schroedinger, 1982, second part, p.14, etc).

"Let us consider the general problem of conservative systems in classical mechanics. The Hamilton-Jacobi equation runs

$$\frac{\partial S}{\partial t}+T\left(q_k,\frac{\partial S}{\partial q_k}\right)+V(q_k)=0\,, \tag{3.2.16}$$

where S *is* the action function, i.e. the time integral of the Lagrange function *T - V* along a path of the system as a function of the end points and the time, q_k is a representative position co-ordinate; *T* is the kinetic energy as function of the *q's* and momenta, being a quadratic form of the latter, for which, as prescribed, the partial derivatives of *S* with respect to the *q's* $\frac{\partial S}{\partial q_k}$ are written. V is the potential energy….

Suppose that a function S has been found. Then this function can be clearly represented for every definite *t,* if the family of surfaces $S=const$, be described in q-space and to each member a value of S be ascribed….

Let the value S_0 be given in Fig. 3 to an arbitrary surface.

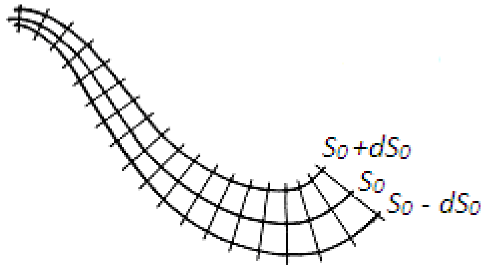

Fig. 3

In order to find the surface $S_0 + dS_0$, take side of the given surface as the positive one, the *step*

$$ds = \frac{dS_0}{\sqrt{2(\varepsilon - V)}}, \tag{3.2.17}$$

The locus of the end points of the steps is the surface $S_0 + dS_0$. Similarly, the family of surfaces may be constructed successively on both sides.

Now it is seen that our system of surfaces $S = const$, can be conceived as the system of wave surfaces of a progressive but stationary wave motion in *q*-space.

The function of action S plays the part of the *phase* of our wave system. The Hamilton-Jacobi equation is the expression of Huygens' principle.

We may sum up that S denotes, apart from a universal constant 1/*h*, the phase angle of the wave function.

We have thus shown: *The point of phase agreement for certain infinitesimal manifolds of wave systems, containing n parameters, moves according to the same laws as the image point of the mechanical system*".

2.2.4. *The description of the electron motion by means the Hamilton-Jacobi equations* (Landau and Lifshitz, 1980, p.100)

"We consider the motion of a particle with mass and charge in the field produced by a second charge e'; we assume that the mass of this second charge is so large that it can be considered as fixed. Then our problem becomes the study of the motion of a charge e in a centrally symmetric electric field with potential $\varphi = e'/r$. The total energy of the particle is equal to

$$\varepsilon = c\sqrt{p^2 + m^2c^2} + \frac{\alpha}{r}, \tag{3.2.18}$$

where $\alpha = ee'$. If we use polar coordinates in the plane of motion of the particle, then as we know from mechanics,

$$p^2 = \frac{M^2}{r^2} + p_r^2, \tag{3.2.19}$$

Where p_r is the radial component of the momentum, and M is the constant angular momentum of the particle. Then

$$\varepsilon = c\sqrt{\frac{M^2}{r^2} + p_r^2 + m^2c^2} + \frac{\alpha}{r}, \tag{3.2.20}$$

A complete determination of the motion of a charge in a Coulomb field starts most conveniently from the Hamilton- Jacobi equation. We choose polar coordinates r, φ in the plane of the motion.

The Hamilton-Jacobi equation has the form

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + \frac{\alpha}{r}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2, \tag{3.2.21}$$

We seek an S of the form

$$S = -\varepsilon t + M\varphi + f(r), \tag{3.2.22}$$

where ε, M are the constant energy and angular momentum of the moving particle. The result is

$$S = -\varepsilon t + M\varphi + \int\sqrt{\frac{1}{c^2}\left(\varepsilon - \frac{\alpha}{r}\right)^2 - \frac{M^2}{r^2} - m^2c^2} \cdot dr, \tag{3.2.22}$$

The trajectory is determined by the equation $\frac{\partial S}{\partial M} = const$. Integration of (3.2.22) leads to the trajectory. The integration constant is contained in the arbitrary choice of the reference line for measurement of the angle φ". (In detail see the book)

2.2.5. *Waves and particles: dualism of description*

We will write down some of the results of the Schrödinger research (including generalizations made later.)

