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Article

$M^8 - M^4 \times CP_2$ duality, preferred extremals, criticality, and Mandelbrot fractals

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Abstract

$M^8 - M^4 \times CP_2$ duality represents an intriguing connection between number theory and TGD but the mathematics involved is extremely abstract and difficult so that I can only represent conjectures. In the following the basic duality is used to formulate a general conjecture for the construction of preferred extremals by iterative procedure. What is remarkable and extremely surprising is that the iteration gives rise to the analogs of Mandelbrot fractals and space-time surfaces can be seen as fractals defined as fixed sets of iteration. The analogy with Mandelbrot set can be also seen as a geometric correlate for quantum criticality.

1 $M^8 - H$ duality briefly

$M^8 - M^4 \times CP_2$ duality [5, 4] states that certain 4-surfaces of M^8 regarded as a sub-space of complexified octonions can be mapped in a natural manner to 4-surfaces in $M^4 \times CP_2$: this would mean that $M^4 \times CP_2$ and therefore also the symmetries of standard model would have purely number theoretical meaning.

Consider a distribution of two planes $M^2(x)$ integrating to a 2-surface $\tilde{M}^2$ with the property that a fixed 1-plane M^1 defining time axis globally is contained in each $M^2(x)$ and therefore in $\tilde{M}^2$. M^1 defines real axis of octonionic plane M^8 and M^2(x) a local hyper-complex plane. Quaternionic subspaces with this property can be parameterized by points of CP_2: this leads to $M^8 - H$ duality as can be shown by a simple argument.

1. Hyper-octonionic subspace of complexified octonions is obtained by multiplying octonionic imaginary units by commuting imaginary unit. This does not bring anything new as far as automorphisms are considered so that it is enough to consider octonions (so that M^2 is replaced with C). Octonionic frame consists of orthogonal octonionic units. The space of octonionic frames containing sub-frame spanning fixed C is parameterized by $SU(3)$. The reason is that complexified octonionic units can be decomposed to the representations of $SU(3) \subset G_2$ as $1 + 1 + 3 + \overline{3}$ and the sub-frame 1+1 spans the preferred C.

2. The quaternionic planes H are represented by frames defined by four unit octonions spanning a quaternionic plane. Fixing $C \subset H$ means fixing the 1+1 part in the above decomposition. The sub-group of $SU(3)$ leaving the plane H invariant can perform only a rotation in the plane defined by two quaternionic units in 3. This sub-group is $U(2)$ so that the space of quaternionic planes $H \supset C$ is parameterized by $SU(3)/U(2) = CP_2$.

3. Therefore quaternionic tangent plane $H \supset C$ can be mapped to a point of CP_2. In particular, any quaternionic surface in E^8, whose tangent plane at each point is quaternionic and contains C, can be mapped to $E^4 \times CP_2$ by mapping the point $(e_1, e_2) \in E^4 \times E^4$ to $(e_1, s) \in e^4 \times CP_2$. The generalization from E^8 to M^8 is trivial. This is essentially what $M^8 - H$ duality says.

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This can be made more explicit. Define quaternionic surfaces in M^8 as 4-surfaces, whose tangent plane is quaternionic at each point x and contains the local hyper-complex plane $M^2(x)$ and is therefore labelled by a point $s(x) \in CP_2$. One can write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4 = \cup_{x \in \tilde{M}^2} X^2(x) \subset E^6 \ .$$

These surfaces can be mapped to surfaces of $M^4 \times CP_2$ via the correspondence $(m(x), e(x)) \rightarrow (m, s(T(X^4(x)))$. Also the image surface contains at given point x the preferred plane $M^2(x) \supset M^1$. One can also write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4 = \cup_{x \in \tilde{M}^2} X^2(x) \subset E^2 \times CP_2 \ .$$

One can also ask what are the conditions under which one can map surfaces $X^4 = \cup_{x \in \tilde{M}^2} X^2 \subset E^2 \times CP_2$ to 4-surfaces in M^8. The map would be given by $(m, s) \rightarrow (m, T^4(s)$ and the surface would be of the form as already described. The surface X^4 must be such that the distribution of 4-D tangent planes defined in M^8 is integrable and this gives complicated integrability conditions. One might hope that the conditions might hold true for preferred extremals satisfying some additional conditions.

One must make clear that the conditions discussed above do not allow most general possible surface.

1. The point is that for preferred extremals with Euclidian signature of metric the M^4 projection is 3-dimensional and involves light like projection. Here the fact that light-like line $L \subset M^2$ spans M^2 in the sense that the complement of its orthogonal complement in M^8 is M^2. Therefore one could consider also more general solution ansatz for which one has

$$X^4 = \cup_{x \in L(x) \subset \tilde{M}^2} X^3(x) \subset E^2 \times CP_2 \ .$$

2. One can also consider co-quaternionic surfaces as surfaces for which tangent space is in the dual of a quaternionic subspace. This says that the normal bundle rather than tangent bundle is quaternionic. The space-time regions with Euclidian signature of induced metric correspond naturally to co-quaternionic surfaces. Quaternionic surfaces are maximal associative sub-manifolds of octonionic space and one of the key ideas of the number theoretic vision about TGD is that associativity (co-associativity) defines the dynamics iof space-time surfaces. That this dynamics gives preferred extremals of Kähler action remains to be proven.

2 The integrability conditions

The integrability conditions are associated with the expression of tangent vectors of $T^(X^4)$ as a linear combination of coordinate gradients ∇m^k, where m^k denote the coordinates of M^8. Consider the 4 tangent vectors $e_{i)}$ for the quaternionic tangent plane (containing $M^2(x)$) regarded as vectors of M^8. $e_{i)}$ have components $e^k_{i)}$, $i = 1, .., 4$, $k = 1, ..., 8$. One must be able to express $e_{i)}$ as linear combinations of coordinate gradients ∇m^k:

$$e^k_{i)} = e^\alpha_{i)} \partial_\alpha m^k \ .$$

Here x^α and e^k denote coordinates for X^4 and M^8. By forming inner products of of $e_{i)}$ one finds that matrix $e^\alpha_{i)}$ represents the components of vierbein at X^4. One can invert this matrix to get $e^{i)}_\alpha$ satisfying $e^{i)}_\alpha e^\beta_{i)} = \delta^\beta_\alpha$ and $e^{i)}_\alpha e^\alpha_{j)} = \delta^i_j$. One can solve the coordinate gradients ∇m^k from above equation to get

$$\partial_\alpha m^k = e^{i)}_\alpha e^k_{i)} \equiv E^k_\alpha \ .$$

The integrability conditions follow from the gradient property and state

$$D_\alpha E^k_\beta = D_\beta E^k_\alpha \ .$$

One obtains $8 \times 6 = 48$ conditions in the general case. The slicing to a union of two-surfaces labeled by $M^2(x)$ reduces the number of conditions since the number of coordinates m^k reduces from 8 to 6 and one has 36 integrability conditions but still them is much larger than the number of free variables- essentially the six transversal coordinates m^k.

For co-quaternionic surfaces one can formulate integrability conditions now as conditions for the existence of integrable distribution of orthogonal complements for tangent planes and it seems that the conditions are formally similar.

3 How to solve the integrability conditions and field equations for preferred extremals?

The basic idea has been that the integrability condition characterize preferred extremals so that they can be said to be quaternionic in a well-defined sense. Could one imagine solving the integrability conditions by some simple ansatz utilizing the core idea of $M^8 - H$ duality? What comes in mind is that M^8 represents tangent space of $M^4 \times CP_2$ so that one can assign to any point (m, s) of 4-surface $X^4 \subset M^4 \times CP_2$ a tangent plane $T^4(x)$ in its tangent space M^8 identifiable as subspace of complexified octonions in the proposed manner. Assume that $s \in CP_2$ corresponds to a fixed tangent plane containing $M^2(x)$, and that all planes $M^2(x)$ are mapped to the same standard fixed hyper-octonionic plane $M^2 \subset M^8$, which does not depend on x. This guarantees that s corresponds to a unique quaternionic tangent plane for given $M^2(x)$.