1. The connection between the path and the wave equations is the connection between the movement of the wave surface and the motion of a point of surface along a certain line.

2. The motion of a particle as a wave, is described by the wave equation. The motion of a particle as a massive body (ray or trajectory) is described by the equation of motion of the point of the wave front - HJE.

3. The main function of HJE is a function $S(x, y, z, t)$, called "action."

4. Function $S(x,y,z,t)$ is instantaneously a function, which describes the surface of the wave front, or, equivalently, the surface of constant value of the phase of the wave.

5. In terms of the vector field theory, the function $S(x,y,z,t)$ can be regarded as a vector, the direction of which at each point is determined by the normal to the surface $\vec{S} = S(x,y,z,t)\vec{n}$.

6. With a accuracy up to the Planck constant, the action S is the phase of the wave $\Phi = \Phi_o e^{-i(\omega t - \vec{k}\,\vec{r})} = \Phi_0 e^{-\frac{i}{\hbar}(\varepsilon t - \vec{p}\,\vec{r})}$, i.e. $S = \hbar(\omega t - \vec{k}\,\vec{r}) = (\varepsilon t - \vec{p}\,\vec{r})$. In the case of electron $\dfrac{\Phi}{\Phi_0} = \psi$.

7. From this follows the transformation $S(x,y,z,t) = i\hbar \ln \dfrac{\Phi(x,y,z,t)}{\Phi_0}$, by which Schrödinger went from HJE to the wave equation.

8. The trajectory (ray) of the point of the surface $S(x,y,z,t)$ can be considered as the radius vector $\vec{r} = s\vec{n}$, where s is the path traveled by the set point $\vec{r}_0$.

9. Denote the phase of the wave with letter theta: $\vartheta = \dfrac{1}{\hbar}(\varepsilon t - \vec{p}\vec{r}) = (\omega t - \vec{k}\vec{r})$. It is obvious that $\vartheta = \dfrac{S}{\hbar} = i \ln \psi$. Using the equation of energy conservation $\varepsilon^2 - c^2 p^2 = m^2 c^4$, we can get the wave front equation for a massive particle:

$$\left(\frac{1}{c}\frac{\partial \vartheta}{\partial t}\right)^2 - (grad\,\vartheta)^2 = \frac{m^2 c^4}{\hbar^2}.$$

For massless particles (EM waves), when $m = 0$, this equation is:

$$\left(\frac{1}{c}\frac{\partial \vartheta}{\partial t}\right)^2 - (grad\,\vartheta)^2 = 0.$$

Characteristically, the values $\hbar \cdot \dfrac{\partial \vartheta}{\partial t} = \varepsilon$ and $\hbar \cdot grad\,\vartheta = -\vec{p}$ constitute a 4-vector. This indicates that the four-dimensional world of the theory of relativity is a consequence of the wave origin of the material particles.

10. From dimensional analysis follows that HJE is described by integral physical values, and the wave equation - by differential physical values. Specifically, the action has the dimension of angular momentum = product of momentum on the path = product of the energy on the time. According to NTEP the square of the wave function is, in absolute terms, the dimensions of density of energy-momentum (energy / volume of space) = pressure or tension (force / area), and in relative terms, the square of wave function is the probability density (dimensionless).

3.0. Wave and a particle in the gravitational theory (Fock, 1964).

3.1. The Law of Propagation of an Electromagnetic Wave Front

"The laws of propagation of light in empty space are thoroughly understood. They find their expression in the well-known equations of Maxwell

However, we are not interested in the general case of light propagation, but only in the propagation of a signal advancing with maximum speed, i.e. the propagation of a wave front. Ahead of the front of the wave all components of the field vanish. Behind it some of them are different from zero. Therefore, some of the field components must be discontinuous at the front.
On the other hand, given the field on some surface moving in space, the derivatives of the field on the surface are, in general, determined by Maxwell's equations.

Such a surface is called a characteristic surface or, briefly, a characteristic. Thus, discontinuities of the field can occur only on a characteristic, but since there must certainly be discontinuities at a wave front, such a front is clearly a characteristic.