Consider the map $T \circ s$. The map takes the tangent plane T^4 at point $(m, e) \in M^4 \times E^4$ and maps it to $(m, s_1 = s(T^4)) \in M^4 \times CP_2$. The obvious identification of quaternionic tangent plane at (m, s_1) would be as T^4. One would have $T \circ s = Id$. One could do this for all points of the quaternion surface $X^4 \subset E^4$ and hope of getting smooth 4-surface $X^4 \subset H$ as a result. This is the case if the integrability conditions at various points $(m, s(T^4)(x)) \in H$ are satisfied. One could equally well start from a quaternionic surface of H and end up with integrability conditions in M^8 discussed above. The geometric meaning would be that the quaternionic surface in H is image of quaternionic surface in M^8 under this map.

Could one somehow generalize this construction so that one could iterate the map $T \circ s$ to get $T \circ s = Id$ at the limit? If so, quaternionic space-time surfaces would be obtained as limits of iteration for rather arbitrary space-time surface in either M^8 or H. One can also consider limit cycles, even limiting manifolds with finite-dimension which would give quaternionic surfaces. This would give a connection with chaos theory.

1. One could try to proceed by discretizing the situation in M^8 and H. One does not fix quaternionic surface at either side but just considers for a fixed $m_2 \in M^2(x)$ a discrete collection X $\{(T^4_i\} \supset M^2(x)$ of quaternionic planes in M^8. The points $e_{2,i} \subset E^2 \subset M2 \times E^2 = M^4$ are not fixed. One can also assume that the points $s_i = s(T^4_i)$ of CP_2 defined by the collection of planes form in a good approximation a cubic lattice in CP_2 but this is not absolutely essential. Complex Eguchi-Hanson coordinates ξ^i are natural choice for the coordinates of CP_2. Assume also that the distances between the nearest CP_2 points are below some upper limit.

2. Consider now the iteration. One can map the collection X to H by mapping it to the set $s(X)$ of pairs $((m_2, s_i)$. Next one must select some candidates for the points $e_{2,i} \in E^2 \subset M^4$ somehow. One can define a piece-wise linear surface in $M^4 \times CP_2$ consisting of 4-planes defined by the nearest neighbors of given point $(m_2, e_{2,i}, s_i)$. The coordinates $e_{2,i}$ for $E^2 \subset M^4$ can be chosen rather freely. The collection $(e_{2,i,i}\,)$ defines a piece-wise linear surface in H consisting of four-cubes in the simplest case. One can hope that for certain choices of $e_{2,i}$ the four-cubes are quaternionic and that there

is some further criterion allowing to choose the points $e_{2,i}$ uniquely. The tangent planes contain by construction $M^2(x)$ so that the product of remaining two spanning tangent space vectors (e_3, e_4) must give an element of M^2 in order to achieve quaternionicity. Another natural condition would be that the resulting tangent planes are not only quaternionic but also as near as possible to the planes T_i^4. These conditions allow to find $e_{2,i}$ giving rise to geometrically determined quaternionic tangent planes as near as possible to those determined by s^i.

3. What to do next? Should one replace the quaternionic planes T_i^4 with geometrically determined quaternionic planes as near as possible to them and map them to points s_i slightly different from the original one and repeat the procedure? This would not add new points to the approximation, and this is an unsatisfactory feature.

4. Second possibility is based on the addition of the quaternionic tangent planes obtained in this manner to the original collection of quaternionic planes. Therefore the number of points in discretization increases and the added points of CP_2 are as near as possible to existing ones. One can again determine the points $e_{2,i}$ in such a manner that the resulting geometrically determined quaternionic tangent planes are as near as possible to the original ones. This guarantees that the algorithm converges.

5. The iteration can be stopped when desired accuracy is achieved: in other words the geometrically determined quaternionic tangent planes are near enough to those determined by the points s_i. Also limit cycles are possible and would be assignable to the transversal coordinates e_{2i} varying periodically during iteration. One can quite well allow this kind of cycles, and they would mean that e_2 coordinate as a function of CP_2 coordinates characterizing the tangent plane is many-valued. This is certainly very probable for solutions representable locally as graphs $M^4 \to CP_2$. In this case the tangent planes associated with distant points in E^2 would be strongly correlated which must have non-trivial physical implications. The iteration makes sense also p-adically and it might be that in some cases only p-adic iteration converges for some value of p.

It is not obvious whether the proposed procedure gives rise to a smooth or even continuous 4-surface. The conditions for this are geometric analogs of the above described algebraic integrability conditions for the map assigning to the surface in $M^4 \times CP_2$ a surface in M^8. Therefore $M^8 - H$ duality could express the integrability conditions and preferred extremals would be 4-surfaces having counterparts also in the tangent space M^8 of H.

One might hope that the self-referentiality condition $s \circ T = Id$ for the CP_2 projection of (m, s) or its fractal generalization could solve the complicated integrability conditions for the map T. The image of the space-time surface in tangent space M^8 in turn could be interpreted as a description of space-time surface using coordinates defined by the local tangent space M^8. Also the analogy for the duality between position and momentum suggests itself.

Is there any hope that this kind of construction could make sense? Or could one demonstrate that it fails? If s would fix completely the tangent plane it would be probably easy to kill the conjecture but this is not the case. Same s corresponds for different planes $M^2(x)$ to different point tangent plane. Presumably they are related by a local G_2 or $SO(7)$ rotation. Note that the construction can be formulated without any reference to the representation of the imbedding space gamma matrices in terms of octonions. Complexified octonions are enough in the tangent space of M^8.

4 Connection with Mandelbrot fractal and fractals as fixed sets for iteration

The occurrence of iteration in the construction of preferred extremals suggests a deep connection with the standard construction of 2-D fractals by iteration - about which Mandelbrot fractal [2, 3] is the canonical example. $X^2(x)$ (or $X^3(x)$ in the case of light-like $L(x) \subset M^2(x)$) could be identified as a union of orbits for the iteration of $s \circ T$. The appearance of the iteration map in the construction of solutions of field equation would answer positively to a long standing question whether the extremely beautiful mathematics of 2-D fractals could have some application at the level of fundamental physics according to TGD.

X^2 (or X^3) would be completely analogous to Mandelbrot set in the sense that it would be boundary separating points in two different basis of attraction. In the case of Mandelbrot set iteration would take points at the other side of boundary to origin on the other side and to infinity. The points of Mandelbrot set are permuted by the iteration. In the recent case $s \circ T$ maps X^2 (or X^3) to itself. This map need not be diffeomorphism or even continuous map. The criticality of X^2 (or X^3) could be seen as a geometric correlate for quantum criticality.

In fact, iteration plays a very general role in the construction of fractals. Very general fractals can be defined as fixed sets of iteration and simple rules for iteration produce impressive representations for fractals appearing in Nature. The book of Michael Barnsley [1] gives fascinating pictures about fractals appearing in Nature using this method. Therefore it would be highly satisfactory if space-time surfaces would be in a well-defined sense fixed sets of iteration. This would be also numerically beautiful aspect since fixed sets of iteration can be obtained as infinite limit of iteration for almost arbitrary initial set. This construction recipe would also give a concrete content for the notion measurement resolution at the level of construction of preferred extremals.

What is intriguing is that there are several very attractive approaches to the construction of preferred extremals [4]. The challenge of unifying them still remains to be met.

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