Let us determine the equation of a characteristic for the system of Maxwell's equations.
Let the value of the field be given for those points and instants whose coordinates are related by the equation

$$t = \frac{1}{c} f(x, y, z), \qquad (3.3.1)$$

In particular, if $f = 0$ this amounts to stating initial conditions. Equation (3.3.1) may be looked upon as the equation of a certain hypersurface in the four-dimensional space-time manifold. When $(gradf)^2 > 1$ the same equation can be considered as the equation of an ordinary surface moving through space. Assume that on the hypersurface (3.3.1) the values of a certain function u are given

$$u\left(x, y, z, \frac{f}{c}\right) = u_0(x, y, z), \qquad (3.3.2)"$$

3.2. The photon trajectory (equations for rays)

"The equation describing the propagation of a wave front can be written in the **linear** form

$$\frac{\partial S}{\partial t} = c\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2}, \qquad (3.3.3)$$

(for definiteness we have chosen the plus sign before the square root).
In mechanics S play the role of the action function and the derivatives $\vec{p} = \frac{\partial S}{\partial \vec{r}}$ - the momenta , p_x, p_y, p_z. Corresponding to the Hamiltonian we have here the expression $\hat{H} = -\frac{\partial S}{\partial t}$ or

$$\hat{H} = c\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2} \ , \qquad (3.3.4)$$

To the trajectories of mechanics there correspond light rays. The equations for them are analogous to Hamilton's equations. They can be written

$$\frac{dx}{dt} = \frac{\partial \hat{H}}{\left(\frac{\partial S}{\partial x}\right)} = c\frac{\left(\frac{\partial S}{\partial x}\right)}{\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2}} \text{, etc. ,} \qquad (3.3.5)$$

$$\frac{d\left(\frac{\partial S}{\partial x}\right)}{dt} = -\frac{\partial \hat{H}}{\partial x} \text{, etc ,} \qquad (3.3.6)$$

Equation (3.3.6) shows that the quantities $\left(\frac{\partial S}{\partial x}\right), \left(\frac{\partial S}{\partial y}\right), \left(\frac{\partial S}{\partial z}\right)$ are constant along a ray, though they can, of course, vary from one ray to another. Therefore the rays will be straight

$$x - x_0 = c\frac{\left(\frac{\partial S}{\partial x}\right)}{\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2}}(t - t_0) \ , \quad \text{etc.,} \qquad (3.3.7)$$

which, according to mechanical-optical analogy, is the equation of motion of a point along the ray. If the sign of S, and hence of $\left(\frac{\partial S}{\partial x}\right), \left(\frac{\partial S}{\partial y}\right), \left(\frac{\partial S}{\partial z}\right)$, is changed, the direction of the ray is reversed ; the sign must be chosen according to the given sense of direction of the ray".

3.3. Connection of action with space-time interval (Fock, 1964)

"Any wave surface can be considered as formed of points moving along the rays with the speed of light according to (3.3.7).

We thus have the possibility of constructing a wave surface at time t when its form at time t is known.

Let the equation of the wave surface at time t_0 have the form

$$S^0(x_0, y_0, z_0) = 0 \ , \qquad (3.3.8)$$

where x_0, y_0, z_0 are coordinates varying over this surface. Knowing the equation of the surface we can calculate the quantities

$$\alpha(x_0, y_0, z_0) = \left(\frac{\left(\frac{\partial S}{\partial x}\right)^0}{\sqrt{\left(\frac{\partial S}{\partial x}\right)^{0\,2} + \left(\frac{\partial S}{\partial y}\right)^{0\,2} + \left(\frac{\partial S}{\partial z}\right)^{0\,2}}} \right)_0 , \text{ etc.,} \qquad (3.3.9)$$

Here the sign of the right-hand sides is determined by the given direction of wave propagation. The equation of the ray passing through the point (x_0, y_0, z_0) of the initial wave surface is

$$\left. \begin{aligned} x - x_0 &= c\alpha(t - t_0) \\ y - y_0 &= c\beta(t - t_0), \quad (\alpha^2 + \beta^2 + \gamma^2 = 1) \\ z - z_0 &= c\gamma(t - t_0) \end{aligned} \right\}, \qquad (3.3.10)$$

The quantities x, y, z give the positions of the point to which the point (x_0, y_0, z_0) moves at time t. Allowing x_0, y_0, z_0 to take on all values which satisfy (3.3.8), we obtain from (3.3.10) all points which at time t lie on the wave surface.

If we solve (3.3.10) for x_0, y_0, z_0 and insert the functions

$$x_0 = x_0(x, y, z, t - t_0), \text{ etc.,} \qquad (3.3.11)$$

into the wave surface equation (3.3.8), we get the relation

$$S(x, y, z, t - t_0) = 0, \qquad (3.3.12)$$

which is the explicit form of the equation of the wave surface at time t. At $t - t_0$ obviously, $x_0 = x, y_0 = y, z_0 = z$ and equation (3.3.12) reduces to (3.3.8), which is the equation of the initially given wave surface.

From the ray equation (3.3.7) there follows the relation

$$c^2(t - t_0)^2 - \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right] = 0, \qquad (3.3.13)$$

which connects the coordinates of the initial and final points on each ray. It is the equation of a sphere centred at the point x_0, y_0, z_0 and of a radius $R = c(t - t_0)$ that increases linearly with time. Just as Hamilton-Jacobi equation, from which we started, this equation expresses the fact that the velocity of light propagation is constant.

For points infinitesimally separated relation (3.3.13) takes on the form

$$c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right) = 0, \qquad (3.3.13')$$

In this form the equation follows directly from Hamilton's equation (3.3.5)".
A frame for which

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial y}\right)^2-\left(\frac{\partial S}{\partial z}\right)^2=0, \tag{3.3.14}$$

is valid may be called inertial in the electromagnetic sense".

3.4. Features of the gravitational field

"The principle of the universal limiting velocity can be made mathematically precise as follows:
For any kind of wave advancing with limiting velocity and capable of transmitting signals the equation offront propagation is the same as the equation for the front of a light wave.

Thus the equation (3.3.14) acquires a general character; it is more general than Maxwell's equations from which we derived it.

The presence of a gravitational field somewhat alters the appearance of the equation of the characteristics from the form (3.3.14), but in this case one and the same equation still governs the propafgation of all kinds of wave fronts travelling with limiting velocity, including electromagnetic and gravitational ones.

Let us considere the expressions

$$(\nabla S)^2=\sum_{\mu,\nu=0}^{3} g^{\mu\nu}\frac{\partial S}{\partial x_\mu}\frac{\partial S}{\partial x_\nu}, \tag{3.3.15}$$

$$ds^2=\sum_{\mu,\nu=0} g^{\mu\nu}\partial x_\mu \partial x_\nu , \tag{3.3.16}$$

which were obtained from the usual expressions of Relativity Theory by introducing variables x_1, x_2, x_3 and x_0 in place of the space and time coordinates x, y, z, t. We established the conditions subject to which the variable x_0 can characterize a sequence of events in time and the variables x_1, x_2, x_3 their location in space.

By itself, the introduction of new variables can naturally not influence the physical consequences of the theory; it is merely a mathematical device.

We shall call equations generally covariant, if they are valid for any arbitrary choice of independent variables.

The most essential characteristic of the gravitational field by which it differs from all other fields known to physics reveals itself in the effect of the field on the motion of a freely moving body or mass point. In a gravitational field all otherwise free bodies move in the same manner, provided the initial conditions of their motion, i.e. their initial positions and velocities, are the same.

According to Newton the gravitational field can be characterized by the gravitational potential $U(x, y, z)$. The gravitational potential produced by an solated spherically symmetric mass M at points exterior to itself is

$$\varphi_g = \frac{\gamma M}{r}, \tag{3.3.17}$$

where r is the distance from the centre of the mass. The quantity γ is the Newtonian constant of gravitation—in c.g.s. units it has the value

$$\gamma = \frac{1}{15000000} \frac{cm^3}{g \cdot cec^2}, \tag{3.3.18}$$

Thus φ_g has the dimensions of the square of a velocity. We note immediately that in all cases encountered in Nature, even on the surface of the Sun or of super-dense stars, the quantity φ_g is very small compared to the square of the speed of light

$$\varphi_g << c^2 , \tag{3.3.19}$$

In the general case of an arbitrary mass distribution the Newtonian potential U it produces satisfies Poisson's equation

$$\Delta\varphi_g = -4\pi\gamma\rho_m , \tag{3.3.20}$$

where ρ_m is the mass density. The Newtonian potentiall φ_g is fully determined by Poisson's equation together with continuity and boundary conditions which are as follows : the function φ_g and its first derivatives must be finite, singlevalued and continuous throughout space and must tend to zero at infinity.

As a result of the equality of inertial and gravitational mass the equation of motion

$$w = grad\varphi_g , \tag{3.3.21}$$

where w is acceleration, has universal character".

3.5. The space-time interval and the space-time metric

"The phenomenon of universal gravitation forces us to widen the framework of the theory of space and time which was the subject of the Newton theory. The necessity of this widening becomes clear from the following considerations. It follows from the equation of wave front propagation, which can be stated in the form

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = 0, \tag{3.3.22}$$

that light is propagated in straight lines. But light possesses energy and by the law of proportionality of mass and energy all energy is indissolubly connected with mass. Therefore fight must possess mass. On the other hand, by the law of universal gravitation, any mass located in a gravitational field must experience the action of that field and in general its motion will therefore not be rectilinear, Hence it follows that in a gravitational field the law of wave front propagation must have a form somewhat different from the one given above.

But the equation of wave front propagation is a basic characteristic of the properties of space and time. Hence it follows that the presence of the gravitational field must affect the properties of space and time and their metric is then not a rigid one. This is indeed the conclusion reached in the theory of gravitation which we now begin to construct.

As was shown, the equation of wave front propagation (3.3.22) with some additional assumptions, leads to the following expression for the square of the interval:

$$ds^2 = c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right), \tag{3.3.23}$$

The influence of the gravitational field on the properties of space and time must have the consequence that the coefficients in the equation of wave front propagation and in the expression for the square of the interval will differ from the constant values appearing in (3.3.22) and (3.3.23). We must now find an approximate form for the square of the interval in a gravitational field of Newtonian potential φ_g .

We shall thus now assume that space-time is in the main Euclidean, or rather pseudo-Euclidean, and that any deviation of space-time geometry from Euclidean geometry is a result of the presence of a gravitational field. Whereever there is no gravitational field, geometry must be Euclidean. For an insular distribution of masses the gravitational field must tend to zero at infinity and therefore we have to assume that at points far removed from the masses the geometry of space-time becomes Euclidean.

We shall now try to find a metric such that these equations coincide approximately with the Newtonian equations of motion for a free body in a given gravitational field. If this attempt is successful it will enable us to introduce the hypothesis that in a space-time with given metric a free body (mass point) moves along a geodesic ; in this way the connection between the law of motion and the metric will be established.

As we know, the equation of a geodesic may be derived from the variational principle

$$\delta \int ds = 0 , \tag{3.3.24}$$

If the squared interval is of the form (3.3.23) we have

$$ds = \sqrt{\left(c^2 - \upsilon^2\right)}dt , \tag{3.3.25}$$

or, for small velocities,

$$ds = \left(c - \frac{\upsilon^2}{2c}\right)dt , \tag{3.3.26}$$

Inserting (3.3.25) or (3.3.26) into (3.3.24) gives equations that describe motion with constant velocity, which indeed is free motion in the absence of a gravitational field. We can now assume that for small velocities and weak gravitational fields ($U << c^2$) the expression for the interval takes the form

$$ds = \sqrt{\left(c^2 - 2\varphi_g - \upsilon^2\right)}dt , \tag{3.3.27}$$

or

$$ds = \left[c - \frac{1}{c}\left(\frac{1}{2}\upsilon^2 + \varphi_g\right)\right]dt\,, \qquad (3.3.28)$$

in place of (3.3.25) or (3.3.26). Since neither an additive constant nor a constant multiplier are of any importance in a Lagrangian the variational principle (3.3.24), with ds taken from (3.3.28), gives the same result as the variational principle

$$\delta\int\left(\frac{1}{2}\upsilon^2 + \varphi_g\right)dt = 0\,, \qquad (3.3.29)$$

but this did indeed describe free motion of a body in a gravitational field. It is true that just because additive constants and multiplicative factors in a Lagrangian are immaterial equation (3.3.29) could be obtained from (3.3.24) and (3.3.27) with any sufficiently large value of the constant c

These arguments give us good reason to assume that under the conditions

$$\varphi_g << c^2; \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \upsilon^2 << c^2\,, \qquad (3.3.30)$$

the square of the interval differs little from the form

$$ds^2 = \left(c^2 - 2\varphi_g\right)dt^2 - \left(dx^2 + dy^2 + dz^2\right), \qquad (3.3.31)$$

The theory of gravitation gives the more exact expression

$$ds^2 = \left(c^2 - 2\varphi_g\right)dt^2 - \left(1 + \frac{2\varphi_g}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right), \qquad (3.3.31)”$$

4.0. The description methods of motion of bodies in gravitational field

“We consider (Fock, 1964). a problem of an astronomical type, relating to the motion of celestial bodies in empty space.

Our problem is simplified in the first place by the fact that the metric nowhere deviates greatly from the Euclidean ; the table given below gives an idea of how small the deviation is.

	Sun	Earth	Moon
α	1·48 km	0·443 cm	0·0053 cm
L	696,000 km	6,370 km	1,738 km
$\alpha : L$	2×10^{-6}	7×10^{-10}	3×10^{-11}

where $\alpha = \frac{\gamma M}{c^2}$ is the gravitational radius of the mass M. For the Sun, and even more so for the planets, the gravitational radius α is much smaller than the geometric radius L, which may be defined as the radius of a sphere of volume equal to that of the body.

A further simplifying circumstance is that at all significant distances from the bodies, the metric does not depend on the detailed internal structure of the latter, but only on certain overall characteristics. Such characteristics are the total mass of the body, its moments of inertia, the position and velocity of its mass centre and so on. The Newtonian potential of a body depends on these same quantities.
To solve Einstein's equations we shall use a method of approximation. It is based on an expansion of all required functions in inverse powers of the speed of light. An expansion that can formally be so described will, in fact, be an expansion in powers of certain dimensionless quantities, such as φ_g / c^2 and υ^2 / c^2 , where φ_g is the Newtonian potential and υ the square of some velocity, say the velocity of one of the bodies...

If we solve wave equations by introducing corrections for retardation we imply that the dimensions of the system are small compared to the wavelength of the waves emitted, which in this case are gravitational waves."

Let us refer once again to the analysis of the structure of GR, made by M.-A. Tonnela (Tonnelat, 1966):

"All the predictions of general relativity follow from the field equations and the laws of geodesic motion:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi T_{\mu\nu} \to g_{\alpha\beta}\,, \tag{3.4.1}$$

$$\delta \int ds = 0\,, \tag{3.4.2}$$

The first allow us to define $g_{\mu\nu}$ and put this value in (3.4.2). All the present-day predictions follow from the below mentioned values $g_{\mu\nu}$:

$$ds^2 = \left(1 - \frac{2\gamma\, M}{c^2 r}\right) c^2 dt^2 - \frac{1}{1 - \dfrac{2\gamma\, M}{c^2 r}} dr^2 - r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2\right), \qquad (3.4.3)\text{”}$$

From our analysis it follows that the equation (3.4.2) is equivalent to HJE; HJE and the expression for the interval ds are connected between them: according to Fock (see above) from HJE can be obtained ds. But then we must conclude that HJE in some sense is equivalent to HEE (3.4.1). The question arises, if both equivalences can take place, and if so, in what way?

HJE describes the motion of a body in an external field. In a Cartesian coordinate system, it has the form (3.2.5)

$$\frac{1}{c^2}\left(\frac{\partial\, S}{\partial\, t} + \varepsilon_{ex}\right)^2 - \left(\frac{\partial\, S}{\partial\, x} - p_{x\ ex}\right)^2 - \left(\frac{\partial\, S}{\partial\, y} - p_{y\ \mathrm{ex}}\right)^2 - \left(\frac{\partial\, S}{\partial\, z} - p_{z\ \mathrm{ex}}\right)^2 = m^2 c^2\,,$$

where the external field is given by functions of energy ε_{ex} and momentum $\vec{p}_{ex}$ (note that, generally, the equation (3.2.5) should be written in covariant form). But the function ε_{ex} and $\vec{p}_{ex}$ must be pre-found from the equation of a field source. Therefore, the equation of the field source in some sense must be equivalent to the HEE. But there are differences here: the HEE gives the metric tensor, while the quantities ε_{ex} and $\vec{p}_{ex}$ are not expressed directly through the metric tensor. To compare with the solutions of general relativity equation, it is necessary to show that this expression is possible, though it, for the solution of HJE this may not be necessary.

4.2. Description of the gravitational field as a perturbation

Let us leave aside the question of the field source equation to the next section, and try to find out if there are in physics, methods of problem solving, which would provide the necessary trajectory, corresponding to (3.4.2) and (3.4.3).

For our search of solution of this problem, the following assumption of M.-A. Tonnela is noteworthy (Tonnela, 1966):

"We assume that the laws of motion can be derived from the geodesic law. The experiment would allow then to recreate with a consistent approximation the structures ($R_{\mu\nu} = 0$ in a vacuum)of non-Euclidean space. But we can, on the contrary, suggests that this geodesic law is invalid, or at least, is shown in a simple Euclidean space, which has phenomenological properties, that is, some distortion, or, if desired, a polarization of empty space by means of gravitational field. Then the action of the gravitational field on light will be the result of special interaction, not the propagation of light in the empty space (but a curvilinear one) .

One might think that in this empty Euclidean space - though polarized by matter - there is an "index of emptiness," which justifies the expressions (3.4.1) and (3.4.2), in case of change of the interval $ds' = nds''$".

In other words (see also (Kyriakos, 2012b)), we can consider the gravitational field as a perturbation of the empty (in the absence of fields) physical vacuum, described by Euclidean geometry. In this case, we can use perturbation theory to calculate the motion of bodies in this field.

How does then the statement of the problem look like?

Here we can use the method of Fock (see above), which "is based on an expansion of all required functions in inverse powers of the speed of light. An expansion that can formally be so described will, in fact, be an expansion in powers of certain dimensionless quantities, such as φ_g / c^2 and υ^2 / c^2 , where φ_g is the Newtonian potential and υ the square of some velocity, say the velocity of one of the bodies".

In the general formulation, this method can be described as follows (Tonnela, 1966):
Assume that the field equations are unknown, i.e. structural conditions are unknown that must be prescribed by a non-Euclidean space. We could search in the reverse order the structure of space,

based on experimental results. For this, it would be sufficient to use the power series expansion with the parameter $\varphi_g / c^2 = \dfrac{\gamma M}{c^2 r}$, characterizing the influence of sources. We obtain:

$$\begin{aligned} ds^2 = &\left(1+\frac{a_1 M}{c^2 r}+\frac{a_2 M^2}{c^4 r^2}+...\right)c^2 dt^2 - \left(1+\frac{b_1 M}{c^2 r}+\frac{b_2 M^2}{c^4 r^2}+...\right)dr^2 - \\ &-\left(1+\frac{c_1 M}{c^2 r}+\frac{c_2 M^2}{c^4 r^2}+...\right)r^2 d\vartheta^2 - \left(1+\frac{d_1 M}{c^2 r}+\frac{d_2 M^2}{c^4 r^2}+...\right)r^2 \sin^2\vartheta d\varphi^2 \end{aligned}, \qquad (3.4.4)$$

$$\delta \int ds = 0, \qquad (3.4.5)$$

In this case the coefficients $a_1, a_2, ...; b_1, b_2, ...; c_1, c_2, ...; d_1, d_2, ...,$ should be consistently determined on the basis of perturbation theory: in first approximation - the first coefficient, then the second, and so on. In this case the test can serve the values from (3.4.3) according to general relativity. In particular:

1) Newton's law of gravity and gravitational shift, dictate a_1 ($a_1 = -2$);

2) The bending of light rays in a gravitational field, dictates $b_1 = -a_1$ ($a_1 = 2,\ b_1 = 2$).

3) the precession of the perihelion of Mercury's, dictates $a_1(b_1 - a_1) + a_2 = 0$ ($a_1 = 2,\ b_1 = 2,\ a_2 = 0$).

All other factors in these problems are equal to zero. It is clear that they are not necessarily equal to zero for other tasks. For example, in the case of the experiments with the gyroscope, the combination ($(2b_1 - a_1)$) arises. Also the new coefficients arise, which are introduced by the form , which does not have spherical symmetry. Such is, for example, the effect of the rotating central body, which introduces the term (*xdy - ydx*), the influence of which can be foreseen by theory and measured with experiment.

Our problem can be formulated as follows: how to build HJE, in such a form that on the basis of the perturbation theory, it would be possible to obtain in the interval, the abovementioned terms, which are other than the pseudo-Euclidian interval?

Since the time of Poincaré, the planetary motion in the solar system is considered on the basis of perturbation theory. Recall also that on the basis of perturbation theory many problems of quantum field theory are solved . Is it possible to use this method in this case, selecting as the initial state the one that is given by Newton's theory and adding members, which follow from the relativistic corrections?

To the analysis of this pathway we will devote a separate study. In the next section we will consider the question about the source equation of gravitational field, which can replace HEE equation.

5.0. The equation of gravity of Hilbert-Einstein as a generalization of the wave equation

Thus, we have seen that the motion of bodies in a gravitational field is described by the Hamilton-Jacobi equation if the source is calculated according to the theory of Hilbert-Einstein
Now remember that according to optical-mechanical analogy it is assumed that the HJE associated with wave equation.

On the other hand, in quantum theory the particle-wave duality implies that the Hamilton-Jacobi equation is associated with a wave of an elementary particle, and hence - with the wave equation of particles.

At the same time, we noted (see (Kyriakos, 2012a)), in the theory of Maxwell-Lorentz that the d'Alembert wave equation contains a source of the electromagnetic field and allows to calculate the electromagnetic field of this source. Similarly, the gravity equation of Hilbert-Einstein contains as a source of the gravitational field, the generalization of mass in the form of the energy-momentum tensor, which allows to calculate the corresponding gravitational field.

The question arises**: is it possible to compare in the gravitational theory, the Hamilton-Jacobi equation with a wave equation?**

By analogy with the abovementioned facts we can assume that the Hilbert-Einstein equations is a generalization of the tensor wave equation in covariant record. In other words, we can assume that the covariant HJE and the HEE are two sides of the optical-mechanical analogy (or of dualism wave-particle) in Gravity.

Are there any results to prove this assumption? Yes, indeed, such results exist.
According to (Fock, 1964, p. 194) “the equation:

$$g^{\mu\nu} \frac{\partial S}{\partial x_\mu} \frac{\partial S}{\partial x_\nu} = 0 , \tag{3.5.1}$$

for the propagation of a gravitational wave-front is the same as the corresponding equation for the front of a light wave in empty space on which the whole theory of space and time, starting from the generally covariant form of Maxwell's equations. Briefly one can say that gravitation is propagated with the speed of light as EM waves”.

Thus (Fock, 1964) “*we see that Einstein's equations are of the type of the wave equation, because their main terms involve the d'Alembert operator*”.

In many textbooks on the theory of gravity is shown that the equation of the HEE in the Newtonian approximation is the inhomogeneous wave equation of D'Alembert:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi_g}{\partial x^2} - \frac{\partial^2 \varphi_g}{\partial y^2} - \frac{\partial^2 \varphi_g}{\partial z^2} = 4\pi\gamma\rho_g , \tag{3.5.2}$$

where φ_g is connected with g_{00} by the relationship

$$g_{00} = 1 - \frac{2\gamma M}{c_2 r} \equiv 1 - \frac{2\varphi_g}{c_2}, \qquad (3.5.3)$$

This assumption does not also conflict the alternative gravitation theory of A. Logunov, which gives the same results as the Hilbert-Einstein theory. Indeed, the basic equation of this theory can be represented in the form of a wave equation with a source, like a wave equation of the EM field (see (Logunov, 2002; Kyriakos, 2012a)).

Let us remember also that from the nonlinear theory of elementary particles (NTEP) follows that the sources of the EM field and the gravitational field (electric charge or mass) arise in a nonlinear wave equation of particles. This gives us a reason to look for the equation of gravity as a generalization of the nonlinear wave equations of elementary particles.

Further research on this issue, will be continue in future articles.

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